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Lecture notes for FYS610 Many particle Quantum Mechanics

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Additions and comments to *Quantum Field Theory and the Standard Model* by Matthew D. Schwartz (2014)

Lorentz and Poincaré transformations

In the previous notes we discussed the rotation group. If we add the translations, we obtain the 3-dimensional Euclidean group. We already know from classical mechanics that a free point particle, which by definition has no properties, and so transforms as a scalar ($s = 0$) under rotations about its own position, in general has a conserved orbital angular momentum $\mathbf{L} = \mathbf{X} \times \mathbf{P}$ when moving through space. When we quantize the theory, we find from $[X^i, P^j] = \mathrm{i}\delta_{ij}$ that the components of the orbital angular momentum operator satisfy the Lie algebra of the rotation group, eq. (12.16):

$$[L^i, L^j] = \mathrm{i}\epsilon_{ijk} L^k \, . \tag{12.17}$$

$\mathbf{L}$ operates on space time coordinates, and in standard spherical coordinates r, θ, ϕ we have the following operator expressions:

$$L^3 = \frac{1}{\mathrm{i}} \frac{\partial}{\partial \phi}\, , \qquad \mathbf{L}^2 = -\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] . \tag{14.1}$$

Since the L^i's satisfy the rotation group Lie algebra, the eigenvalues of $\mathbf{L}^2$ must be of the form $l(l+1)$, with integer or half integer values for l, and L^3 has eigenvalues $m \in -l, -l+1 \ldots l$. But it is a well known result of the theory of differential equations that $\mathbf{L}^2$ only have normalizable eigenfunction for *integer* l. These eigenfunctions are the well known *spherical harmonics*, $Y_l^m(\theta, \phi)$. Their normalized form with the Condon–Shortley phase convention most commonly used in quantum mechanics is:

$$Y_l^m(\theta,\phi) = \langle \theta, \phi | l, m\rangle = \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\, P_l^{|m|}(\cos\theta) e^{\mathrm{i}m\phi} \begin{cases} (-1)^m & m \geq 0\, ; \\ 1 & m < 0\, ; \end{cases} \tag{14.2}$$

Here $P_l^m(\cos\theta)$ is an associated Legendre polynomial, which is a polynomial in $\cos\theta$ and $\sin\theta$.

For a particle with spin, the orbital angular momentum operator $\mathbf{L}$ acts on the spatial basis vectors $|\,\theta, \phi\,\rangle$ or $|\,l, m_l\,\rangle$, while the spin operator, $\mathbf{S}$, acts on the internal spin states, with basis vectors $|\,s, m_s\,\rangle$, so $[L^i, S^j] = 0$. Thus, the composite system can be described in the direct product basis $\{|\,l, m_l\,\rangle|\,s, m_s\,\rangle\}$, disregarding other degrees of freedom. But, as noted in the previous lecture, this is not irreducible under rotations. According to eq. (13.26), these states can be expanded in Clebsh–Gordan series as:

$$|\,l, m_l\,\rangle|\,s, m_s\,\rangle = \sum_{j=|l-s|}^{l+s} \sum_{m_j=-j}^{j} \langle l s m_l m_s | l s j m_j \rangle |\,j, m_j\,\rangle\, . \tag{14.3}$$

Here, the basis states $|\, j, m_j \,\rangle$ are eigenstates of the *total angular momentum* operators $\mathbf{J}^2$ and J^3:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}\,, \qquad \mathbf{J}^2 |\, j, m_j \,\rangle = j(j+1)\,, \quad J^3 |\, j, m_j \,\rangle = m_j |\, j, m_j \,\rangle\,. \tag{14.4}$$

Note that the Clebsh–Gordan series contains contributions from total angular momenta $|l - s| \leq j \leq l + s$. Since the Lie algebra is unchanged, the subgroup of rotations under the Euclidean group is still Abelian, so the transformation operator in the Euclidean group reads (cf. eqs. (12.9) and (13.4)):

$$U(\boldsymbol{\phi}, \mathbf{a}) = e^{\mathrm{i}\mathbf{a}\cdot\mathbf{P}} e^{\mathrm{i}\boldsymbol{\phi}\cdot\mathbf{J}}\,. \tag{14.5}$$

But beware that the parameterization depends on the ordering of the non-commuting operators (see below). The last basic commutator of the Euclidean group is $[J^i, P^j]$. Since $\mathbf{S}$ is independent of space variables, we have:

$$[J^i, P^j] = [L^i, P^j] = \epsilon_{ikl}[X^k, P^j]P^l = \mathrm{i}\epsilon_{ikl}\delta_{kj}P^l = \mathrm{i}\epsilon_{ijl}P^l\,, \tag{14.6}$$

which expresses that $\mathbf{P}$ transforms as a vector under rotations. In the same way we also find $[J^i, X^j] = \mathrm{i}\epsilon_{ijk}X^k$.

To recover the formulas finite transformations from a Lie algebra, we can use a form of the *Baker–Campbell–Hausdorff* or *iterated commutator* formula, which can be written for any matrices or operators A and B:

$$e^{\lambda B} A e^{-\lambda B} = \sum_n \frac{\lambda^n}{n!}\,[B, A]^{(n)}\,. \tag{14.7}$$

Here A and B are operators, and $[A, B]^{(n)}$ is an *iterated commutator*, defined recursively by:

$$\begin{aligned} &[B, A]^{(0)} = A\,, \qquad [B, A]^{(1)} = [B, A]\,, \\ &[B, A]^{(n)} = [B, [B, A]^{(n-1)}] = [B, [B, \ldots [B, A] \ldots]]\,. \end{aligned} \tag{14.8}$$

Now the rules of manipulating functional series are essentially the same as those for analytical functions, except that we must respect the commutation relations. Of course, the series on the right hand side must converge to be useful, but for our purposes this is not a problem, since we shall always be able to sum it — in many cases it is even finite. This is really a combinatorial result, but an analytical proof, although more restrictive, is simpler. We define an operator-valued function as:

$$f(\lambda) = e^{\lambda B} A e^{-\lambda B} \quad \Longrightarrow \quad f'(\lambda) = \frac{\mathrm{d}f}{\mathrm{d}\lambda} = e^{\lambda B}(BA - AB)e^{-\lambda B} = e^{\lambda B}[B, A]e^{-\lambda B}\,.$$

We can then prove by induction that $f^{(n)}(\lambda) = e^{\lambda B}[B, A]^{(n)}e^{-\lambda B}$, because making this assumption, we find:

$$f^{(n+1)}(\lambda) = e^{\lambda B}[B, [B, A]^{(n)}]e^{-\lambda B} = e^{\lambda B}[B, A]^{(n+1)}e^{-\lambda B}.$$

Since the formula is valid for $n = 0$, it follows by induction for all n. Eq. (14.6) is then an immediate consequence of Taylor's theorem:

$$f(\lambda) = \sum_n \frac{f^{(n)}(0)}{n!} \lambda^n = \sum_n \frac{\lambda^n}{n!} [B, A]^{(n)} ,$$

provided the series converges, which is eq. (14.6).

To see how we use this formula, let us apply it to find the transformation properties of $\mathbf{J}$ under translations. From the translation law of operators, eq. (12.1) (note the placing of $U^\dagger$!):

$$\mathbf{J}' = U(0, \mathbf{a})^\dagger J^i \, U(0, \mathbf{a}) \, \mathbf{e}_i = \mathbf{e}_i \sum_n \frac{(-\mathrm{i})^n}{n!} [a^j P^j, J^i]^{(n)} .$$

But from eq. (14.6) and $[P^i, P^j] = 0$:

$$\begin{aligned} [a^j P^j, J^i] &= -\mathrm{i}\epsilon_{jik} a^j P^k = \mathrm{i}(\mathbf{a} \times \mathbf{P})^i , \\ [a^j P^j, J^i]^{(2)} &= [a^l P^l, \mathrm{i}(\mathbf{a} \times \mathbf{P})^i] = 0 = [a^j P^j, J^i]^{(n)} \quad \text{for } n \geq 2 . \end{aligned}$$

Thus the whole infinite series has only two non-vanishing terms, and we have:

$$\mathbf{J}' = U(0, \mathbf{a})^\dagger \mathbf{J} \, U(0, \mathbf{a}) \mathbf{e}_i = \mathbf{J} + \mathbf{a} \times \mathbf{P} . \tag{14.9}$$

This yields a group theoretical proof of the formula $\mathbf{J} = \mathbf{L} + \mathbf{S}$: Consider a rotational invariant system at a fixed point, where $\mathbf{J} = \mathbf{S}$, as found in lecture notes 13. Then introduce translations and translate the system from the origin to $\mathbf{a} = \mathbf{x}$. Similarly one finds, not unexpectedly, that $\mathbf{P}$ and $\mathbf{J}$ transform as vectors under rotations. We must also beware that since the operators do not commute, the parameterization in eq. (14.5) depends on the operator order:

$$U(\boldsymbol{\phi}, \mathbf{a}) = e^{\mathrm{i}\mathbf{a}\cdot\mathbf{P}} e^{\mathrm{i}\boldsymbol{\phi}\cdot\mathbf{J}} = e^{\mathrm{i}\boldsymbol{\phi}\cdot\mathbf{J}} e^{-\mathrm{i}\boldsymbol{\phi}\cdot\mathbf{J}} e^{\mathrm{i}\mathbf{a}\cdot\mathbf{P}} e^{\mathrm{i}\boldsymbol{\phi}\cdot\mathbf{J}} = e^{\mathrm{i}\boldsymbol{\phi}\cdot\mathbf{J}} e^{\mathrm{i}\mathbf{a}\cdot\mathbf{P}'} = e^{\mathrm{i}\boldsymbol{\phi}\cdot\mathbf{J}} e^{\mathrm{i}\mathbf{a}''\cdot\mathbf{P}} , \tag{14.10}$$

where $P^i = R^{ij} P^j$, and we have used the invariance of the scalar product under rotation to write $\mathbf{a} \cdot \mathbf{P}' = \mathbf{a}'' \cdot \mathbf{P}$ where $a''^i = (R^{-1})^{ij} a^j = a^j R^{ji}$.

We now turn to the Lorentz group. It has two types of elements, the rotations, which forms a subgroup, which we have already analyzed, and the *boosts*, transformations that change the velocity. From the course in classical mechanics, we already know that the boosts do *not* form a subgroup, because the product of two boosts in different directions is not a pure boost, but involves a rotation (see *Goldstein*, sec. 7.3). In discussing boosts, it is useful to introduce the *rapidity*, β, as a parameter. If v is we have $\beta = \mathrm{artanh}(v)$. To see why, it suffices to consider a boost in the 3-direction in the defining, or fundamental, representation of the group. We then have the boost matrix (see S eq. (2.14)):

$$\Lambda^3(\beta) = \begin{pmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \gamma v & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\beta & 0 & 0 & \sinh\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta & 0 & 0 & \cosh\beta \end{pmatrix} . \tag{14.11}$$

This matrix has eigenvalues $e^{\beta}, 1, 1, e^{-\beta}$, *i.e.* the same as the corresponding rotation matrix, except that there is one dimension more, and, more importantly, that the rotation angle has become imaginary. We can still diagonalize Λ^3:

$$\Sigma^3(\beta) = V^{-1}\Lambda^3(\beta)V = \mathrm{Diag}[e^{\beta}, 0, 0, e^{-\beta}]\,. \tag{14.12}$$

But since Λ^3 is neither Hermitean, nor unitary, the matrix of eigenvectors, V, is not a unitary matrix:

$$V = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \qquad V^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \tag{14.13}$$

From the diagonal form of the matrix it is trivial to verify that boosts in the 3-direction are additive in rapidity, and hence form an Abelian subgroup of the Lorentz group: $\Sigma^3(\beta_1)\Sigma\ 3(\beta_2) = \Sigma^3(\beta_1 + \beta_2)$. By transforming back this is true also for Λ^3, and any other representation obtained by a similarity transformation. And since the choice of 3-axis is arbitrary, this is true for boosts in an any direction. But it turns out that it is impossible to find a transformation that transforms all the representation matrices into unitary matrices. And that is not only true for the defining representation, but for representations of any dimension larger than 2: *There are no finite-dimensional unitary representations of the Lorentz group, except for the trivial one.* This turns out to have important consequences for the structure of relativistic quantum field theories.

Nevertheless, it is useful to consider the generators the Lorentz group. An arbitrary Lorentz transformation can be parameterized by a rotation vector, $\boldsymbol{\phi} = \phi\mathbf{n}$, describing rotations about an axis $\mathbf{n}$, and boosts, $\boldsymbol{\beta} = \beta\mathbf{e}$, describing velocity changes of rapidity β in the direction $\mathbf{e}$. A general transformation can thus be written $\Lambda(\boldsymbol{\beta}, \boldsymbol{\phi})$. The generators for rotations are unchanged, except for getting a trivial extension for the time coordinate. Following the convention of writing J^i for the rotation generators of the Lorentz group, eqs. (13.2a) become:

$$J^i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & T^i & \\ 0 & & & \end{pmatrix}. \tag{13.2$'$}$$

For boosts the generators are:

$$K^i = -\mathrm{i}\left(\frac{\partial\Lambda(\boldsymbol{\beta}, \boldsymbol{\phi})}{\partial\beta_i}\right)_{\boldsymbol{\beta}=0, \boldsymbol{\phi}=0}. \tag{14.14}$$

This yields:

$$K^1 = -\mathrm{i}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^2 = -\mathrm{i}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^3 = -\mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{14.15}$$

This shows explicitly that the $K's$ are not Hermitean, and hence not observables, although they are symmetric, and $\mathrm{i}K^i$ are Hermitean. The commutator relations involving J^i are, of course unchanged. For those involving the K^i's we find:

$$[J^i, K^j] = \mathrm{i}\epsilon_{ijk} K^k\,, \qquad [K^i, K^j] = \mathrm{i}\epsilon_{ijk} J^k\,. \tag{14.16}$$

The first of these shows that $\mathbf{K}$ actually transforms like a vector under rotations, which could be expected. The second reflects the fact that two boosts in different directions do not commute, and generate a rotation, as noted above. Since $[J^3, K^3] = 0$, we could now proceed to generate finite dimensional representations of the Lorentz group by extending the construction for the rotation group, but as these representations are not unitary, they are of a somewhat restricted usefulness.

Since $\mathbf{K}$ is a vector and we have found that boosts in a fixed direction form an Abelian subgroup of the Lorentz group, we can write any boost in the defining representation as:

$$\Lambda(\boldsymbol{\beta}, 0) = e^{\mathrm{i}\boldsymbol{\beta}\cdot\mathbf{K}}\,. \tag{14.17}$$

As noted, this operator is not unitary, and so of restricted usefulness for quantum systems.

We can finally attack the full Poincaré group, by adding the translations. These will now include the time translations, which looks like the spatial translations, although it is rather special in that it is parameterized by p^0, the eigenvalue of the Hamiltonian for a non-interacting particle. Although it is irrelevant for the translations alone, when we combine the translations with the Lorentz transformations, we switch the sign of the energy phase relative to the spatial terms, on order to have a Lorentz scalar in the exponent, $e^{-\mathrm{i}p\cdot x}$, with $p \cdot x = p^\mu x_\mu = p^0 x^0 - \mathbf{p}\cdot\mathbf{x}$.

From the composition law of Poincaré transformations, eq. (11.7): $x' = \Lambda x + a$, one can work out the remaining commutators of the Poincaré group, which are:

$$[P^\mu, P^\nu] = 0\,, \qquad [K^i, P^0] = \mathrm{i}P^i\,, \qquad [K^i, P^j] = \mathrm{i}\delta_{ij} P^0\,. \tag{14.18}$$

The second of these equations tells us that a boost changes the momentum, the last that when it changes the momentum in the direction of motion, the energy changes, but when the change in velocity is perpendicular to the momentum (since then the work vanishes).

In summary, any Poincaré transformation on a physical system can be performed by a unitary transformation, and from the above analysis we know that it must be writeable as:

$$U(\boldsymbol{\beta}, \boldsymbol{\phi}, a) = e^{-\mathrm{i}a_\mu P^\mu}\, e^{\mathrm{i}\boldsymbol{\phi}\cdot\mathbf{J}}\, e^{\mathrm{i}\boldsymbol{\beta}\cdot\mathbf{K}}\,, \tag{14.19}$$

but we have not yet found a suitable Hermitean form for $\mathbf{K}$. Again different operators do not commute so the values of the parameters a^μ, $\boldsymbol{\phi}$, $\boldsymbol{\beta}$ depend on the order of the operators. In this case the situation is actually worse than for the rotation group, because the combinations $\boldsymbol{\phi}\cdot\mathbf{J}$ and $\boldsymbol{\beta}\cdot\mathbf{K}$ are scalars under rotations, but have non-trivial transformation properties under boosts. This can be taken care of by introducing a new set of generators, written as an antisymmetric tensor, $M^{\mu\nu} = -M^{\nu\mu}$, defined by:

$$M^{0k} = -M^{k0} = K^k\,, \qquad M^{kl} = \epsilon_{klm} J^m\,. \tag{14.20}$$

This construction is analogous to the formation of $F^{\mu\nu}$ from **E** and **B** in electromagnetism. The corresponding transformation parameters are $\omega^{\mu\nu} = -\omega^{\nu\mu}$:

$$\omega^{0k} = -\omega^{k0} = \frac{1}{2}\beta^k \,, \qquad \omega^{kl} = \frac{1}{2}\epsilon_{klm}\phi^m \,. \tag{14.21}$$

Then one can write:

$$U(\omega^{\mu\nu}, a) = U(\boldsymbol{\beta}, \boldsymbol{\phi}, a) = e^{-\mathrm{i}a_\mu P^\mu}\, e^{\mathrm{i}\omega_{\mu\nu}M^{\mu\nu}} \,. \tag{14.22}$$

Here $\omega_{\mu\nu}M^{\mu\nu}$ transforms as a scalar under Lorentz transformations.

Expressed in terms of $M^{\mu\nu}$ and P^μ the Lie algebra of the Poincaré group can be written:

$$[M^{\mu\nu}, M^{\rho\sigma}] = \mathrm{i}(g^{\mu\sigma}M^{\nu\rho} - g^{\mu\rho}M^{\nu\sigma} + g^{\nu\rho}M^{\mu\sigma} - g^{\nu\sigma}M^{\mu\rho}) \,, \tag{14.23a}$$

$$[M^{\mu\nu}, P^\rho] = \mathrm{i}(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu) \,, \tag{14.23b}$$

$$[P^\mu, P^\nu] = 0 \,. \tag{14.23c}$$