UNIVERSITETET I STAVANGER
Institutt for matematikk og naturvitenskap

Lecture notes for FYS610 Many particle Quantum Mechanics

Note 12, 21.3 2017

Additions and comments to *Quantum Field Theory and the Standard Model* by Matthew D. Schwartz (2014)

Some group theory

In lecture note 11 we found that under a symmetry transformation, S, of space $\mathbf{x} \to \mathbf{x}'$, leaving probabilities unchanged, the state $|\,\psi\,\rangle$ and the wave function $\psi(\mathbf{x})$ must transform as:

$$|\,\psi\,\rangle' = U(S)^{\dagger}|\,\psi\,\rangle\,; \tag{11.2'}$$

$$\psi(\mathbf{x}') = U(S)\psi(\mathbf{x})\,, \tag{11.8b'}$$

respectively, where $U(S)$ is an unitary (anti-)linear operator on the Hilbert space, $\mathcal{H}$. Note in particular the difference in transformation law for the state and for the wave-function. Furthermore, the symmetry operators must have the group property, *i.e.* in particular is S_1 and S_2 are symmetry operations, so is S_i^{-1} and S_2S_1, in standard notation. But because physical states are represented by rays in $\mathcal{H}$, we cannot demand that $U(S_2)U(S_1) = U(S_2S_1)$, but only that:

$$U(S_2)U(S_1) = e^{\mathrm{i}\delta(S_2,S_1)}U(S_2S_1)\,, \tag{11.10'}$$

where $\delta(S_2, S_1)$ is a real function of the group elements S_1 and S_2. We say that the set $\{U(S_i)\}$ forms a projective representation of the symmetry group, while if we can choose the operators $U(S_i)$ such that $\delta(S_2, S_1) = 0$ for all S_1, S_2, we just say that we have a representation. This is the case normally considered in the mathematical theory of group representations. Note that eq. (11.10') remains valid in the case of internal symmetries, with $\mathbf{x}' = \mathbf{x}$.

In order for scalar products to remain unchanged, the transformation of eq. (11.2') corresponds to the operators transforming as:

$$O' = U^{\dagger}(S)OU(S)\,. \tag{12.1}$$

We note that when we apply this formula for $S = S_1S_2$, the phase $\delta(S_2, S_1)$ in eq. (11.10') drops out, so the operators always transform according to a (true) representation:

$$O'' = U(S_1)^{\dagger}O'U(S_1) = U(S_1)^{\dagger}U(S_2)^{\dagger}OU(S_2)U(S_1) = U(S_2S_1)^{\dagger}OU(S_2S_1)\,. \tag{12.2}$$

Note the switch in the order of the operators, which can be traced back to the manipulations in eq. (11.6). This disappearance of the phase ambiguity for operators is actually important when we consider canonical quantizations. This is derived from the Poisson brackets of Hamiltonian dynamics (see lecture note 1), which means that the operators of classical mechanics and Quantum Mechanics have the same symmetry properties. But in Classical Mechanics there is no room for the phases of a projective transformation. This, of course, also means that symmetry properties of quantal systems can be, and often are, more complex than for classical systems.

The operator form of the transformation laws also lets us introduce transformation laws in quantum field theories which include transformations of the time coordinate, in particular Poincaré transformations, since in relativistic field theories space and time are treated on an equal footing. Thus the relativistic version of eq. (11.8b) for a Poincaré transformation $x' = \Lambda x + a$ reads:

$$O(\Lambda x + a) = U(\Lambda, a) O(x) U(\Lambda, a)^\dagger \, . \tag{12.3}$$

In order to use the above formalism, we need to know how to find the operators U. This is actually a large field, with separate textbooks. We shall just present some important results. We shall also first consider true representations, with $\delta = 0$ in eq. (11.10'), and then come back to the extension to projective representations.

First we note that there is no requirement in quantum mechanics that the representation must be *faithful*, *i.e.* that different symmetry transformations are represented by different operators. Thus any group has the trivial representation, $U(S) = \mathbb{1}$ for all elements, S, in the group. This representation obviously satisfies $U(S_2)U(S_1) = U(S_2 S_1)$, and is called the trivial representation of the group. Quantities transforming according to this representation transforms as $\psi(\mathbf{x}') = \psi(\mathbf{x})$, are often called *scalars* in physics.

Before we continue, let us first observe that the eigenvectors of a unitary operator are orthogonal, and hence can be assumed orthonormal. Indeed, if U is unitary, we have:

$$\begin{aligned}
&C = \frac{1}{2}(U + U^\dagger) = C^\dagger \quad S = \frac{1}{2\mathrm{i}}(U - U^\dagger) = S^\dagger \\
&[C, S] = \frac{1}{4i}\left([U, U] - [U, U^\dagger] + [U^\dagger, U] - [U^\dagger, U^\dagger]\right) = 0 \\
&U = C + \mathrm{i}S \quad U^\dagger = C - \mathrm{i}S \, .
\end{aligned} \tag{12.4}$$

Thus, since U can be written as the sum of two commuting hermitean operators, which can be diagonalized simultaneously, it is diagonalized with them, having the same eigenvectors. Furthermore, the eigenvalues are complex numbers of modulus 1, since:

$$U|\,\lambda\,\rangle = \lambda|\,\lambda\,\rangle \quad \Longrightarrow \quad 1 = \langle\lambda|U^\dagger U|\,\lambda\rangle = |\lambda|^2 \, . \tag{12.5}$$

Hence we have shown that one can always write $U = e^{\mathrm{i}H}$ where H is Hermitean, because we can write $\lambda = e^{\mathrm{i}\phi}$, and define H by $H|\,\lambda\,\rangle = \phi|\,\lambda\,\rangle$.

From this construction an important result for *Abelian groups*, *i.e.* groups where all elements commute, follows. Since for such groups $U(S_i)U(S_j) = U(S_j)U(S_i)$ for all S_i, S_j, the matrices $\{U(S_i)\}$ all commute, and can be diagonalized simultaneously. Thus, for an Abelian symmetry group, we can chose a basis in $\mathcal{H}$ such that the basis vectors satisfy:

$$U(S_i)|\,\alpha; k\,\rangle = e^{\mathrm{i}\phi_i(k)}|\,\alpha; k\,\rangle \, . \tag{12.6}$$

If S_0 is the unit element in the group, we have the normalization $\phi_0(k) = 0$ for all k. There will in general be several, or even infinitely many, states with the same k, and these are labelled by α and describes other properties of the system. In this basis group multiplication is simply an addition, so if $s_l S_m = S_n$, we find:

$$\begin{aligned} U(S_n)|\,\alpha;k\,\rangle &= e^{\mathrm{i}\phi_n(k)}|\,\alpha;k\,\rangle = U(S_l S_m)|\,\alpha;k\,\rangle \\ &= U(S_l)U(S_m)|\,\alpha;k\,\rangle = e^{\mathrm{i}[\phi_l(k)+\phi_m(k)]}|\,\alpha;k\,\rangle\,, \end{aligned} \tag{12.7}$$

so $\phi_n(k) = \phi_m(k) + \phi_n(k)$. This is simply eq. (11.10') applied to the basis vectors, with $\delta = 0$. Assuming that we have also orthonormalized the basis in the additional quantum number(s) α, so $\langle\alpha', k'|\alpha; k\rangle = \delta_{\alpha\alpha'}\delta_{kk'}$, we can write $U(S_i)$ as:

$$\begin{aligned} U(S_i) &= \sum_{\alpha' k'}\sum_{\alpha k}|\,\alpha',k'\,\rangle\langle\alpha',k'|U(S_i)|\,\alpha;k\rangle\langle\alpha;k| = \sum_{\alpha' k'}\sum_{\alpha k} e^{\mathrm{i}\phi_i(k)}|\,\alpha',k'\,\rangle\delta_{\alpha\alpha'}\delta_{kk'}\langle\alpha,k| \\ &= \sum_k e^{\mathrm{i}\phi_i(k)}\sum_\alpha |\,\alpha;k\,\rangle\langle\alpha,k| = \sum_k e^{\mathrm{i}\phi_i(k)}P_k = \sum_k U_k(S_i)P_k\,. \end{aligned} \tag{12.8}$$

Here $P_k = \sum_a |\,\alpha;k\,\rangle\langle\alpha,k|$ is simply the projection operator onto the subspace of fixed k in $\mathcal{H}$. Thus we have written $U(S_i)$ as the direct sum over such subspaces. This is called a *reduction* of the unitary representation $U(S)$ of the group to a direct sum of *irreducible representations*, which are labelled by k, which is a quantum number to a physicist. All the irreducible representations are of the form $U_k(S_i) = e^{\mathrm{i}\phi(k)}$ for some function $\phi_i(k)$ of i and k, which remains to be found. This is just a complex number, so we have shown that for any Abelian group the unitary irreducible representations are all one-dimensional. This is a special case of a fundamental result of the theory of group representations, which states that any linear representation of a group can be written as a direct sum of irreducible representations, like in eq. (12.8). The difference for non-Abelian groups is that $U_k(S_i)$ will be matrices, in the case of the Poincaré group and other non-compact groups even infinite-dimensional matrices, *i.e.* operators.

The most important examples of Abelian groups for us are groups of translations. Let us first consider translations in one dimension, so eq. (11.9') takes the form $\psi(x + a) = U(a)\psi(x)$. We clearly have $U(a)U(b) = U(b)U(a) = U(a + b)$, so this is an Abelian group. Let us only consider continuous and differentiable representations — this is more or less required in Quantum Mechanics anyhow. Then $f(a, k) = \phi_a(k)$ is a differentiable function of a satisfying $f(0, k) = 0$ and $f(a + b, k) = f(a, k) + f(b, k)$. Differentiating the last equation, we have for any b:

$$\frac{\partial f}{\partial a}(a,k) + \frac{\partial f}{\partial a}(b,k) = \frac{\partial f}{\partial a}(a,k) = \frac{\partial f}{\partial a}(a+b,k) = c_k$$

In the first step we have used that $f(b, k)$ does not depend on a, and in the last that since $a + b$ is arbitrary, neither does $\partial f/\partial a(a + b, k)$. Hence $f(a, b) = c_k a$. But k is just a label for the eigenstates of $U(a)$, and we are free to redefine it as long as different eigenstates get different labels. Thus, we may chose $c_k = k$, so we simply have $\phi_a(k) = f(a, k) = ak$. Hence, the irreducible representations of the one-dimensional translation group are simply:

$$U_k(a) = e^{\mathrm{i}ka}\,. \tag{12.9}$$

The simplest transformation properties have, not unexpectedly, those wavefunctions which are eigenstates of $U(a)$ itself. Assuming a fixed α, for $|\,\psi\,\rangle = |\,k\,\rangle$ we have:

$$\psi_k(x) = \langle x|k\rangle = \langle 0|U(x)|\,k\rangle = e^{\mathrm{i}kx}\langle 0|k\rangle = \psi_k(0)e^{\mathrm{i}kx}\,. \tag{12.10}$$

Here $\psi_k(0)$ is just a normalization constant. The correct transformation properties follow trivially:

$$\psi_k(x+a) = \psi_k(0)e^{\mathrm{i}k(x+a)} = U_k(a)\psi_k(x)\,. \tag{12.11}$$

We see that this state $|\,\psi_k\,\rangle$ transforms according to an irreducible representation of the translation group. Furthermore, if we calculate the conserved momentum from Noether's theorem, for a translation invariant Lagrangian, we indeed find $P = k$. Thus we may interpret k as the momentum of the state ψ_k. It is often taken as a postulate that the simplest physical systems, like elementary particles, *must* transform according to irreducible representations of all their symmetry groups, because otherwise it is always possible to consider a reducible system to be composed of several simpler ones.

If a system has several commuting abelian symmetries, one can repeat the above construction for each of them. For translations, one arrives at the generalization of (12.9), with ka interpreted as a scalar product. It is also clear that for translations, the group transformation law (11.10') is satisfied for $\delta = 0$, and there are no problems with projective representations. But for rotations in two dimensions, which we discussed in lecture note 11, such representations appear. By repeating the construction above, we find the irreducible representations for any real k:

$$U_k(\phi) = e^{\mathrm{i}k\phi}\,. \tag{12.12}$$

But because ϕ and $\phi \pm 2n\pi$ is the same physical angle for any integer n, and $U_k(\phi + 2n\pi) = e^{\mathrm{i}2nk\pi}U(\phi) \neq U(\phi)$ for any irrational k an even most rational values, this is a projective representation unless k is an integer. This is therefore the only values that appear in classical theories, but quantum theories are not so restrictive.

We are now turning to non-abelian symmetries. In that case the operators for different symmetry transformations in general do not commute. This makes it impossible to represent all transformations as a direct sum of one-dimensional representations. But a generalization of eq. (12.8) still applies. One can always find bases in $\mathcal{H}$ of the direct product form $\{|\,\alpha;k\,\rangle|\,k;\kappa\,\rangle\}$, such that the symmetry operators only transform the states $\{k;\kappa\}$ among themselves. In such a basis $U(S_i)$ can be written as:

$$U(S_i) = \sum_k \sum_{\kappa,\kappa'} |\,k;\kappa'\,\rangle D^k_{\kappa'\kappa}(S_i)\langle k;\kappa\,|P_k\,. \tag{12.13}$$

Here $D^k_{\kappa\kappa'}$ is a square matrix. Furthermore, the set of these matrices forms a representation of the group:

$$D^k(S_2)D^k(S_1) = D^k(S_2S_1)\,, \tag{12.14}$$

where matrix multiplication is understood. The index k again labels the representation. For finite groups, *i.e.* groups with a finite number of elements, the matrices are always finite dimensional, and there are always unitary representations of the group.

For our purposes, it is the *continuous groups* that are of the largest interest. We shall restrict ourselves to *Lie groups*, which have group elements that are differentiable functions of a finite number of parameters. Thus, a rotation in the (proper) rotation group in 3 dimensions, SO(3), has 3 parameters, which can be taken to be the rotation angle and the direction of the rotation axis, or alternatively three Euler angles. The Lorentz group, SO(3,1), is an extension of the rotation groups, with three additional parameters, which can be taken to be the component of the boost parameter along three orthogonal axes. Finally, the Euclidean group in 3 dimensions and the Poincaré group are extensions of the rotation group and the Lorentz group to include translations in three or four dimensions, respectively. There is one important difference between the rotation group and the Lorentz group. The parameter space of the rotation group is a *compact* space, because the parameters, which can be chosen as $\boldsymbol{\phi} = \phi\mathbf{n}$, with $0 \leq \phi < 2\pi$ and $\mathbf{n}$ is a unit vector, so $|\phi|$ fits into a sphere in 3-space of radius 2π. For compact groups like this, there always exists finite-dimensional unitary representations and it turns out that these are all we need in physics. But the Lorentz, and the Poincaré groups are not compact, and for them there exist no finite dimensional unitary representations.

When analyzing Lie groups, it is very useful to introduce *generators*. We have already noted that any unitary operator U can be written $U = e^{\mathrm{i}H}$, where H is Hermitean. It will be convenient to write the parameters of the group as a vector, $\mathbf{a}$. Then, if $U(\mathbf{a}) = U(S(\mathbf{a}))$ represents some element of the group, $S(\mathbf{a})$, we can write $U(\mathbf{a}) = e^{\mathrm{i}H(\mathbf{a})}$. Furthermore, it is convenient to parameterize the group so that the unit element is $S_0 = S(0)$, so $U(0) = \mathbb{1}$. The *generators* of the group are then Hermitean operators defined as:

$$T^i = \frac{1}{\mathrm{i}}\left(\frac{\partial U(\mathbf{a})}{\partial a_i}\right)_{\mathbf{a}=0} = \left(\frac{\partial H}{\partial a_i}\right)_{\mathbf{a}=0} = T^{i\dagger}\,. \tag{12.15}$$

Since T^i is a unitary operator, it is a quantum mechanical observable, and its eigenvectors can be used as basis vectors. In the particular case of the translation group, which is also a Lie group in any number of dimensions, one immediately finds from eq. (12.9) that $T^i = k_i$, the components of the momentum operator.

Generators are convenient, because they can be used to express the structure of a non-Abelian Lie group in a compact way. One can work out that they satisfy a non-associative commutator algebra, called a *Lie Algebra*, which can be written:

$$[T^i, T^j] = \mathrm{i}c_{ijk}T^k\,, \tag{12.16}$$

where the constants c_{ijk}, called the *structure constants* of the group, can be calculated from the group composition rule. Of course, the particular form of the generators depends both on the choice of the parametrization, *i.e.* on α, and on the particular choice of $U(\alpha)$. But one can show that different choices just results in linear transformations among the T^i's. For Abelian groups one of course has $c_{ijk} = 0$, since all the U's, and hence the generators, commute. But for non-Abelian groups this is not the case. In particular, for the rotation group, with $\boldsymbol{\phi}$ as parameters, where one mostly writes $T^i = J^i$, of $T^i = S^i$, one finds:

$$[T^i, T^j] = \mathrm{i}\epsilon_{ijk}T^k \quad \Longleftrightarrow \quad \mathbf{J} = \frac{1}{\mathrm{i}}\mathbf{J}\times\mathbf{J}\,, \tag{12.17}$$

where ϵ_{ijk} is the Levi–Civita tensor.