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Lecture notes for FYS610 Many particle Quantum Mechanics

Note 11, 15.3 2017

Additions and comments to *Quantum Field Theory and the Standard Model* by Matthew D. Schwartz (2014)

Symmetry transformations

A *symmetry transformation* on a quantum mechanical Hilbert space, $\mathcal{H}$, is a mapping $|\Psi\rangle \to |\psi'\rangle$ such that all probabilities, *i.e.* squared scalar products, are invariant:

$$|\langle\phi'|\psi'\rangle|^2 = |\langle\phi|\psi\rangle|^2\,, \tag{11.1}$$

for all vectors $|\phi\rangle$ and $|\psi\rangle$ in $\mathcal{H}$. Wigner's theorem states that if this is the case, there exist a unitary linear or antilinear operator on $\mathcal{H}$ such that for all $|\psi\rangle$ we have:

$$|\psi'\rangle = U^\dagger|\psi\rangle\,. \tag{11.2}$$

We write $U^\dagger = U^{-1}$ here to have slightly moe convenient formulas later.

An operator, A, is called antiunitary if it satisfies:

$$A(\alpha|\psi\rangle + \beta|\phi\rangle) = \alpha^* A|\psi\rangle + \beta^*|\phi\rangle\,, \tag{11.3}$$

for all complex numbers α and β. In practice antilinear unitary, or *antiunitary*, operators only occur in connection with time reversal, $t \to -t$. The reason is most simply seen by observing that under time reversal an energy eigenstate in the Schrödinger picture should transform as:

$$e^{-\mathrm{i}Et}|\psi_E\rangle \quad\longrightarrow\quad Ae^{-\mathrm{i}Et}|\psi_E\rangle = e^{+\mathrm{i}Et}|\psi_E\rangle\,, \tag{11.4}$$

but no operators in $\mathcal{H}$ act directly on t. But an antiunitary A does switch the sign of the phase.

From the above, it immediately follows that symmetry transformations are invertible, and that two successive such transformations is also a symmetry transformation, with an operator that is the product of the operators for each transformation. Hence the set of all symmetry transformations on a system forms a *group*. But this general group is far too large to be interesting, and in practice, we are only interested in specific subgroups.

It is important to note, though, that a symmetry transformation U is not unique, because physical systems are defined by *rays* in $\mathcal{H}$ (see lecture notes #2), so $e^{\mathrm{i}\phi}U$ for some real number ϕ represents the same physical transformation as U. This has important consequences.

A symmetry transformation need not transform all vectors in $\mathcal{H}$. In particular, a transformation is called local, or *internal*, if it leaves the position eigenstates unchanged (up to a phase):

$$|\,\mathbf{x}'\,\rangle = U^\dagger|\,\mathbf{x}\,\rangle = e^{\mathrm{i}\phi}|\,\mathbf{x}\,\rangle\,, \tag{11.5}$$

This means that the wavefunction satisfies:

$$\psi'(\mathbf{x}') = \langle\mathbf{x}'|\psi'\rangle = e^{-\mathrm{i}\phi}\langle\mathbf{x}|U^\dagger|\,\psi\rangle = e^{-\mathrm{i}\phi}U^\dagger\psi(\mathbf{x})\,. \tag{11.6}$$

For a single transformation we can trivially redefine U so as to make $\phi = 0$. However, in interesting cases, U is member of a set of operators which has a group structure, and then we shall see that one cannot always remove this phase factor.

For our purposes, we are most interested in the case where the symmetries are space-time symmetries, representing Poincaré transformations, *i.e.* Lorentz transformations and translations. Under such a transformation, the 4-coordinates transforms as (see *Goldstein* ch. 7.2, *Schwartz* ch. 2.1):

$$x^\mu \;\to\; x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu\,, \tag{11.7}$$

where $\Lambda^\mu{}_\nu$ is a Lorentz transformation matrix, satisfying $\Lambda^{\mathsf{T}} g \Lambda = g$ with g as the metric, while a^μ is a translation. Because boosts mix space and time, Lorentz transformations are not easily implemented in a quantum mechanical setting, where positions are represented by operators while time is a parameter. For the moment we shall therefore restrict ourselves to just rotations and space translations, which together form the *Euclidean group* in 3 dimensions. Then the transformation law is $\mathbf{x} \;\to \mathbf{x}' = R\mathbf{x} + \mathbf{a}$, with R as a rotation matrix, $R^{-1} = R^{\mathsf{T}}$, we have:

$$\psi'(\mathbf{x}') = \langle\mathbf{x}'|\psi'\rangle = \langle R\mathbf{x}{+}\mathbf{a}|U^\dagger(R,\mathbf{a})|\,\psi\rangle = U^\dagger(R,\mathbf{a})\psi(R\mathbf{x}{+}\mathbf{a}) = \langle\mathbf{x}|\psi\rangle = \psi(\mathbf{x})\,. \tag{11.8a}$$

Since U is unitary, this can be written

$$\psi(R\mathbf{x} + \mathbf{a}) = U(R,\mathbf{a})\psi(\mathbf{x})\,. \tag{11.8b}$$

Performing too successive transformations we find:

$$\mathbf{x}'' = R_2\mathbf{x}' + \mathbf{a}_2 = R_2R_1\mathbf{x} + R_1\mathbf{a}_2 + \mathbf{a_1}\,, \tag{11.9a}$$

so we have the group multiplication rule:

$$(R_2,\mathbf{a}_2)(R_1,\mathbf{a}_1) = (R_2R_1, R_1\mathbf{a}_2 + \mathbf{a_1})\,. \tag{11.9b}$$

At this stage it is tempting to conclude that from eq. (1.1) in the form $|\langle\phi''|\psi''\rangle|^2 = |\langle\phi'|\psi'\rangle|^2 = |\langle\phi|\psi\rangle|^2$ it follows that:

$$U(R_2,\mathbf{a_2})U(R_1,\mathbf{a_1}) \stackrel{?}{=} U(R_2R_1, R_1\mathbf{a}_2 + \mathbf{a_1})\,.$$

If this is the case, we say that the set of matrices $U(R,\mathbf{a})$ forms a *representation of the group*. But we cannot conclude this, because physical states are only represented by rays in $\mathcal{H}$. All we can be sure of so far is that this relation is true up to a phase:

$$U(R_2,\mathbf{a_2})U(R_1,\mathbf{a_1}) = e^{\mathrm{i}\delta(R_2,\mathbf{a}_2;R_1,\mathbf{a}_1)}U(R_2R_1, R_1\mathbf{a}_2 + \mathbf{a_1})\,, \tag{11.10}$$

where $\delta(R, \mathbf{a}; 1, 0) = \delta(1, 0; R, \mathbf{a}) = 0$ if we fix $U(1, 0) = \mathbb{1}$. We see that the matrices $U(R, \mathbf{a})$ only reproduce the group multiplication up to a phase. We say that they form a *projective representation* of the group.

The question is if it is possible to chose a phase convention for the U's such that δ can be simplified further. It is not difficult to find an example where this phase cannot be transformed away. Consider the group SO(2), the group of rotations in 2-dimensional space. It is parameterized by a single angle, the rotation angle ϕ, which takes values in the interval $[0, 2\pi\rangle$. The group multiplication rule is simply $(\phi_2)(\phi_1) = (\phi_1 + \phi_2 \bmod 2\pi)$. This group is isomorphous to the group U(1) of complex numbers of modulus 1, as is seen by just mapping $\phi \to e^{\mathrm{i}\phi}$. Now consider the operators:

$$U_\alpha(\phi) = e^{\mathrm{i}\alpha\phi}\,\mathbb{1}\,. \tag{11.11}$$

Here α is the eigenvalue of the angular momentum of the state. It is well known that for $\alpha = m = 0, \pm1, \pm2\ldots$, these form representations of O(2) and U(1). If $\phi_1 + \phi_2 = \phi_3 + 2k\pi$, with $0 \le \phi_1, \phi_2, \phi_3 < 2\pi$ and $k = 0$ or 1, we have:

$$U_m(\phi_2)U_m(\phi_1) = e^{\mathrm{i}m(\phi_1+\phi_2)} = e^{\mathrm{i}m\phi_3}e^{\mathrm{i}2\pi mk}\mathbb{1} = e^{\mathrm{i}m\phi_3}\mathbb{1}\,, \tag{11.12}$$

which shows that the operators $U_m(\phi)$ forms a representation of U(1). If $m = 0$ we have the *trivial representation*, where all transformations are represented by the unit operator. But if α is non-integer, we only have a representation up to a phase:

$$U_m(\phi_2)U_m(\phi_1) = e^{\mathrm{i}m(\phi_1+\phi_2)} = e^{\mathrm{i}m\phi_3}e^{\mathrm{i}2\pi\alpha k}\mathbb{1} = e^{\mathrm{i}\delta(\phi 2,\phi_1)}e^{\mathrm{i}\alpha\phi_3}\mathbb{1}\,,$$

where $\delta = (2k\pi - \phi_2 - \phi_1)\alpha \neq 2n\pi$ for $k = 1$ and any integer n for most values of $\phi_1 + \phi_2$. This phase cannot be removed by any phase conventions. Indeed *anyons*, two-dimensional particles with statistics intermediary between fermions and bosons, originally introduced by Jon Magne Leinås and Jan Myrheim, can have angular momentum of any α. However, in three dimensions, it turns out that only half-integer and integer values of α is allowed. Furthermore, the spin-statistics theorem (see *Schwartz*, ch. 12) requires that integer spin particles must be bosons, half-integer spin particles fermions. We shall come back to the simplest case, $m = \frac{1}{2}$.

To extend this analysis to quantum field theories, we observe that since U is unitary, the transformation of an operator O under the transformation of eq. (11.2) is:

$$O' = U^\dagger O U\,.$$

In quantum field theory we can freely mix positions and times, since they are parameters on an equal footing. For a space-time symmetry, a similar analysis to the one leading to eqs. (11.8) above then yields, for a Poincaré transformation of a field operator

$$\phi(x') = U(\Lambda, a)\phi(x)U^\dagger(\Lambda, a)\,.$$

It remains to find how these operators actually can be implemented.