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# Lecture notes for FYS610 Many particle Quantum Mechanics

**Note 10, 14.3 2017**

**Additions and comments to *Quantum Field Theory and the Standard Model* by Matthew D. Schwartz (2014)**

## An introduction to partial summation and renormalization

The Feynman diagrams are first and foremost a tool for doing perturbation theory. But there are many cases that it would be very useful, or even necessary, to be able to go beyond perturbation theory. In Problem 15 (*Schwartz* problem 7.4) we saw that if we add a mass term as a perturbation to the Lagrangian density, we can sum the perturbation series for the propagator *exactly*, arriving back at the Schwinger-Dyson equation. In this case, we can even solve this, and find the massive propagator from the massless one. This technique of summing the perturbation series, at least partially, has some very useful generalization.

The starting point for the derivation of the Feynman rules in the Hamiltonian formulation was the Dyson series, *i.e.* the expansion of the exponential functions in the formula

$$\langle \Omega | T\{\phi(x_1)\cdots\phi(x_n)\}|\,\Omega\rangle = \frac{\langle 0|T\{\phi_0(x_1)\cdots\phi_0(x_n)e^{\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}_I[\phi_0]}\}|\,0\rangle}{\langle 0|T\{e^{\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}_I[\phi_0]}\}|\,0\rangle}\,, \qquad S\,(7.63)$$

in powers of the coupling constant(s). We further know that we can take care of the denominator in this expression by simply disregard *all* disconnected diagrams in the ensuing series, so we will just drop these in the following. If we use Wick's theorem and then Fourier transform the resulting series term by term, we find:

$$\mathcal{T}(p_1 \ldots p_n) = \sum_{\text{Connected diagrams}} (\text{Feynmandiagrams})\,. \qquad (10.1)$$

Here $p_1 \ldots p_n$ are the *external momenta*, and we have dropped the disconnected diagrams in accordance with the discussion in *Schwartz* sec. 7.3.2. In this expansion the propagators on the external legs are retained, they are removed when using the LSZ formula.

In a translation-invariant theory, momentum will be conserved in the 2-point function to all order in perturbation theory according to Noether's theorem, so we can write

$$\mathcal{T}(p_1, p_2) = (2\pi)^4 \delta^4(p_1 - p_2) D_F(p_1)\,, \qquad (10.2)$$

where we call $D_F(p)$ the full propagator. To the zeroth order in perturbation theory we then have:

$$D_F(p) \approx D_F^0(p) = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}\,, \qquad (10.3)$$

where $F_F^0(p)$ is called the *free propagator*. Diagrammatically:

$$\underset{p}{\Longrightarrow} \approx \underset{p}{\longrightarrow} \, .$$

The arrows on the propagators here are just decorative in the case of neutral fields. This diagrammatic notation simplifies many manipulation with Feynman series. Let us redo a slightly generalized version of problem 15 (*Schwartz* problem 7.4), where we add a term $\frac{1}{2}\delta m^2\phi^2$ to $\mathcal{L}_I$. This perturbation gives rise to the additional Feynman diagram:

$$\underset{\delta m^2}{-\!\bullet\!-} \, .$$

Note that there are no propagators on the "legs" of such an interaction diagram. It just contributes a factor $\delta m^2$ to the adjoining propagators. We then recover the Schwinger-Dyson equation from the Feynman rules:

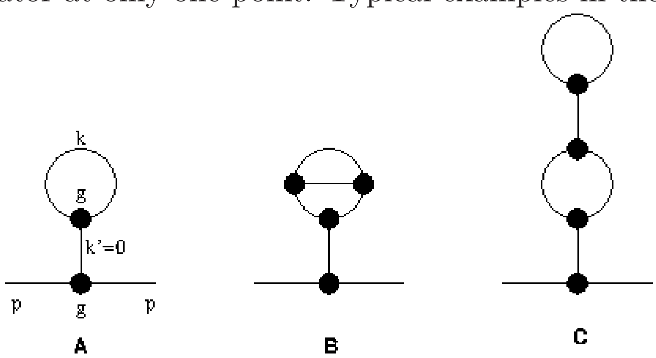
$$\begin{aligned}
\underset{p}{\Longrightarrow} &= \underset{p}{\longrightarrow} + \underset{p}{\longrightarrow}\underset{\delta m^2}{\bullet}\underset{p}{\longrightarrow} + \underset{p}{\longrightarrow}\underset{\delta m^2}{\bullet}\underset{p}{\longrightarrow}\underset{\delta m^2}{\bullet}\underset{p}{\longrightarrow} \cdots \\
&= \longrightarrow\left[\sum_n \left(\bullet\!\!\longrightarrow\right)^n\right] = \frac{\longrightarrow}{1 - \ -\!\bullet\!\!\longrightarrow} \\
&= \frac{1}{\left(\longrightarrow\right)^{-1} - -\!\bullet\!-} = \frac{\mathrm{i}}{p^2 - m^2 - \delta m^2 + \mathrm{i}\epsilon} \, .
\end{aligned} \tag{10.4}$$

Thus, we see that the mass has been *renormalized*, $m^2 \to m^2 + \delta m^2$ by this interaction. As in problem 15, we could even have avoided summing the geometric series, if we had noted that from the Feynman rules it follows that except for the free propagator, any diagram in the perturbation series has a first vertex after the initial propagator. When exiting that vertex, the remaining possible diagrams are exactly the same as for the full series. Hence, $\Longrightarrow$ satisfies the Schwinger Dyson equation, which can be immediately solved.

$$\begin{aligned}
\Longrightarrow &= \longrightarrow + \longrightarrow\!\bullet\!\!\Longrightarrow = \longrightarrow + \Longrightarrow\!\bullet\!\!\longrightarrow \\
\Longrightarrow &= \frac{\longrightarrow}{1 - \ \longrightarrow\!\bullet} = \frac{\longrightarrow}{1 - \ -\!\bullet\!\!\longrightarrow} \, .
\end{aligned} \tag{10.5}$$

The last set of equations are derived by starting with the last free propagator, instead of the first.

The same approach can be used to sum over all *tadpole diagrams*, *i.e.* diagrams connecting to a propagator at only one point. Typical examples in the $\phi^3$ theory are:



We see that momentum conservation ensures that the line connecting these diagrams to the propagator carries zero momentum. Hence, if the diagrams are finite, they all have a constant value of the form:

$$\mathrm{i}gD_F^0(0)b_i = \frac{g}{m^2 - \mathrm{i}\epsilon}b_i\,, \tag{10.6}$$

for some constant $b_i$, which depends on the diagram. Thus, for diagram A in the figure, we have:

$$b_A = ig\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{i}{k^2 - m^2 + \mathrm{i}\epsilon}\,. \tag{10.7}$$

This integral is actually divergent in all space-time dimensions, not only 4, even if $m^2 > 0$. This is actually a critical observation, for it seems to make the Feynman diagram approach doomed to fail. However, we shall disregard this catastrophic failure for a moment, and assume that all the $b_i$'s to be finite. This can actually happen in non-relativistic many-particle theories. The contribution from *all* the tadpole diagrams inserted at a specific point in a Feynman diagram will then give a total contribution:

$$B = \frac{g}{m^2 - \mathrm{i}\epsilon}\sum_i b_i\,. \tag{10.8}$$

If we make the even stronger assumption than before, that this infinite sum over $i$ also converges, so $B$ is finite, and calculable, we easily find the contribution from all tadpole diagrams to the propagator. This sum is precisely the same as the one calculated in eqs. (10.3) and (10.5), with the trivial substitution $\delta m^2 \to B$. This is because if we consider any number of tadpoles of any type, they will be connected to the propagator at the black dots in one of the diagrams in eq. (10.4). This means that the effect of the tadpole diagrams is simply to *renormalize* the particle mass, $m^2 \to m_R^2 = m^2 + B$. In many-body theories this renormalization can be finite, showing that a particle passing through an interacting medium has a different effective mass than it has outside, which is an experimentally observed effect for electrons in solids.

The case when $b_i$, or more accurately $B$ diverges, is more tricky. To handle this case we must first introduce a *regularization procedure*. This is a prescription which modifies the Feynman integrals in some way by introducing a parameter, such that when the parameter goes to some limit, the original integral is recovered, while in some other range of the parameter, the integral is finite. The idea is to carry out the evaluation of the regularized Feynman diagrams, and then let the parameter go to its limit where the regularization is removed, hoping that the physical quantities calculated remain finite, and independent of the renormalization procedure. Many such schemes are or have been in use, all having advantages and disadvantages. *Schwartz* prefers *dimensional regularization*, where the momentum integrals are continued analytically into a *complex* number of space time dimensions, $4 \to d$ (!). This is preferably done after the integration over $k^0$ is *Wick rotated*, *i.e.* rotated in the complex plane to run paarallell to the imaginary axis (see *Schwartz*, appendix B.3.

For illustrative purposes we shall just introduce a simple momentum cut-off, a so-called *ultraviolet cut-off*, assuming that all momenta and energies are smaller than some maximal momentum parameter $\Lambda \gg m$, which ultimately should be taken to infinity. It is often used for simple qualitative considerations but it has the obvious disadvantage that translation-invariance is lost, which makes it unsuitable for general use. In that case the parameter $B = B_\Lambda$ will depend on $\Lambda$, but be finite. The renormalized mass will then also depend on lambda, $m_R^2(\Lambda) = m^2 + B_\lambda$. But this will not solve any problems, because when we let $\lambda \to \infty$, $m_R(\Lambda)$ will diverge, and nothing is gained. But in a fundamental theory, the *bare mass*, $m$, is not accessible to experiments anyhow, in contrast to in condensed matter physics or nuclear physics, where our particle can leave the surrounding medium, and have their masses measured. We may therefore assume that also the bare mass depends on $\lambda$, so $m^2 \to m_\lambda^2$, and then take the limit such that:

$$m_R^2 = m_\lambda^2 + B_\lambda \to m_R^2 \qquad \text{as} \qquad \Lambda \to \infty\,.$$

Thus we assume that the unobservable $m_\lambda$ also diverges as $\lambda \to \infty$. The price we have paid is that there is no way actually to calculate the value of the mass in the theory.

Based on the above, the presription for handling the tadpole diagrams is then simply to ignore them, but to insert the renormalized, and hence observed, mass in all propagators.

It turns out that in general, not only the mass, but also at least the coupling constant(s) and the wavefunction of a relativistic quantum theory needs to be renormalized. However, it has been rigorously shown that in favorable cases, including QED and the standard model, only a finite number of parameters need to be renormalized. When this has been done, all other observable quantities are, in principle, calculable and finite. Theories with this property are called *renormalizable.* It turns out that renormalizability is a very stringent constraint on possible quantum field theories. When it was introduced in QED around 1950, the renormalization procedure was met with much scepticism, in spite of the phenomenological success of that theory. However, it has been extremely successful in selecting theories which agree with experiments. Only General Relativity of the currently accepted fundamental theories is not renormalizable, needing an infinite number of renormalization conditions as a quantum field theory.

In addition to ultraviolet divergences, field theories may also have infrared-divergences, in particular when the momentum goes to zero. This in particular plagues massless theories, like QED. These divergences are connected to real physics accessible to experiments, and must be circumvented by special methods.