UNIVERSITETET I STAVANGER

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Suggested solutions, FYS 500 — Classical Mechanics Theory 2017 fall

Set 7 for 6. October 2017

PROBLEM 39:

Exam problem 1i, 2014 spring. See separate solution sheet.

PROBLEM 40:

a) The orbits of a particle of mass m in a potential $V = -k/r$ for a circle and a parabola of fixed angular momentum ℓ follows from the general solution of *Goldstein* sect. 3.6 for eccentricities $e = 0$ and $e = 1$ as:

$$r = \frac{\ell^2}{mk}\frac{1}{1+e\cos\theta} = \frac{\ell^2}{mk}\begin{cases} 1 & \text{circle, } e=0; \\ \frac{1}{1+\cos\theta} & \text{parabola, } e=1. \end{cases}$$

Here we have assumed that $\theta = 0$ at the turning point for the parabola. For the latter we have $r_t = r_0/2$, where $r_0 = l^2/mk$ is the radius of the circular orbit.

b) From the definition of eccentricity, $e = \sqrt{1 + 2E\ell^2/mk^2}$, one sees that a parabolic orbit with $e = 1$ has $E = 0$. This is also easily seen by evaluating $E = m\dot{r}^2/2 + \ell^2/2mr^2 - k/r$ at the turning point, $r = r_t$. Since E is conserved, we have at any point of a parabolic orbit ($v_p = |\dot{\mathbf{r}}|$):

$$E = \frac{1}{2}m\,v_p(r)^2 - \frac{k}{r} = 0 \qquad \Longrightarrow \qquad v_p(r) = \sqrt{\frac{2k}{mr}}\,.$$

For a circular orbit we have $r(t) = r_0$ and $\dot{r} = 0$, so:

$$v_c^2 = r_0^2\dot{\theta}^2 = \frac{l^2}{m^2 r_0^2} = \frac{k}{mr_0}\,,$$

with r_0 from part a, so we have $v_p(r_0) = \sqrt{2}v_c$. [Since most comets are moving in almost parabolic orbits, this means that they all have essentially the same speed as they cross the Earth's orbit, namely 42.1 km/s, since the Earth's orbit is almost circular, with $v_c = 29.8$ km/s.]

PROBLEM 41:

a) The orbit equation reads in this case:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{m}{\ell^2}\frac{\mathrm{d}}{\mathrm{d}u}V\left(\frac{1}{u}\right) = \frac{m}{\ell^2}\frac{\mathrm{d}}{\mathrm{d}u}\left(ku - \frac{1}{2}hu^2\right) = \frac{m}{\ell^2}(k - hu)\,,$$

or

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + (1 + \frac{mh}{\ell^2})u = \frac{mk}{\ell^2}.$$

This is an inhomogeneous harmonic equation, with homogeneous solution $u = A\cos(\beta(\theta - \theta'))$ and a particular solution $u = mk/l^2\beta^2$. Here $\beta = \sqrt{1 + mh/\ell^2}$, and A and θ' are constants of integrations. Thus we have:

$$u = A\cos(\beta(\theta - \theta')) + \frac{mk}{\ell^2\beta^2} \qquad \Longrightarrow \qquad r = \frac{c}{1 + e\cos(\beta(\theta - \theta'))}\,,$$

where $c = \ell^2\beta^2/km$, while $e = A\ell^2\beta^2/km$ is another unknown constant. We see that r reaches its minimum value, *i.e.* the inner turning point, when $\theta = \theta'$. Measuring θ from this point, we have $\theta' = 0$.

b) We have already found c and β. The discussion of the qualitative nature of the orbits is essentially identical to that of the Kepler problem, see sect. 3.7 of *Goldstein*. For $e \geq 1$ we have that $u = 0$ ($r = \infty$) for $\cos(\beta\theta) = -1/e$, and we have unbound motion, with a single turning point. If $0 < e < 1$ we have bound orbits with two turning points, and if $e = 0$ we have a circular orbit of radius c. However, the orbits for $e > 0$ are *not* conical sections, unless $\beta = 1$, *i.e.* $h = 0$.

c) For an orbit to close, it must be bound, so $0 \leq e < 1$. If $e = 0$ it is a circle, which is certainly closed. Otherwise it is closed if $r(\theta + 2n\pi) = r(\theta)$, for some integer n, which means that $\cos(\beta\theta + 2n\beta\pi) = \cos(\beta\theta)$, But the cosine function is periodic with period 2π, so this happens if and only if $2n\beta\pi = 2m\pi$ for some integer m, *i.e.* if $\beta = m/n$, a rational number.

PROBLEM 42:

a) See the proposed solution for the previous problem with $k = 0$. We still have $\beta = \sqrt{1 + mh/\ell^2}$.

b) The angle Ψ between the asymptote of the incoming particle and the turning point, which we have fixed at $\theta = 0$ in this problem. We thus need to find the value of the angle $\Psi > 0$, such that $r(\Psi) = \infty$ (we also will have $r(-\Psi) = \infty$, by the symmetry of the orbit about the turning point). The scattering angle is $\Theta = \pi - 2\Psi$ (see *Goldstein* sect. 3.10). This means that, since $\ell = msv_0$ and $\beta = \sqrt{1 + h/ms^2v_0^2}$:

$$\cos(\beta\Psi) = 0 \quad \Leftrightarrow \quad \Psi = \frac{\pi}{2\beta} \quad \Rightarrow \quad \Theta(s) = \pi - 2\Psi = \pi\left(1 - \frac{1}{\beta}\right) = \pi\left(1 - \frac{1}{\sqrt{1 + h/ms^2v_0^2}}\right).$$

If $h = 0$, we have $\beta = 1$ and therefore $\Theta = 0$, as we should. Furthermore, $l = 0$ when $s = 0$, so $\beta \to \infty$ when $s \to 0$, and thus $\Theta \to \pi$ when $s \to 0$, so in this case we find the correct solution also for $l = s = 0$.

c) Solving $\Theta = \Theta(s)$ for $s(\Theta)$ we find after some algebra:

$$s(\Theta) = \frac{1}{v_0}\sqrt{\frac{h}{m}}\frac{\pi - \Theta}{\sqrt{\Theta(2\pi - \Theta)}}.$$

Since $\Theta(s)$ is monotonous, $\mathrm{d}\Theta/\mathrm{d}s \neq 0$ for any finite s, we find the cross section from the definition (see *Goldstein* sect. 3.10):

$$\sigma(\Theta) = \frac{s}{\sin\Theta}\left|\frac{\mathrm{d}s}{\mathrm{d}\Theta}\right| = \frac{h}{mv_0^2}\frac{1}{\sin\Theta}\frac{\pi^2(\pi - \Theta)}{\Theta^2(2\pi - \Theta)^2}.$$

[We see that $\sigma(\Theta)$ diverges for $\Theta \to 0$, but not quite as strongly as the Rutherford cross section. We have no divergence (no glory scattering) for $\theta -> \pi$.]

PROBLEM 43:

a) The orbit equation for $u(\theta) = 1/r(\theta)$ is unchanged from Problem 42:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + \beta^2 u = 0,$$

with $\beta^2 = 1 + mh/l^2 = 1 - |h|/mv_0^2s^2$. For $\beta^2 > 0$ we thus have the same solution:

$$r = \frac{r_t}{\cos(\beta\theta)}.$$

The critical value $s = s_c$ is given by $\beta = \beta(s_c) = 0$, which leads to:

$$s_c = \sqrt{\frac{|h|}{m}}\frac{1}{v_0}.$$

We see that we can write $\beta^2 = 1 - (s_c/s)^2$.

b) We proceed exactly as in the previous problem to find the angle where $r(\Psi) = \infty$ for $\Psi > 0$. As before $\Psi = \pi/2\beta$, so the scattering angle is:

$$\Theta(s) = \pi - 2\Psi = \pi\left(1 - \frac{1}{\beta}\right) = \pi\left(1 - \frac{1}{\sqrt{1 - |h|/ms^2v_0^2}}\right).$$

Now, since $\beta < 1$ we have $\Theta < 0$, as expected in an attractive potential. To have orbiting, the particle must go completely around the center of force at least once, which means that $\Theta \leq -2\pi$. Using $\beta^2 = 1 - (s_c/s)^2$ we find that this will happen if:

$$1 - 1/\beta < -2 \qquad \Longleftrightarrow \qquad \beta < \frac{1}{3} \qquad \Longleftrightarrow \qquad s = \frac{s_c}{\sqrt{1-\beta^2}} < \frac{3\sqrt{2}}{4} s_c = \frac{3}{2v_0}\sqrt{\frac{|h|}{2m}}\,.$$

Thus for impact parameters $s_c < s \leq (3\sqrt{2}/2)\, s_c \approx 1.06\, s_c$ we have orbiting.

c) If $s = s_c$, we have $\beta = 0$, and the solution of the orbit equation is simply:

$$u = a(\theta - \theta_0) \qquad \Longleftrightarrow \qquad r = \frac{c}{\theta - \theta_0}\,,$$

where $c = 1/a$ and θ_0 are constants of integration. We can always choose coordinates such that $\theta_0 = 0$. Then $\theta = 0$ corresponds to $r = \infty$. We see that r decreases monotonously with increasing $|\theta|$, and reaches $r = 0$ when $\theta = \pm\infty$, *i.e.* after infinitely many orbits around the center of force.

[It is not difficult to show that if the particle starts with angular momentum $l = mv_0 s_c$ at a *finite* distance from the center, it will reach $r = 0$ in a *finite* time, although it circles the center of force an infinite number of times, and so reaches an infinite *angular* velocity.]

d) In this case we have $\beta^2 = -\gamma^2 < 0$, and the solution of the orbit equation can be written:

$$u(\theta) = \frac{1}{r(\theta)} = A\exp(\gamma\theta) + B\exp(-\gamma\theta)\,.$$

To have a scattering solution, we must allow $r \to \infty$, *i.e.* $u = 0$. Choosing coordinates such that $\theta \to 0$ as $r \to \infty$ leads to $A + B = 0$, so we find:

$$r(\theta) = \frac{c}{\sinh(\gamma\theta)}\,,$$

where $c = 1/2A$. This also yields a single-valued $\theta(r)$, with orbits generally very similar to those discussed in the previous part.