UNIVERSITETET I STAVANGER

Institutt for matematikk og naturvitenskap

Suggested solutions, FYS 500 — Classical Mechanics Theory 2017 fall

Set 6 for 29 September 2017

Problem 32:

It is assumed that the frictional force on a particle is given by $\mathbf{f_i} = -k_i\dot{\mathbf{r}}_\mathbf{i}$, If we write the total force $\mathbf{F}_i = \mathbf{F}'_i + \mathbf{f}'_i$, the derivation of eq. (3.24) in *Goldstein* remains unchanged, and we find:

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \mathbf{p}_i \cdot \mathbf{r}_i = 2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i = 2T + \sum_i \mathbf{F}'_i \cdot \mathbf{r}_+ \sum_i \mathbf{f}_i \cdot \mathbf{r}_i = 2T + \sum_i \mathbf{F}'_i \cdot \mathbf{r}_+ \sum_i k_i\dot{\mathbf{r}}_i \cdot \mathbf{r}_i \,.$$

Taking the average of this equation over a time τ, we find:

$$\frac{1}{\tau}\int_0^\tau \frac{\mathrm{d}G}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{\tau}[G(\tau) - G(0)] = 2\overline{T} + \overline{\sum_i \mathbf{F}' \cdot \mathbf{r}_i} + \overline{\sum_i k_i\dot{\mathbf{r}}_i \cdot \mathbf{r}_i} \,.$$

The last average can be evaluated as:

$$\overline{\sum_i k_i\dot{\mathbf{r}}_i \cdot \mathbf{r}_i}\,.' = \frac{1}{\tau}\sum_i k_i \int_0^\tau \dot{\mathbf{r}}_i \cdot \mathbf{r}_i\,\mathrm{d}t = \frac{1}{2\tau}\sum_i k_i \int_0^\tau \frac{\mathrm{d}\mathbf{r}_i^2}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{2\tau}\sum_i k_i\left[\mathbf{r}_i(\tau)^2 - \mathbf{r}_i(0)^2\right] \,,$$

which approaches zero as $\tau \to \infty$ if $r_i(\tau)$ is bounded. Thus the result is proven.

Problem 33:

The turning points are at the perihelion distance, $r_1 = a(1-e)$, and the aphelion distance, $r_2 = a(1+e)$, where a is the semimajor axis. Eliminating a between these two relations we have:

$$r_2 = r_1\frac{1+e}{1-e} = 35.2\,\mathrm{AU} \,.$$

The period follows from Kepler's third law $T^2 = 4\pi^2a^3/GM_\odot$, where $a = r_1/(1-e)$, G Newton's constant of gravitation and $M_\odot$ the Sun's mass. Strictly speaking, $M_\odot$ should be replaced by $M_\odot + m$ with m as the comet mass, but this is a negligible correction. Looking up $M_\odot$ and G we easily calculate T. If we also neglect the Earth's mass, we can make an even simpler calculation, since we know the Earth's period of revolution, $T_\oplus = 1\,\mathrm{year}$ and semimajor axis, $a_\oplus = 1\,\mathrm{AU}$. Thus:

$$T = \left(\frac{a}{a_\oplus}\right)^{3/2} T_\oplus = \left(\frac{r_1}{(1-e)\,a_\oplus}\right)^{3/2} T_\oplus = 76\,\mathrm{years} \,.$$

Problem 34:

In the absence of external forces, the transformed equation for the orbit can be written:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = 0 \,,$$

which is the equation for harmonic motion. The general solution is:

$$u(\theta) = A\cos(\theta - \delta) = B\cos\theta + C\sin\theta \,,$$

where A and δ are constants of integration, $B = A\cos\delta$ and $C = B\sin\delta$. Since $x = r\cos\theta$, $y = r\sin\theta$ and $r = 1/u$, we find

$$Bx + Cy = \frac{B\cos\theta + C\sin\theta}{B\cos\theta + C\sin\theta} = 1 \,,$$

which is the equation for a straight line. The geometric solution is left to the students.

Problem 35:

Exam problem 1, 2014 fall. See separate solution sheet.

Problem 36:

Exam problem 1, 2014 spring. See separate solution sheet.

Problem 37:

With

$$x = r\cos\theta = \frac{(1-e^2)\cos\theta}{1+e\cos\theta}a\,, \qquad y = r\sin\theta = \frac{(1-e^2)\sin\theta}{(1+e\cos\theta)}a$$

we find

$$\begin{aligned}\frac{(x+ea)^2}{a^2} - 1 &= \left(\frac{(1-e^2)\cos\theta}{1+e\cos\theta} + e\right)^2 - 1 = \left(\frac{(1-e^2)\cos\theta + e + e^2\cos\theta}{1+e\cos\theta}\right)^2 - 1 = \left(\frac{e+\cos\theta}{1+e\cos\theta}\right)^2 - 1 \\ &= \frac{e^2 + 2e\cos\theta + \cos^2\theta - 1 - 2e\cos\theta - e^2\cos^2\theta}{(1+e\cos\theta)^2} = \frac{(1-e^2)(\cos^2\theta - 1)}{(1+e\cos\theta)^2} \\ &= -\frac{(1-e^2)\sin^2\theta}{(1+e\cos\theta)^2} = -\frac{y^2}{b^2}\,.\end{aligned}$$

where $b = \sqrt{1-e^2}\,a$ is the *semiminor* axis of the ellipse.

Problem 38:

We have a circular orbit if the effective potential $V'(r) = -k/r + l^2/2\mu r^2$ has its minimum, where μ is the reduced mass of the two particles. This leads to Kepler's third law which can be solved for the orbital period as;

$$\tau = 2\pi\sqrt{\frac{\mu r^3}{k}}\,.$$

When the particles are stopped, which we take to be at the time $t = 0$, they start to fall toward each other with initial conditions $r(0) = r_0$, $\dot r(0) = 0$, $\theta(0) = 0$ and $\dot\theta(0) = 0$. We thus have the angular momentum as $l = \mu r_0^2\dot\theta(0) = 0$ while the energy is $E = \frac{1}{2}\mu\dot r^2 - k/r = -k/r_0$. This yields:

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{1}{\dot r} = -\frac{1}{\sqrt{\frac{2}{\mu}\left(E + \frac{k}{r}\right)}} = -\frac{1}{\sqrt{\frac{2k}{\mu}\left(\frac{1}{r} - \frac{1}{r_0}\right)}}\,.$$

Note that we need the *negative* square root here, because r decreases as t increases, so $\dot r < 0$. The two particles hit each other when $r = 0$, which takes a time:

$$\tau' = -\sqrt{\frac{\mu}{2k}}\int_{r_0}^{0}\frac{\mathrm{d}r}{\sqrt{\left(\frac{1}{r} - \frac{1}{r_0}\right)}} = \sqrt{\frac{\mu r_0}{2k}}\int_0^{r_0}\frac{\sqrt{r}\,\mathrm{d}r}{\sqrt{r_0 - r}}$$

If we substitute $r = r_0\sin^2 w$ we have $\mathrm{d}r = 2r_0\sin w\cos w$, so

$$\tau' = \sqrt{\frac{2\mu r_0^3}{k}}\int_0^{\pi/2}\sin^2 w\,\mathrm{d}w = \frac{\pi}{2}\sqrt{\frac{\pi\mu r_0^3}{2k}} = \frac{\tau}{4\sqrt{2}}\,.$$

[If this had been an exam problem, the r-integral would have been given.]