UNIVERSITETET I STAVANGER

Institutt for matematikk og naturvitenskap

Suggested solutions, FYS 500 — Classical Mechanics Theory 2017 fall

Set 5 for 22. September 2017

Problem 27:

A string can only support stretching, *i.e.* a positive tension, but not a negative one. Thus if we introduce a Lagrangian multiplier to constrain the length of the string, the corresponding constraining force on the particle, Q_r, must act inward as a centripetal force, so $Q_r < 0$ (a string cannot push the particle). Using cylindrical coordinates r, θ in the obvious manner, a Lagrangian multiplier λ is introduced to enforce the constraint $r - \ell = 0$. Taking the zero of the potential energy at $r = 0$, we find the constrained Lagrangian $\widehat{L}$ as:

$$\widehat{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta - \lambda(r - \ell)$$

The variation with respect to r then yields:

$$-\frac{\delta\widehat{L}}{\delta r} = m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0\,.$$

The constraint $r = \ell$ implies $\dot{r} = \ddot{r} = 0$, so this equation reduces to:

$$Q_r = \lambda = mg\cos\theta + m\ell\dot{\theta}^2\,,$$

and the string tension vanishes when $Q_r = 0$. Thus λ is indeed equal to the string tension, derived by elementary means as the sum of the component of gravity along the string and the centrifugal “force”. We see that Q_r only becomes negative if $\theta > \pi/2$ for small enough $\dot{\theta}$. As we increase $\omega = \dot{\theta}(0)$ from zero, this will happen for a value of ω just large enough for the particle to reach the horizontal position, *i.e.* $Q_r = 0$ for $\dot{\theta} = 0$ at $\theta = \pi/2$. Since the energy,

$$E = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta\,,$$

is conserved, calculating E both at $\theta = 0$ and $\theta = \pi/2$ we find that this happens for a value $\omega = \omega_-$ given by:

$$E = \frac{1}{2}m\ell^2\omega_-^2 - mg\ell = -mg\ell\cos\frac{\pi}{2} = 0 \qquad \Longrightarrow \qquad \omega_- = \sqrt{\frac{2g}{\ell}}\,.$$

[Remark: If ω is further increased, the particle gets higher and higher before $Q_r = 0$, until it can reach the vertical position, $\theta = \pi$ with an angular velocity $\omega_m = \dot{\theta}\big|_{\theta=\pi}$ such that $Q_r\big|_{\theta=\pi} = 0$, or $m\ell\omega_m^2 = -mg\cos\pi = mg$. From energy conservation this happens when $\omega = \omega_+$, given by:

$$E = \frac{1}{2}m\ell^2\omega_+^2 - mg\ell = \frac{1}{2}m\ell^2\omega_m^2 - mg\ell\cos\pi = \frac{3}{2}mg\ell \qquad \Longrightarrow \qquad \omega_+ = \sqrt{\frac{5g}{\ell}}\,.$$

For $\omega_- < \omega \leq \omega_+$ the string will be slack at some point. For $\omega > \omega_+$ the particle will circulate indefinitely with a stretched string, in the absence of friction. Note that we do not need to solve for the motion of the pendulum, which is fortunate, since the exact solution requires some advanced mathematical analysis.]

Problem 28:

a) The potential energy is the sum of the elastic and the gravitational energy. With the z-axis pointing upward, we have the total potential and the associated force as:

$$V(z) = \frac{1}{2}k(z-z_0)^2 + mgz \qquad \Longrightarrow \qquad F(z) = -\frac{\partial V}{\partial z} = -k(z-z_0) - mg\,.$$

We have equilibrium at $z = z_e$ if the total force vanishes, *i.e.* if:

$$0 = F(z_e) = -k(z_e - z_0) - mg \qquad \Longrightarrow \qquad z_e = z_o - \frac{mg}{k} = z_0 - \frac{g}{\omega^2}\,,$$

where we have introduced $\omega = \sqrt{k/m}$ for later use. The Lagrangian is then:

$$L = T - V = \frac{1}{2}m\dot{z}^2 - \frac{1}{2}k(z-z_0)^2 - mgz\,.$$

This gives rise to the equation of motion:

$$-\frac{\delta L}{\delta z} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = m\ddot{z} + k(z-z_0) + mg = 0 \qquad \Longrightarrow \qquad \ddot{z} + \omega^2 z = \omega^2 z_0 - g = \omega^2 z_e\,.$$

This, of course, also immediately follows from Newton's second law. The solution of this inhomogeneous linear differential equation is the sum of the well known general solution $z_g = A\cos(\omega t) + B\sin(\omega t)$ of the corresponding homogenous harmonic equation, with A and B being integration constants, and any particular solution, z_p, of the inhomogeneous equation. Since the inhomogeneous term is constant, we see by inspection that $z_p = z_e$ is such a solution. Thus the general solution for z is:

$$z(t) = z_g + z_p = A\cos(\omega t) + B\sin(\omega t) + z_e\,,$$

This means that the particle oscillates harmonically about z_e [with amplitude $\sqrt{A^2+B^2}$]. From the boundary conditions $z(0) = 0$, $\dot{z}(0) = 0$ one finds:

$$\begin{aligned}
z(0) &= A + z_e = 0 \qquad \Longrightarrow \qquad A = -z_e = -z_0 + \frac{g}{\omega^2} \\
\dot{z}(0) &= [-A\omega\sin(\omega t) + B\omega\cos(\omega t)]_{t=0} = B\omega = 0 \qquad \Longrightarrow \qquad B = 0\,, \\
z &= z_e\left(1 - \cos(\omega t)\right)\,.
\end{aligned}$$

b) As seen in the original coordinate system, T is unchanged, while the length of the spring is $z - z_0 - \frac{1}{2}at^2 = z - z_1$.

$$L = T - V = \frac{1}{2}m\dot{z}^2 - \frac{1}{2}k(z-z_1)^2 - mgz$$

The corresponding Euler–Lagrange equation is also essentially unchanged:

$$-\frac{\delta L}{\delta z} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = m\ddot{z} + k(z-z_1) + mg = 0 \qquad \Longrightarrow \qquad \ddot{z} + \omega^2 z = \omega^2 z_1 - g = -g + \omega^2 z_0 + \frac{1}{2}a\omega^2 t^2\,.$$

The only change from the previous part is in the inhomogeneous term. We easily verify that the solution given in the problem indeed provides a particular solution, provided:

$$\begin{aligned}
z_p(t) &= C + Dt^2 \qquad \Longrightarrow \qquad \ddot{z}_p + \omega^2 z_p = 2D + \omega^2(C + Dt^2) = -g + \omega^2 z_0 + \frac{1}{2}a\omega^2 t^2 \\
\omega^2 D &= \frac{1}{2}a\omega^2 \qquad \Longrightarrow \qquad D = \frac{1}{2}a\,, \\
2D + \omega^2 C &= -g + \omega^2 z_e \qquad \Longrightarrow \qquad C = z_0 - \frac{g+2D}{\omega^2} = z_0 - \frac{g+a}{\omega^2} = z_e'.
\end{aligned}$$

Thus, we have the general solution:

$$z(t) = A\cos(\omega t) + B\sin(\omega t) + C + Dt^2 = A\cos(\omega t) + B\sin(\omega t) + z_e' + \frac{1}{2}at^2\,.$$

This looks like the previoius result, except for the added acceleration term $\frac{1}{2}at^2$ and that the equilibrium position z_e has been changed to $z_e' = z_0 - (g+a)/\omega^2$, as if the gravitational acceleration has been changed to $g' = g + a$. The boundary conditions can be calculated exactly as before, leading to $A = -z_e'$, $B = 0$, so the solution is:

$$z(t) = z_e' \left(1 - \cos(\omega t)\right) + \frac{1}{2}at^2 \, .$$

c) Introducing $\zeta = z - \frac{1}{2}at^2$, we have $\dot{\zeta} = \dot{z} - at$. Inserting this in the Lagrangian, we find:

$$L = T - V = \frac{1}{2}m(\dot{\zeta} + at)^2 - \frac{1}{2}k(\zeta - z_0)^2 - mg(\zeta + \tfrac{1}{2}at^2) \, .$$

The equation of motion for ζ is thus:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\zeta}} - \frac{\partial L}{\partial \zeta} = m\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\zeta} + at) + k(\zeta - z_0) + mg = m\ddot{\zeta} + k(\zeta - z_0) + mg' = 0$$

We see that we have *exactly* the same equation as in part a above, except that the gravitattional acceleration has been replaced by $g' = g + a$. The new equilibrium position as seen from the rocket, is found from setting $\ddot{z} = 0$, and is thus z_e' found in the previous part. The solution of the initial value problem is also the same, with this substitution:

$$\zeta(t) = z_e' \left(1 - \cos(\omega t)\right) \, .$$

Since $z = \zeta + \frac{1}{2}at^2$, this is the same as the solution of the previous problem.

d) The momentum conjugate to ζ is:

$$p_\zeta = \frac{\partial L}{\partial \dot{\zeta}} = m(\dot{\zeta} + at) \neq m\dot{\zeta} \, .$$

This shows what generally happens when the canonical momentum is calculated in non-inertial coordinate systems. The corresponding Hamiltonian is:

$$H = \dot{\zeta}p_\zeta - L = m\dot{\zeta}(\dot{\zeta} + at) - \frac{1}{2}m(\dot{\zeta} + at)^2 + V = \frac{1}{2}m(\dot{\zeta} + at)^2 + V - mat(\dot{\zeta} + at) = T + V - mat(\dot{\zeta} + at) \, ,$$

so we do not have $H = T + V$ in this coordinate system. H, like L, has become explicitly time dependent, and is therefore not conserved.

e) We shall modify L to $L' = L + \mathrm{d}F/\mathrm{d}t$, with $F = F(\zeta, t)$, so that we have:

$$m\dot{\zeta} = p_\zeta' = \frac{\partial L'}{\partial \dot{\zeta}} = \frac{\partial L}{\partial \dot{\zeta}} + \frac{\partial}{\partial \dot{\zeta}}\frac{\mathrm{d}F}{\mathrm{d}t} = m(\dot{\zeta} + at) + \frac{\partial}{\partial \dot{\zeta}}\frac{\mathrm{d}F}{\mathrm{d}t} \, .$$

Hence, the condition $m\dot{\zeta} = p_\zeta'$ leads to, using the chain rule:

$$\frac{\partial}{\partial \dot{\zeta}}\frac{\mathrm{d}F}{\mathrm{d}t} = -mat \qquad \Longrightarrow \qquad \frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial z}\dot{z} + \frac{\partial F}{\partial t} = -mat\dot{\zeta} + f(\zeta, t) = -mat\dot{z} + ma^2t^2 + f(z - \tfrac{1}{2}at^2, t) \, ,$$

where $f(\zeta, t)$ is an unknown function of ζ and t. Since F does not depend on $\dot{\zeta}$, and hence not on $\dot{z}$, the coefficients of $\dot{z}$ in this equation must agree, so $\partial F/\partial z = -mat$, or $F = -matz + g(t) = -mat\zeta + g(t) - m\frac{1}{2}a^2t^3$, where $g(t)$ can be chosen freely. By a suitable choice of $g(t)$ we have $F = -mat\zeta$, and find:

$$L' = L + \frac{\mathrm{d}F}{\mathrm{d}t} = L + \frac{\partial F}{\partial \zeta}\dot{\zeta} + \frac{\partial F}{\partial t} = L + \frac{1}{2}m(\dot{\zeta} + at)^2 - V - mat\dot{\zeta} - ma\zeta = \frac{1}{2}m\dot{\zeta}^2 - \frac{1}{2}k(\zeta - z_0)^2 - m(g + a)\zeta \, ,$$

which immediately leads to both $p_\zeta' = m\dot{\zeta}$ and the equation of motion in part c.

f) We find the Hamiltonian derived from L' as:

$$H' = \frac{\partial L'}{\partial \dot{\zeta}}\dot{\zeta} - L' = p_\zeta\dot{\zeta} - L' = \tfrac{1}{2}\dot{\zeta}^2 + \frac{1}{2}k(\zeta - z_0)^2 + m(g + a)\zeta \, .$$

This H' *is* independent of time, and hence conserved. But it is not the total energy of the system of particle + rocket, since it does not contain the increasing kinetic energy of the accelerating rocket, which is not conserved, but is provided by the burning of the rocket fuel.

PROBLEM 29:

a) In standard notation, $Z = (m_1 z_1 + m_2 z_2)/(m_1 + m_2)$ and $z = z_2 - z_1$, we have:

$$\begin{aligned} L &= \frac{1}{2} m_1 \dot{z}_1^2 + \frac{1}{2} m_2 \dot{z}_2^2 - \frac{1}{2} k (z_2 - z_1 - l)^2 - m_1 g z_1 - m_2 g z_2 \\ &= \frac{1}{2} (m_1 + m_2) \dot{Z}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \dot{z}^2 - (m_1 + m_2) g Z - \frac{1}{2} k (z - l)^2 \,. \end{aligned}$$

b) The equations of motion are found as:

$$\begin{aligned} Z: &\quad (m_1 + m_2) \ddot{Z} + (m_1 + m_2) g = 0 \,, \\ z: &\quad \frac{m_1 m_2}{m_1 + m_2} \ddot{z} + k(z - l) = 0 \,. \end{aligned}$$

We recognize that the problem has been separated into the free fall of the center of mass and harmonic relative motion. The solutions of these equations are:

$$\begin{aligned} Z(t) &= -\frac{1}{2} g t^2 + At + B \,, \\ z(t) &= l + a \sin(\omega t) + b \cos(\omega t) \,, \qquad \omega = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} \,. \end{aligned}$$

where A, B, a and b are constants of integration. The initial conditions at $t = 0$ are $z_1(0) = 0$, $z_2(0) = l$ and $\dot{z}_1(0) = 0$, $\dot{z}_2(0) = v_0$. This yields:

$$\begin{aligned} B = Z(0) &= \frac{m_1 z_1(0) + m_2 z_2(0)}{m_1 + m_2} = \frac{m_2}{m_1 + m_2} l \,. \\ b &= z(0) - l = z_2(0) - z_1(0) - l = 0 \,. \\ A = \dot{Z}(0) &= \frac{m_1 \dot{z}_1(0) + m_2 \dot{z}_2(0)}{m_1 + m_2} = \frac{m_2}{m_1 + m_2} v_0 \,. \\ a\omega = \dot{z}(0) &= \dot{z}_2(0) - \dot{z}_1(0) = v_0 \qquad \Longrightarrow \qquad a = \frac{v_0}{\omega} \,. \end{aligned}$$

Thus the two particles oscillate harmonically about the freely falling center of mass with a frequency that does not depend on v_0.

PROBLEM 30:

a) From the lectures we have, in standard notation, that the angular momentum $l = mr^2 \dot{\theta}$ is conserved, so if r is constant, so is $\omega = \dot{\theta}$.

b) The condition for circular motion is that the *effective* potential, $V'(r)$ has an extremum. In the present case we have:

$$V'(r) = V(r) + \frac{l^2}{2mr^2} = \frac{-GmM}{r} + \frac{l^2}{2mr^2} \,.$$

This is the condition for a circular orbit of fixed angular momentum l is:

$$\left. \frac{\partial V'(r)}{\partial r} \right|_{r=r_0} = -f'(r_0) = \frac{GmM}{r_0^2} - \frac{l^2}{mr_0^3} = 0 \qquad \Longrightarrow \qquad \frac{l^2}{mr_0^3} = \frac{GmM}{r_0^2} \qquad \Longleftrightarrow \qquad r_0 = \frac{l^2}{Gm^2 M} \,.$$

If we replace $l = mr_0^2 \omega$ by the orbital period $\tau = 2\pi/\omega$, we find:

$$r_0 = \frac{(mr_0^2 \omega)^2}{2Gm^2 M} = \frac{r_0^4 \omega^2}{GM} = \frac{4\pi^2 r_0^4}{GM\tau^2} \qquad \Longleftrightarrow \qquad \frac{r_0^3}{\tau^2} = \frac{GM}{4\pi^2} \,.$$

which is Kepler's third law (for arbitrary M) in the simplest case of a circular orbit. Note that this formula does not contain the mass m, or the reduced mass μ in the case of two orbiting objects.

c) We have:

$$V'(r) = \frac{-Km}{r^2} + \frac{l^2}{2mr^2} \qquad \left.\frac{\partial V'(r)}{\partial r}\right|_{r=r_0} = -f'(r_0) = \frac{2Km}{r_0^3} - \frac{l^2}{mr_0^3} = 0 \qquad \Longrightarrow$$

$$K = \frac{l^2}{2m^2} = \frac{1}{2}r_0^4\omega^2 \qquad \Longrightarrow \qquad r_0 = \frac{(2K)^{\frac{1}{4}}}{\sqrt{\omega}} \qquad \Longleftrightarrow \qquad \frac{r_0^2}{\tau} = \frac{1}{\pi}\sqrt{\frac{K}{2}}\,.$$

The energy follows as:

$$E = T + V = \frac{1}{2}mr_0^2\omega^2 - \frac{Km}{r_0^2} = 0\,.$$

That $E \geq 0$ is a strong hint that the motion is unstable.

<u>PROBLEM 31</u>:

a) See *Goldstein*, fig. 3.11, with $\mu = m/2$, the reduced mass of the two particles.

b) The effective potential (for constant l) is:

$$V'(r) = \frac{1}{2}kr^2 + \frac{l^2}{2\mu r^2}\,.$$

The circular orbit is at the minimum of $V'(r)$ is given by:

$$\frac{\partial V'}{\partial r}(r_0) = -f'(r_0) = kr_0 - \frac{l^2}{\mu r_0^3} = 0 \qquad \Longrightarrow \qquad r_0 = \left(\frac{l^2}{\mu k}\right)^{1/4}.$$

c) We have $\partial^2 V'/\partial r^2 = k + 3l^2/\mu r^4$, so the Taylor series becomes:

$$\begin{aligned} V'(r) &= V'(r_0) + \frac{\mathrm{d}V'}{\mathrm{d}r}(r_0)\,(r-r_0) + \frac{1}{2}\frac{\mathrm{d}^2V'}{\mathrm{d}r^2}(r_0)\,(r-r_0)^2 + O\left((r-r_0)^3\right) \\ &= \sqrt{\frac{k}{\mu}}l + 2k(r-r_0)^2 + O\left((r-r_0)^3\right)\,. \end{aligned}$$

Neglecting the higher order terms, and introducing $\rho = r - r_0$ as a new variable, we have the equation of motion for ρ as:

$$\mu\ddot{\rho} = -\frac{\mathrm{d}V'}{\mathrm{d}\rho} = -4k\rho \qquad \Longrightarrow \qquad \ddot{\rho} + \frac{4k}{\mu}\rho = 0\,.$$

This is the equation for harmonic oscillations with angular frequency $\omega = 2\sqrt{k/\mu}$. We note that this is *twice* the frequency of linear radial oscillations in the same potential.