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Set 5. Exercises for 22. September 2017

Problem 27: *Goldstein*, exercise 2.26. Note that the particle should be hanging in a massless **string** (not a spring!). Also use ℓ as symbol for the length of the string, so as not to confuse it with the symbol for the Lagrangian.

Problem 28: A particle of mass m is suspended in a vertical spring of spring constant k in a gravitational field.

a) Use a coordinate z for the particle such that $z = z_0$ when the spring is unstretched. Find the equilibrium position of the particle. Furthermore write down the Lagrangian, L, and find the equation of motion. Solve the equation if the particle is released from rest at $z = 0$ when $t = 0$. The acceleration of gravity is g.

b) The particle is placed in a rocket which is fired vertically from rest at $t = 0$ with constant acceleration a, such that $z_0 \to z_0 + \frac{1}{2}at^2$. Find the equation of motion for z, and show that the solution in the case that the particle is released from rest at $z = 0$ when t=0 can be written:

$$z(t) = A\cos(\omega t) + B\sin(\omega t) + C + Dt^2 \,,$$

where $\omega = \sqrt{k/m}$, and A, B, C and D are constant which shall be determined.

c) Introduce a rocket-fixed coordinate $\zeta = z - \frac{1}{2}at^2$, rewrite the Lagrangian in terms of ζ, and find the resulting equation of motion. Find the equilibrium position for ζ. Also solve the equation with the same boundary conditions as in the previous part, and show that the motion is the same, just expressed in the new coordinate.

d) Find the canonical momentum, p_ζ, conjugate to ζ. Do we have $p_\zeta = m\dot{\zeta}$? Also find the resulting Hamiltonian. Does it have the form $H = T + V$, with T and V expressed in terms of ζ?.

e) We know from the lectures and from Problem 21 (derivation 1.8 in *Goldstein*) that we can add a term $\mathrm{d}F/\mathrm{d}t$ to L without changing the Euler-Lagrange equations. Find F such that $p'_\zeta = m\dot{\zeta}$ when calculated from the modified Lagrangian. Verify that the equation of motion for ζ is unchanged.

f) Find the Hamiltonian H' corresponding to L', and show that it is conserved. Why is it nevertheless not the *total energy* of the system (particle plus rocket).

Problem 29: Two point particles with masses m_1 and m_2 are connected by a vertical elastic spring of unstretched length l and spring constant k. The acceleration of gravity is g. The particles are constrained to only move one above the other in the z-direction.

a) Write down the Lagrangian and find the equations of motion, using the z-components of the center of mass and the relative separation as generalized coordinates. [The potential energy of an elastic spring of length z and rest length l is $V(z) = \frac{1}{2}k(z - l)^2$.]

b) At $t = 0$ particle 2, positioned a height l above particle 1, is thrown upward with a velocity v_0. Solve the equations of motion both for the center of mass and the relative coordinate.

Problem 30: Consider a particle of mass m moving in a circular orbit in the field of a fixed attractive central potential.

a) Show that the particle moves with constant angular velocity $\omega = \dot{\phi}$.

b) Newton's law of gravitation in this case reads $V(r) = -GmM/r$, where M is the central mass, and G is the gravitational constant. Find the equation for the radius r of a circular orbit, and show that it satisfies:
$$\frac{r^3}{\tau^2} = \frac{GM}{4\pi^2} ,$$
where $\tau = 2\pi/\omega$ is the period of revolution. This a particular case of Kepler's 3. law.

c) Now consider a central potential of the form $V(r) = -Km/r^2$. Find the radius of a circular orbit. Also show that the total energy always vanishes for such an orbit. [This strongly indicates that such an orbit is always unstable.]

Problem 31: Two particles of equal mass m interact via a harmonic potential $V(r) = \frac{1}{2}kr^2$, where r is their separation.

a) Make a sketch shoving $V(r)$, the centrifugal potential and the effective potential $V'(r)$.

b) Find the radius, r_0 of a circular orbit of angular momentum ℓ.

c) Make a series (Taylor) expansion of $V'(r)$ around $r = r_0$, neglecting all terms of order $(r - r_o)^3$ and higher. Find the frequency of small oscillations about the circular orbit if the separation is changed slightly from r_0, without changing ℓ.