UNIVERSITETET I STAVANGER

Institutt for matematikk og naturvitenskap

**Suggested solutions, FYS 500 — Classical Mechanics Theory 2017 fall**

**Set 4 for 15. September 2017**

Problem 21:

The modified action integral is:

$$
\begin{aligned}
I' &= \int_1^2 L'(\dot{q}_1, \ldots, \dot{q}_n; q_1, \ldots, q_n; t)\,\mathrm{d}\,t \\
&= \int_1^2 L(\dot{q}_1, \ldots, \dot{q}_n; q_1, \ldots, q_n; t)\,\mathrm{d}\,t + \int_1^2 \frac{\mathrm{d}F}{\mathrm{d}t}(q_1, \ldots, q_n; t)\,\mathrm{d}\,t = I + F\Big|_1^2 .
\end{aligned}
$$

But the coordinates $\{q_i\}$ at the endpoints are fixed, so $\delta F|_1^2 = 0$ and therefore:

$$\delta I' = \delta I \,,$$

and the Euler–Lagrange equations are unchanged.

Problem 22:

In spherical cylinder coordinates we have $\mathbf{r} = r[\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]$. If we change the coordinates slightly, $\mathbf{r}$ changes by (see Problem. 13):

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial r}\mathrm{d}r + \frac{\partial \mathbf{r}}{\partial \theta}\mathrm{d}\theta + \frac{\partial \mathbf{r}}{\partial \phi}\mathrm{d}\phi = \mathrm{d}r\,\mathbf{e}_r + r\mathrm{d}\theta\,\mathbf{e}_\theta + r\sin\theta\,\mathrm{d}\phi\,\mathbf{e}_\phi \,.$$

For motion on a sphere of radius $a$, we have the constraint $r = a$ so $\mathrm{d}r = 0$. Thus $\mathrm{d}\mathbf{r} = a(\mathrm{d}\theta\,\mathbf{e}_\theta + r\sin\theta\,\mathrm{d}\phi\,\mathbf{e}_\phi)$ with length $\mathrm{d}s = |\mathrm{d}\mathbf{r}|$ given by:

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{r} = a^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,.$$

A (smooth) curve on the surface $r = a$ is described by a relation between the angular variables $\theta$ and $\phi$, say $\theta = \theta(\phi)$, or $\phi = \phi(\theta)$. Choosing the latter form turns out to be simpler, because it makes $\theta$ a cyclical variable. We then have $\mathrm{d}\phi = \phi'(\theta)\,\mathrm{d}\theta$, where $\phi'(\theta) = \mathrm{d}\phi/\mathrm{d}\theta$, so $\mathrm{d}s = \sqrt{1+\sin^2\theta\,\phi'^2}\,\mathrm{d}\theta$. The length of the curve described by the equation $\theta = \theta(\phi)$ between the points with coordinates $(\theta_1, \phi_1 = \phi(\theta_1))$ and $(\theta_2, \phi_2 = \phi(\theta_2))$ is then:

$$s = \int_1^2 \mathrm{d}\,s = a\int_{\theta_1}^{\theta_2}\sqrt{1+\sin^2\theta\,\phi'^{\,2}}\,\mathrm{d}\theta \equiv a\int_{\theta_1}^{\theta_2} f(\phi', \theta)\,\mathrm{d}\theta\,,$$

with $f(\phi', \theta) = \sqrt{1+\sin^2\theta\,\phi'^{\,2}}$. We see that $f$ is indeed independent of $\theta$. To find the *shortest*, or more generally a *stationary*, path connecting the two points is then the variational problem $\delta s = 0$, which is solved by the solutions of the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial f}{\partial \phi'} - \frac{\partial f}{\partial \phi} = \frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\,\phi'^{\,2}}}\right) = 0\,.$$

which immediately yields the first integral:

$$p_\phi = \partial f/\partial\phi' = \frac{\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\,\phi'^{\,2}}} = \text{constant}\,.$$

Since all points on a sphere are equivalent, we can, without loss of generality, choose our coordinates so that the starting point is one of the poles, so that $\sin\theta_1 = 0$. Then we find the constant $p_\phi$ as:

$$p_\phi = \frac{\sin^2\theta_1\,\phi'(\theta_1)}{\sqrt{1+\sin^2\theta_1\,\phi'(\theta_1)^{\,2}}} = 0\,.$$

This means that $\phi' = 0$ for all $\theta$, and the solution of the problem is $\phi(\theta) = \phi_1$. This describes arcs of great circles along the meridians of the sphere. These remain arcs of great circles if we rotate our coordinate system arbitrarily. [If we instead choose to describe the curve as $\theta = \theta(\phi)$ the solution process is a little more involved.]

Problem 23:

A coordinate is cyclic if we can change its value *continuously* in a time-independent manner, without changing the Lagrangian. This will certainly be the case a coordinate transformation leaves the mass distribution unchanged, since then the potential is unchanged, and neither is the kinetic energy affected. In this problem we must thus find those transformations which leave the mass distribution unchanged. In the following $x, y$ and $z$ are cartesian coordinates, $r, \theta$ and $z$ cylindrical coordinates.

In addition to the total energy, which of course is conserved in all cases, since the Lagrangian is time independent, we find:

a) The mass distributions is unchanged for translations parallel to the $xy$-plane, in particular in the $x$- and $y$- directions. Thus $x$ and $y$ are cyclic, and the corresponding conjugate momenta $p_x = m\dot{x}$ and $p_y = m\dot{y}$ are conserved, $\dot{p}_x = \dot{p}_y = 0$. The same is, of course, also any linear combination of them. If we switch to cylindrical coordinates the same argument gives that $p_r = m\dot{r}$ and $p_\theta = l_z = mr^2\dot{\theta}$ are conserved. However, these conservation laws are not independent of the previous ones. Since $\dot{r}^2 = \dot{x}^2 + \dot{y}^2$, we gave $p_r^2 = p_x^2 + p_y^2$ and for $l_z$ we have:

$$\dot{l}_z = \frac{\mathrm{d}}{\mathrm{d}t}(x\dot{p}_y - y\dot{p}_z) = \dot{x}p_y + x\dot{p}_y - \dot{y}p_x - y\dot{p}_x = \frac{1}{m}(p_x p_y - p_y p_x) = 0\,,$$

so $l_z$ is also conserved.

b) Invariance: Translation in the $x$-direction; $x$ is cyclic and $p_x = m\dot{x}$ is conserved.

c) Invariance: Rotation about the cylinder axis and translations along it; $\theta$ and $z$ are cyclical, $l_z$ and $p_z = m\dot{z}$ are conserved.

d) Invariance: Rotation about the cylinder axis; $\theta$ is cyclical and $l_z$ is conserved.

e) Invariance: Translation along the $z$ axis; $z$ is cyclical and $p_z$ is conserved.

f) Invariance: Rotation about the dumbbell axis; $\theta$ is cyclical and $l_z$ is conserved.

g) The helical thread is invariant under a simultaneous rotation and translation, where $\mathrm{d}z = ka\,\mathrm{d}\theta$, where $k$ is the slope of the helix and $a$ its radius. This means that the generalized coordinate $\xi = z - ka\theta$ is cyclical. Expressing $L$ in terms of $r, \xi$ and $z$, using $\dot{\theta} = (\dot{z} - \dot{\xi})/ka$, we find:

$$L = \frac{m}{2}\left(\dot{r}^2 + \frac{r^2}{k^2a^2}(\dot{\xi} - \dot{z})^2 + \dot{z}^2\right) - V(r, z)\,.$$

The conserved generalized momentum is then:

$$p_\xi = \frac{\partial L}{\partial \dot{\xi}} = \frac{mr^2}{k^2a^2}(\dot{\xi} - \dot{z}) = -\frac{mr^2\dot{\theta}}{ka} = -\frac{l_z}{ka}\,.$$

Thus $l_z$ is conserved after all (but not $m\dot{z}$).

Problem 24:

In polar coordinates $\rho, \theta$ in the plane of the rotating hoop the distance of the point mass from the $z$-axis is given by the constraint $\rho = a\sin\theta$. If we measure $\theta$ from the *lowest* point of the hoop, its height above this point is $z = a(1 - \cos\theta)$ $(= 0$ when $\theta = 0)$. The velocity along the hoop is $v_\theta = a\dot{\theta}$. In addition, the mass rotates with the hoop around the $z$ axis with velocity $v_h = \rho\omega = \omega a\sin\theta$. Since the two velocity components are perpendicular, the Lagrangian is:

$$L = T - V = \frac{1}{2}m\left(v_\theta^2 + v_h^2\right) - mgz = \frac{1}{2}ma^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) - mga(1 - \cos\theta)\,.$$

There are no cyclical coordinates, but $L$ does not contain the time explicitly, so the energy is a constant of motion. Since $T$ is quadratic in the velocities, it can be written:

$$E = T + V = \frac{1}{2}ma^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + mga(1 - \cos\theta)\,.$$

The equation of motion is found as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = ma(a\ddot\theta - a\omega^2 \sin\theta\cos\theta + g\sin\theta) = 0\,.$$

If the mass remains stationary in equilibrium at some angle $\theta = \theta_e$, one has $\dot\theta\big|_{\theta_e} = 0$ at all times, so $\ddot\theta\big|_{\theta_e} = 0$. At this point the equation of motion reduces to:

$$\sin\theta_e(a\omega^2\cos\theta_e - g) = 0\,.$$

This equation has two solutions. One is $\sin\theta_e = 0$, which yields the obvious equilibria for $\theta_e = 0$ and $\theta_e = \pi$, where the mass is at the bottom or the top of the hoop. The last position is easily shown to be unstable, indeed the energy has a local maximum there. The other equilibrium position is given by:

$$a\omega^2\cos\theta_e = g \qquad \Longrightarrow \qquad \theta_e = \pm\arccos\left(\frac{g}{\omega^2 a}\right)\,.$$

Because $|\cos\theta_e| \le 1$ for any real angle $\theta_e$, this is only a solution if $|g/\omega^2 a| \le 1$, *i.e.*:

$$\omega \ge \omega_0 = \sqrt{\frac{g}{a}}\,.$$

[It is not hard to show that the energy in this state is *lower* than the one with $\theta_e = 0$, so this represents the stable equilibrium for $\omega > \omega_0$.]

PROBLEM 25:

It is assumed that all (generalized) forces, except the impulsive one, $\mathbf{F}$, can derived from a potential included in the Lagrangian, $L$, and are finite. Let $F_j(t)$ be the component of the generalized impulsive force associated with the generalized momentum $q_j$. This has an impulse:

$$S_j = \int_{\Delta t} F_j\,\mathrm{d}t\,.$$

All $F_k(t)$ vanish for times outside the short interval $\Delta t$. The Lagrange equations for a system which also have forces that are not derived from the potential reads (see *Goldstein* sect. 1.5):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q_j}\right) - \frac{\partial L}{\partial q_j} = F_j\,.$$

Integrating this result over the short interval $t = t_i$ to $t = t_f = t_i + \Delta t$, where $F_j$ is non-vanishing, we find:

$$\int_{t_i}^{t_f} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q_j}\right)\mathrm{d}t - \int_{t_i}^{t_f}\frac{\partial L}{\partial q_j}\,\mathrm{d}t = \int_{t_i}^{t_f} F_j\,\mathrm{d}t$$
$$\left(\frac{\partial L}{\partial \dot q_j}\right)_f - \left(\frac{\partial L}{\partial \dot q_j}\right)_i - \overline{\frac{\partial L}{\partial q_j}}\,\Delta t = S_j\,.$$

Here $\overline{\partial L/\partial q_j}$ is the average of the effective forces derived from $L$, which is assumed to be small compared to $F_j$. It thus will not contribute in the limit $\Delta t \to 0$, and we have:

$$\left(\frac{\partial L}{\partial \dot q_j}\right)_f - \left(\frac{\partial L}{\partial \dot q_j}\right)_i = p_j(t_f) - p_j(t_i) = S_j\,,$$

where we have introduced the generalized momenta $p_j = \partial L/\partial \dot q_j$.