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Institutt for matematikk og naturvitenskap

Suggested solutions, FYS 500 — Classical Mechanics Theory 2017 fall

Set 2 for 1. September 2017

Problem 8:

From the definition of kinetic energy and Newton's second law, we find:

$$T = \frac{1}{2} m\mathbf{v}^2 \qquad \Longrightarrow \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = m\dot{\mathbf{v}}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v}\,.$$

If m is not constant, we use Newton's second law in the form $\mathbf{F} = \dot{\mathbf{p}}$:

$$\frac{\mathrm{d}(mT)}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}(m^2\mathbf{v}^2)}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}(\mathbf{p}^2)}{\mathrm{d}t} = \dot{\mathbf{p}}\cdot\mathbf{p} = \mathbf{F}\cdot\mathbf{p}\,.$$

Problem 9:

With the notation of sections 1.1 and 1.2 of *Goldstein*, we have for two particles of constant mass, labelled 1 and 2:

$$M\ddot{\mathbf{R}} = (m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2) = (\mathbf{F}_{21} + \mathbf{F}_{12}) + \mathbf{F}_1^{(e)} + \mathbf{F}_2^{(e)} = (\mathbf{F}_{21} + \mathbf{F}_{12}) + \mathbf{F}^{(e)}\,.$$

If eq. (1.22), $M\ddot{\mathbf{R}} = \mathbf{F}^{(e)}$, shall be satisfied, we must have $\mathbf{F}_{21} + \mathbf{F}_{12} = 0$, or $\mathbf{F}_{12} = -\mathbf{F}_{21}$, which is the weak form of Newton's third law.

Furthermore, from eq. (1.24) we have:

$$\dot{\mathbf{L}} = \mathbf{r}_1 \times \mathbf{F}_1^{(e)} + \mathbf{r}_2 \times \mathbf{F}_2^{(e)} + \mathbf{r}_1 \times \mathbf{F}_{21} + \mathbf{r}_2 \times \mathbf{F}_{12} + = \mathbf{N}_1^{(e)} + \mathbf{N}_2^{(e)} + (\mathbf{r}_1 - \mathbf{r}_2)\times\mathbf{F}_{21}\,,$$

where we have used $\mathbf{F}_{12} = -\mathbf{F}_{21}$. Since $\mathbf{N}^{(e)} = \mathbf{N}_1^{(e)} + \mathbf{N}_2^{(e)}$, eq. (1.26) is fulfilled only if $(\mathbf{r}_1 - \mathbf{r}_2)\times\mathbf{F}_{21} = 0$. But this is true only if $(\mathbf{r}_1 - \mathbf{r}_2)$ is parallel to $\mathbf{F}_{12}$, and we have the strong form of Newton's third law.

Problem 10:

We may choose coordinates such that the nucleus is initially at rest at the origin. Let the electron be ejected with momentum vector $\mathbf{p}_e$, the neutrino with momentum $\mathbf{p}_\nu$ and the momentum of the recoiling nucleus be $\mathbf{p}_N$, where it is given that $\mathbf{p}_e \cdot \mathbf{p}_\nu = 0$. Choosing coordinates such that before the decay the nucleus is at rest, the total momentum of the system is $\mathbf{P}_i = \mathbf{0}$. Since no external forces are acting, momentum is conserved, hence:

$$\mathbf{P}_f = \mathbf{p}_N + \mathbf{p}_e + \mathbf{p}_\nu = \mathbf{P}_i = \mathbf{0}\,, \qquad \Longleftrightarrow \qquad \mathbf{p}_N = -\mathbf{p}_e - \mathbf{p}_\nu\,.$$

The magnitude of the recoil momentum is found as:

$$p_N = |\mathbf{p}_N| = |\mathbf{p}_e + \mathbf{p}_\nu| = \sqrt{p_e^2 + p_\nu^2 + 2\mathbf{p}_e\cdot\mathbf{p}_\nu} = \sqrt{p_e^2 + p_\nu^2} = \sqrt{1.73^2 + 1.00^2}\,\frac{\mathrm{MeV}}{\mathrm{c}} = 2.00\,\frac{\mathrm{MeV}}{\mathrm{c}}\,.$$

The angle, θ_N, between the recoil momentum and the electron is then found from:

$$\mathbf{p}_N\cdot\mathbf{p}_e = -\left(p_e^2 + \mathbf{p}_\nu\cdot\mathbf{p_e}\right) = -p_e^2 = p_N p_e \cos\theta_N\,, \qquad \Longrightarrow \qquad \theta_e = \arccos\frac{-p_e}{p_N} = 2.62 = 149.9^\circ\,.$$

We must choose the solution with $\theta_N > \pi/2 = 90^\circ$ since the nucleus is recoiling in the opposite direction of the emitted electron. Since the mass m_N of the residual nucleus is given in standard SI units, we have to use these units also for the momentum, $1\,\mathrm{Mev/c}{=}5.34\cdot 10^{-22}\,\mathrm{kg\,m/s^2}$:

$$T_N = \frac{p_N^2}{2m_N} = \frac{\left(2.00\cdot 5.34\cdot 10^{-22}\right)^2}{2\cdot 3.90\cdot 10^{-25}}\,\mathrm{J} = 1.46\cdot 10^{-18}\,\mathrm{J}\;[= 9.13\,\mathrm{eV}]\,.$$

PROBLEM 11:

With the assumptions made, and choosing the direction of motion in the radial direction, we have the kinetic and potential energies, T and V, as:

$$T = \tfrac{1}{2}m\dot{r}^2 \qquad V(r) = -\frac{GMm}{r} = -\frac{gmR^2}{r}\,.$$

Here m is the particle mass, M and R the mass and radius of the Earth, respectively, and we have exploited that the acceleration of gravity at the Earth's surface is given by $g = GM/R^2$. The particle energy is then:

$$E(r) = T + V = m\left(\frac{1}{2}\dot{r}^2 - \frac{gR^2}{r}\right)\,.$$

At fixed r, E(r) is minimal when $\dot{r} = 0$. Thus the minimum energy for a particle at $r = \infty$ is $E(\infty) = 0$. But $E(r)$ is conserved, so the escape velocity $v_e = \dot{r}(R)$ follows as:

$$E(R) = E(\infty) \qquad \Longrightarrow \qquad \tfrac{1}{2}v_e^2 - gR = 0 \qquad \Longrightarrow \qquad v_e = \sqrt{2gR} = 11.2\,\mathrm{km/s}\,.$$

PROBLEM 12:

Newtons 3. law requires that at any time the change of momentum of the rocket caused by the burning of the fuel must be opposite and of equal magnitude of the momentum carried away by the exhaust gases emitted during that time. If the rocket velocity changes by dv, its momentum changes by $m(t)\,\mathrm{d}v$, where $m(t)$ is the instantaneous mass of the rocket. The mass of the exhaust emitted during that time, δm, is equal to the mass loss of the rocket, $\delta m = -\mathrm{d}m$, and so the momentum transferred to the rocket is $\mathrm{d}p = m\mathrm{d}v = v'\delta m = -v'\mathrm{d}m$ (> 0, since the rocket loses mass, so $\mathrm{d}m < 0$). The effective (recoil) force on the rocket is thus:

$$F_R = m\frac{\mathrm{d}v}{\mathrm{d}t} = -v'\frac{\mathrm{d}m}{\mathrm{d}t}$$

In addition the force of gravity acts on the rocket. If it is close enough to the ground, this is simply $-mg$, where g is the acceleration of gravity. The equation of motion for the rocket is thus, as stated:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -v'\frac{\mathrm{d}m}{\mathrm{d}t} - mg\,.$$

If the fuel is burned at a constant rate, $\mathrm{d}m/\mathrm{d}t = -c$ ($c > 0$!), we have $m(t) = m_0 - ct$ (as long as $m/t > 0$), where m_0 is the initial rocket mass. If the rocket starts at rest, we have:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v'c}{m_0 - ct} - g \qquad \Longrightarrow \qquad v(t) = \int_0^t \frac{v'c}{m_0 - ct}\mathrm{d}t - gt = v'\ln\left(\frac{m_0}{m_0 - ct}\right) - gt\,.$$

We cannot invert this equation analytically to find the time t when the rocket reaches a specific velocity. However, we can rewrite it, using $t = (m_0 - m)/c$, as:

$$v(m) = v'\ln\left(\frac{m_0}{m}\right) - \frac{g}{c}(m_0 - m) \approx vs\ln\left(\frac{m_0}{m}\right) - \frac{gm_0}{c}\,,$$

which is approximately valid when the remaining mass of the rocket is small compared to the start mass m_0. We then find the mass, m_e, when $v(m_e) = v_e$ as:

$$\frac{v_e}{v'} + \frac{gm_0}{v'c} \approx \ln\left(\frac{m_0}{m_e}\right) \qquad \Longrightarrow \qquad \frac{m_0}{m_e} \approx \exp\left(\frac{v_e}{v'} + \frac{gm_0}{v'c}\right) = 274\,.$$

where we have used $v_e = 11.2\,\mathrm{km/s}$ from the previous problem and that $m_0/c = 60\,\mathrm{s}$.

Problem 13:

In spherical polar coordinates we can write:

$$\mathbf{r} = r[\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]\,,$$

Following exactly the same procedure as in the lecture notes for 1 September, we find the basis vectors as:

$$\begin{aligned}
\mathbf{e}_r &= \mathbf{E}_r = \frac{\partial\,\mathbf{r}}{\partial r} = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta] = \widehat{\mathbf{r}}\,,\\
\mathbf{e}_\theta &= \frac{1}{r}\mathbf{E}_\theta = \frac{1}{r}\frac{\partial\,\mathbf{r}}{\partial\theta} = [\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta]\,,\\
\mathbf{e}_\phi &= \frac{1}{r\sin\theta}\mathbf{E}_\phi = \frac{1}{r\sin\theta}\frac{\partial\,\mathbf{r}}{\partial\phi} = [-\sin\phi, \cos\phi, 0]\,.
\end{aligned}$$

To translate the vector components between the systems, we use the identity:

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k} = V_r\mathbf{e}_r + V_\phi\mathbf{e}_\phi + V_\theta\mathbf{e}_\theta\,,$$

and find:

$$\begin{aligned}
V_r &= \mathbf{e}_r\cdot\mathbf{V} = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]\cdot[V_x, V_y, V_z] = V_x\sin\theta\cos\phi + V_y\sin\theta\sin\phi + V_z\cos\theta\,,\\
V_\theta &= \mathbf{e}_\theta\cdot\mathbf{V} = V_x\cos\theta\cos\phi + V_y\cos\theta\sin\phi - V_z\sin\theta\,,\\
V_\phi &= \mathbf{e}_\phi\cdot\mathbf{V} = -V_x\sin\phi + V_y\cos\phi\,.
\end{aligned}$$

with inverse:

$$\begin{aligned}
V_x &= \mathbf{i}\cdot\mathbf{a} = V_r\mathbf{i}\cdot\mathbf{e}_r + V_\theta\mathbf{i}\cdot\mathbf{e}_\theta + V_\phi\mathbf{i}\cdot\mathbf{e}_\phi = V_r\sin\theta\cos\phi + V_\theta\cos\theta\cos\phi - V_\phi\sin\phi\,.\\
V_y &= \mathbf{j}\cdot\mathbf{a} = V_r\mathbf{j}\cdot\mathbf{e}_r + V_\theta\mathbf{j}\cdot\mathbf{e}_\theta + V_\phi\mathbf{j}\cdot\mathbf{e}_\phi = V_r\sin\theta\sin\phi + V_\theta\cos\theta\sin\phi + V_\phi\cos\phi\,.\\
V_z &= \mathbf{k}\cdot\mathbf{a} = V_r\mathbf{k}\cdot\mathbf{e}_r + V_\theta\mathbf{k}\cdot\mathbf{e}_\theta + V_\phi\mathbf{k}\cdot\mathbf{e}_\phi = V_r\cos\theta - V_\theta\sin\theta\,.
\end{aligned}$$

Problem 14:

With $\mathbf{A} = \mathbf{c}\times\mathbf{d}$ and using eq. (0.24a) from the lecture notes of 25.08:

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{A} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{A}) = \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})]\,,$$

which is the first equality. The second equality is proven in the same manner, with $\mathbf{A} = \mathbf{a}\times\mathbf{b}$. Using eq. (0.32) for the term in square brackets we then have:

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})] = \mathbf{a}\cdot[\mathbf{c}(\mathbf{b}\cdot\mathbf{d}) - \mathbf{d}(\mathbf{b}\cdot\mathbf{c})] = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\,.$$