UNIVERSITETET I STAVANGER

Institutt for matematikk og naturvitenskap

Suggested solutions, FYS 500 — Classical Mechanics Theory 2017 fall

Set 1 for 25. August 2017

Problem 1:

Since vector functions are differentiated component-wise, we find:

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = [v_{0x}, v_{0y}, v_{0z} + gt] = \mathbf{v}_0 - gt\,\mathbf{k}\,,$$
$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = -g\mathbf{k}\,.$$

.

From Newton's second law it then follows that:

$$\mathbf{F} = m\mathbf{a} = -mg\,\mathbf{k}\,.$$

Thus the force has magnitude mg and is directed in the negative z-direction.

Problem 2:

We find:

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = [-\omega R \sin\omega t, \omega R\cos\omega t, w]\,,$$
$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = [-\omega^2 R\cos\omega t, -\omega^2 R\sin\omega t, 0]\,,$$
$$\mathbf{v}\cdot\mathbf{a} = -\omega^3 R^2(\sin\omega t\cos\omega t - \cos\omega t\sin\omega t + 0) = 0\,.$$

The last line shows that $\mathbf{F}\cdot\mathbf{v} = m\mathbf{a}\cdot\mathbf{v} = 0$, so $\mathbf{F}\perp\mathbf{v}$.

Problem 3:

This problem can be solved either by brute force or by choosing a smart coordinate system.

I) By brute force, calculating backwards and skipping the ugliest part of the algebra, we find:

$$r^2 s^2\sin^2\theta = r^2s^2(1-\cos^2\theta) = r^2s^2 - (\mathbf{r}\cdot\mathbf{s})^2 = (r_1^2+r_2^2+r_3^2)(s_1^2+s_2^2+s_3^2) - (r_1s_1+r_2s_2+r_3s_3)^2$$
$$= \ldots = (r_2s_3 - r_3s_2)^2 + (r_3s_1 - r_1s_3)^2 + (r_1s_2 - r_2s_1)^2 = (\mathbf{r}\times\mathbf{s})^2\ .$$

Taking the (positive) square root of both sides of this equation gives the wanted result.

II) By choosing a smart coordinate system: The formula is trivial if $\mathbf{r}$ or $\mathbf{s}$ is the null vector, or if the two vectors are colinear ($\sin\theta = 0$). If not, we know from elementary geometry that the two vectors span a plane. We chose this plane to be the xy-plane, so $r_3 = s_3 = 0$. In this plane we can chose the x-axis to be along $\mathbf{r}$, so $\mathbf{r} = r\hat{\mathbf{r}} = r\mathbf{e}_1$, *i.e.* $r_1 = r$, $r_2 = 0$. Since then $rs\cos\theta = r_1s_1 + r_2s_2 = rs_1$, we have $s_1 = s\cos\theta$, and $s_2 = \pm\sqrt{s^2 - s_1^2} = \pm s\sqrt{1-\cos^2\theta} = \pm s|\sin\theta|$. Thus:

$$\mathbf{r}\times\mathbf{s} = [r_2s_3 - r_3s_2, r_3s_1 - r_1s_3, r_1s_2 - r_2s_1] = [0,0,r_1s_2] = rs[0,0,\pm|\sin\theta|]\,,$$

and the result trivially follows.

Problem 4:

a) Newton's 2. law on component form yields:

$$m[\ddot{x}, \ddot{y}, \ddot{z}] = [0, 0, -mg] \qquad \Longleftrightarrow \qquad \ddot{x} = 0\,, \qquad \ddot{y} = 0\,, \qquad \ddot{z} = -g\,,$$

with immediate solutions, taking the boundary conditions into account:

$$x = x_0\,, \qquad y = y_0\,, \qquad z = z_0 - \frac{1}{2}gt^2\,,$$

which is the same as in problem 1 for $\mathbf{v}_0 = 0$.

b) Since the basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are time independent, we can integrate a vector by integrating it component-wise and using that if $\mathbf{g}$ is constant, so are its components. Since $\mathbf{F} = m\mathbf{g}$, we find (with the initial time $t_0 = 0$):

$$\mathbf{v}(\mathbf{t}) = \mathbf{v}_0 + \int_0^t \dot{\mathbf{v}}(t)\,\mathrm{d}t = \mathbf{v}_0 + \int_0^t \frac{1}{m}\mathbf{F}(t)\,\mathrm{d}t = \mathbf{v}_0 + \int_0^t \mathbf{g}\,\mathrm{d}t = \mathbf{v}_0 + \mathbf{g}t\,,$$
$$\mathbf{r}(\mathbf{t}) = \mathbf{r}_0 + \int_0^t \mathbf{v}(t)\,\mathrm{d}t = \mathbf{r}_0 + \int_0^t \dot{\mathbf{r}}(t)\,\mathrm{d}t = \mathbf{r}_0 + \int_0^t (\mathbf{v}_0 + \mathbf{g}t)\,\mathrm{d}t = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2\,.$$

Problem 5:

a) The trajectory is the one given in problem 4b, with $x_0 = y_0 = z_0 = 0$ and $v_{0x} = v_0\cos\theta, v_{0y} = 0, v_{0z} = v_0\sin\theta$. Hence $x = v_0\cos\theta\ t$, $y = 0$ and $z = v_0\sin\theta\ t - \frac{1}{2}gt^2$. To find the equation of the trajectory, we eliminate the time from these equations by inserting $t = x/(v_0\cos\theta)$ in the expression for z:

$$z = v_0\sin\theta\ t - \frac{1}{2}gt^2 = x\tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta}\,.$$

This is the equation for a parabola, with axis parallel to the z-axis.

b) At the maximal height the derivative of $z(x)$ is $\mathrm{d}z/\mathrm{d}x = 0$. This maximum, $x = x_m$, is thus found from:

$$0 = \left.\frac{\mathrm{d}z}{\mathrm{d}x}\right|_{x=x_m} = v_0\tan\theta - \frac{gx_m}{v_0^2\cos^2\theta} \qquad \Longrightarrow \qquad x_m = \frac{v_0^2}{g}\cos\theta\sin\theta = \frac{v_0^2}{2g}\sin 2\theta\,.$$

The maximal height is:

$$z_m(\theta) = z(x_m) = \frac{v_0^2}{g}\sin^2\theta - \frac{v_0^2}{2g}\sin^2\theta = \frac{v_0^2}{2g}\sin^2\theta\,.$$

c) The projectile is at the ground level if $z(x) = 0$. This is a quadratic equation, with solutions $x = 0$, the starting point, and $x = x_r$, where the range x_r is the solution of:

$$0 = \tan\theta - \frac{gx_r}{2v_0^2\cos^2\theta} \qquad \Longrightarrow \qquad x_r = \frac{2v_0^2}{g}\cos\theta\sin\theta = \frac{v_0^2}{g}\sin 2\theta = 2x_m\,.$$

Thus, the angle θ that maximizes the range is the same as the one maximizes $\sin 2\theta$, which is for $2\theta = \pi/2$, or $\theta = \pi/4$.

Problem 6:

a) Without loss of generality, we can take the initial direction of motion in the x-direction. Since the drag force $\mathbf{F}_S$, is always directed along $\mathbf{v}$, it cannot change the direction of motion, so the whole motion takes place in this direction. The x-component of Newton's 2. law then yields:

$$m\dot{v} = F_S = -6\pi\eta R v\,, \qquad \Longrightarrow \qquad \frac{\dot{v}}{v} = -\frac{6\pi\eta R}{m} = -\frac{1}{\tau}\,,$$

Here we have introduced the *time constant*, τ, as:

$$\tau = \frac{m}{6\pi\eta R} = \frac{(4\pi/3)\rho R^3}{6\pi\eta R} = \frac{2\rho R^2}{9\eta}\,.$$

where $\rho = 1\,000\,\mathrm{kg/m^3}$ is the density of the water in the drop. The differential equation for $v(t)$ is standard, and we find, if $v(0) = v_0$:

$$\int_{v_0}^{v} \frac{\mathrm{d}v}{v} = -\int_0^t \frac{\mathrm{d}r}{\tau} \qquad \Longrightarrow \qquad \ln\left(\frac{v}{v_0}\right) = -\frac{t}{\tau} \qquad \Longrightarrow \qquad v = v_0 e^{-t/\tau}\,.$$

The numerical value of τ is

$$\tau = \frac{2\pi\rho R^2}{9\eta} = \begin{cases} 0.15\,\mathrm{ms} & R = 1\,\mu\mathrm{m}, \\ 150\,\mathrm{s} & R = 1\,\mathrm{mm}. \end{cases}$$

b) Since there is no initial velocity, all forces will act in the z-direction, and we may choose a coordinate system with the z-axis positive *downward*. In this system the z-component of Newton's 2. law reads, with $v = \dot{z}$ and τ defined as above and the boundary condition $v(0) = 0$:

$$m\dot{v} = mg - 6\pi\eta R v\,, \qquad \Longrightarrow \qquad \dot{v} = g - v/\tau\,,$$

$$\int_0^v \frac{\mathrm{d}v}{g\tau - v} = \frac{1}{\tau}\int_0^t \mathrm{d}t\,, \qquad \Longrightarrow \qquad \ln(g\tau - v)\Big|_0^v = -\frac{t}{\tau} \qquad \Longrightarrow \qquad v = g\tau\left(1 - e^{-t/\tau}\right)\,.$$

We see that

$$v(t) \longrightarrow g\tau = v_\tau \qquad \text{as} \qquad t \longrightarrow \infty\,,$$

where v_τ is the terminal velocity.

Problem 7:

The arguments for this problem follow those of problem 5 closely, so we only the main points are presented.

a) Introducing a constant $K = \frac{1}{2}C\rho A/m$, the equation of motion yields:

$$m\dot{v} = F_R = -mKv^2 \qquad \Longrightarrow \qquad \int_{v_0}^{v} \frac{\mathrm{d}v}{v^2} = -Kt\,,$$

$$\frac{1}{v}\Big|_{v_0}^{v} = Kt \qquad \Longrightarrow \qquad v = \frac{v_0}{1 + Ktv_0}$$

[We see that $v \to 0$ as $t \to \infty$ also in this case, but much more slowly].

b) The terminal velocity, v_t, is reached when the gravitational force and the air resistance are oppositely equal, *i.e.* when

$$mg = \tfrac{1}{2}C\rho A v_t^2 \qquad \Longrightarrow \qquad v_t = \sqrt{\frac{C\rho A g}{2m}} = \sqrt{\frac{g}{K}}\,.$$

c) As in problem 5, we assume for simplicity $v_0 = 0$:

$$m\dot{v} = mg - mKv^2 = mK\left(v_t^2 - v^2\right)\,,$$

$$\int_0^v \frac{\mathrm{d}v}{v_t^2 - v^2} = \frac{1}{v_t}\int_0^v \left(\frac{1}{v_t - v} + \frac{1}{v_t + v}\right)\mathrm{d}\,v = Kt\,,$$

$$\ln\left(\frac{v_t + v}{v_t - v}\right) = v_t K t = \sqrt{Kg}t \qquad \Longrightarrow \qquad \frac{v_t + v}{v_t - v} = e^{\sqrt{Kg}t}\,,$$

$$v(t) = v_t\left(\frac{e^{v_t K t} - 1}{e^{v_t K t} + 1}\right) = v_t\left(\frac{1 - e^{-v_t K t}}{1 + e^{-v_t K t}}\right)\left[= v_t \tanh\left(\frac{v_t K T}{2}\right)\right] \longrightarrow v_t \text{ as } t \to \infty\,.$$