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Set 1. Exercises for 25. August 2017

Problem 1: A particle of mass m is moving on a trajectory:

$$\mathbf{r}(t) = \left[x_0 + v_{0x}t,\, y_0 + v_{0y}t,\, z_0 + v_{0z}t - \frac{1}{2}gt^2\right] .$$

Find the velocity vector as a function of time, $\mathbf{v}(t)$, and also the acceleration, $\mathbf{a}(t)$, and show that the motion is caused by a constant force $\mathbf{F}$, which shall be determined (magnitude and direction).

Problem 2: A particle of mass m moves in a magnetic field on the trajectory:

$$\mathbf{r}(t) = [R\cos\omega t,\, R\sin\omega t,\, wt] ,$$

where R, ω and w are constants. Find the particle's velocity, $\mathbf{v}(t)$, and acceleration, $\mathbf{a}(t)$, as functions of time. Show that the force on the particle is perpendicular to the velocity.

Problem 3 Prove eq. (0.22) of the lecture notes for 25.8: If $\mathbf{r}$ and $\mathbf{s}$ are to vectors, then:

$$|\mathbf{r} \times \mathbf{s}| = rs\,|\sin\theta| , \tag{0.22}$$

where $r = |\mathbf{r}|$, $s = |\mathbf{s}|$ and θ is the angle between them. [Hint: Brute force works, but exploiting your freedom to chose a smart coordinate system simplifies the calculation a lot].

Problem 4: It should be well known that the force on a particle of mass m in a constant gravitational field acting in the negative z-direction is given by $\mathbf{F} = [0, 0, -mg]$, where g is the acceleration of gravity. A particle is falling from rest starting at $\mathbf{r}_0 = [x_0, y_0, z_0]$ in such a field, with no other forces acting.

a) Solve Newton's equation of motion for the particle, and show that the trajectory is the one given in problem 1 above with $\mathbf{v}_0 = [v_{0x},\, v_{0y},\, v_{0z}] = 0$.

b) Solve the same problem on vector form, if the gravitational acceleration, $\mathbf{g}$, points in an arbitrary direction, the initial position is $\mathbf{r}_0$ and the initial velocity $\mathbf{v}_0$.

Problem 5: A projectile is shot from ground level at the point $\mathbf{r}_0 = 0$ with initial velocity $\mathbf{v}_0 = v_0[\cos\theta, 0, \sin\theta]$, with $0 < \theta < \pi/2$, in a coordinate system with the z-axis oriented vertically upwards. Neglect air resistance.

a) Determine the trajectory $\mathbf{r}(t)$ and show that the particle follows a parabola with a vertical axis.

b) Find expressions for the maximal height of the shot and its range as functions of θ.

c) For what angle θ is the range of the projectile maximal? Find an expression for this maximal range.

Problem 6: No exact general formulas exist to determine the fluid dynamical forces acting on an object moving through a gas or a liquid. But for very slow motion one can assume that one has *creeping flow*, with fluid forces proportional to the velocity difference between the moving object and the fluid. For a sphere of radius R moving through a fluid of viscosity η at rest, Stokes' law of resistance states that the force on the sphere, the *drag force*, is:

$$\mathbf{F}_S = -6\pi\eta R\,\mathbf{v} ,$$

where the sign reflects that for a symmetrical object like a sphere the drag acts in the opposite direction of the velocity $\mathbf{v}$.

a) Show that in the absence of gravity a spherical water droplet moving through air with an initial velocity v_0 in some direction at $\mathbf{r}_0 = 0$ will have a velocity:

$$v(t) = v_0 e^{-t/\tau}$$

at time t. Find an expression for the time constant τ, and determine its numerical value for droplets of radii $R = 1\,\mu\text{m}$ and $R = 1\,\text{mm}$. The viscosity of air (at $0\,^\circ\text{C}$) is $\eta = 1.7 \cdot 10^{-6}\,\text{Ns/m}^2$ (SI unit).

b) A water droplet is falling from rest at $t = 0$ in the Earth's gravitational field ($g = 9.81\,\text{m/s}^2$). Solve Newton's equation for the motion in the vertical direction, and show that the velocity approaches a constant, $v(t) \to v_t$, as $t \to \infty$. Find an expression for this terminal velocity, v_t, and determine its numerical value for spheres of radii $R = 1\,\mu\text{m}$ and $R = 1\,\text{mm}$.

Problem 7: Creeping flow, as discussed in problem 6, applies only to very small objects and/or very low velocities. In a much larger range of velocities and sizes the drag on a moving object is much better approximated by a law quadratic in the relative velocity, v. Again for a symmetric object the drag force is directed in the opposite direction of the motion, with a magnitude which can be expressed approximately as.

$$F_R = \frac{1}{2} C \rho A v^2 \,,$$

where A is the cross section of the object perpendicular to the direction of motion, ρ the density of the *fluid* and C a dimensionless constant, called the *drag coefficient*, which depends on the shape of the object. Note that this result is independent of the viscosity of the fluid.

a) Solve the equation of motion for an object of mass m with an initial velocity v_0 and drag coefficient C moving through a fluid of density ρ, if gravity can be neglected.

b) Show that for an object falling in the gravitational field in a fluid under conditions when the above law is valid, the object will reach a terminal velocity, v_t, at which the forces balance ($g = 9.81\,\text{m/s}^2$).

c) Find the velocity as a function of time, $v(t)$, if the object starts from rest. Show that $v(t) \to v_t$ as $t \to \infty$.