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Suggested solutions, FYS 500 — Classical Mechanics Theory 2017 fall

Set 12 for 10. November 2017

Problem 70:

In standard notation, with $\beta = \dot{x}/c$ and $\gamma = 1/\sqrt{1-\beta^2}$, we have the Lagrangian as:

$$L = mc^2 + T - V = -\sqrt{1-\beta^2}mc^2 - \frac{1}{2}kx^2\,,$$

from which the energy follows from the standard Hamiltonian construction:

$$E = h = p\dot{x} - L = \gamma mc^2 + V(x) = \gamma mc^2 + \frac{1}{2}kx^2\,,$$

where the canonical momentum is:

$$p = \frac{\partial L}{\partial \dot{x}} = -mc^2\frac{\mathrm{d}\gamma}{\mathrm{d}\dot{x}} = \gamma mc\beta\,.$$

(see *Goldstein*, eqs. 6.137 and 6.140). Since L (and therefore h) is independent of t, E is conserved. As usual in one-dimensional problems, $x(t)$ is most easily obtained from the expression for the energy. We can rewrite this as:

$$\frac{1}{\gamma} = \sqrt{1-\left(\frac{\dot{x}}{c}\right)^2} = \frac{mc^2}{E-V(x)} \qquad \Longleftrightarrow \qquad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = c\sqrt{\left[1-\left(\frac{mc^2}{E-V(x)}\right)^2\right]}\,.$$

This is the equation of motion, which can be solved by a direct integration, but the answer can only be obtained in terms of elliptic integrals. [For results valid in the limit of small oscillations, se *Goldstein*, p. 316-7.]

Problem 71:

The Lagrangian for a particle in an electromagnetic field is given by *Goldstein* eq. (1.63) as:

$$L = \frac{1}{2}m\mathbf{v}^2 - q\phi + q\mathbf{A}\cdot\mathbf{v}\,,$$

where ϕ is the electric scalar potential ($q\phi$ is the potential energy), and $\mathbf{A}$ the magnetic vector potential. With a gauge transformation, $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\psi$, $\phi \to \phi' = \phi - \partial\psi/\partial t$, the transformed Lagrangian is:

$$L' = \frac{1}{2}m\mathbf{v}^2 - q\phi' + q\mathbf{A}'\cdot\mathbf{v} = \frac{1}{2}m\mathbf{v}^2 - q\phi + q\mathbf{A}\cdot\mathbf{v} + q\left(\frac{\partial\psi}{\partial t} + \boldsymbol{\nabla}\psi\cdot\mathbf{v}\right) = L + \frac{\mathrm{d}\psi}{\mathrm{d}t}\,,$$

where we have used that $\mathbf{v} = \dot{\mathbf{r}}$. But we learned in problem 21 (cf. *Goldstein* derivation 1.8) that adding a total derivative to the Lagrangian does not change the equations of motion, and hence the motion of the particle.

Problem 72:

Exam problem 2, 2013 fall. See separate solution sheet.

Problem 73:

Exam problem 1, 2015 spring. See separate solution sheet.

PROBLEM 74:

a) We must have:

$$\mathbf{a}' = \mathbf{C}'\mathbf{b}' = \mathbf{La} = \mathbf{LCb} = \mathbf{LCL}^{-1}\mathbf{b}' \qquad \Longleftrightarrow \qquad \mathbf{C}' = \mathbf{LCL}^{-1}\,.$$

b) Since $\mathrm{Tr}\,\mathbf{AB} = \mathrm{Tr}\,\mathbf{BA}$, we have:

$$C'^{\mu}{}_{\mu} = \mathrm{Tr}\,\mathbf{LCL}^{-1} = \mathrm{Tr}\,\mathbf{L}^{-1}\mathbf{LC} = \mathrm{Tr}\,\mathbf{C} = C^{\mu}{}_{\mu}\,.$$

c) With the matrix notation $\mathbf{F} = (F^{\alpha}{}_{\beta})$ we have from the explicit representation of $\mathbf{F}$ (see *Goldstein* 7.71):

$$\mathbf{F} = \frac{1}{c}\begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix} \quad \Rightarrow$$

$$\mathbf{F}^2 = \frac{1}{c^2}\begin{pmatrix} \mathbf{E}^2 & -c(\mathbf{E}\times\mathbf{B})_x & -c(\mathbf{E}\times\mathbf{B})_y & -c(\mathbf{E}\times\mathbf{B})_z \\ -c(\mathbf{E}\times\mathbf{B})_x & E_x^2 - c^2(B_y^2+B_z^2) & E_xE_y + c^2B_xB_y & E_xE_z + c^2B_xB_z \\ -c(\mathbf{E}\times\mathbf{B})_y & E_xE_y + c^2B_xB_y & E_y^2 - c^2(B_x^2+B_z^2) & E_yE_z + c^2B_yB_z \\ -c(\mathbf{E}\times\mathbf{B})_z & E_xE_z + c^2B_xB_z & E_yE_z + c^2B_yB_z & E_z^2 - c^2(B_x^2+B_y^2) \end{pmatrix}.$$

According to the previous part, the trace of $c^2\mathbf{F}^2$ is invariant (a scalar):

$$\frac{1}{2}c^2\mathrm{Tr}\,\mathbf{F}^2 = \frac{1}{2}c^2F^{\alpha}{}_{\beta}F^{\beta}{}_{\alpha} = \mathbf{E}^2 - c^2\mathbf{B}^2\,.$$

d) If $|\mathbf{E}| > c\,|\mathbf{B}|$ in some coordinate system, then from the previous part $\mathbf{E}^2 - c^2\mathbf{B}^2 > 0$ in any frame, so $|\mathbf{E}| - c\,|\mathbf{B}| > 0$ in all coordinate systems.

PROBLEM 75:

The first identity follows from the vector triple product property $\mathbf{a}\cdot\mathbf{b}\times\mathbf{e} = \mathbf{a}\times\mathbf{b}\cdot\mathbf{e}$ with $\mathbf{e} = \mathbf{c}\times\mathbf{d}$. The last identity then follows from the formula given.

PROBLEM 76:

a) Since $\mathbf{E}$ and $\mathbf{B}$ are vectors, the spatial scalar product $\mathbf{E}\cdot\mathbf{B}$ is rotation invariant. It therefore suffices to prove that it is also invariant under a boost. This we can take to be the x-direction, $\mathbf{L}_x(\beta)$, as the result for an arbitrary direction can then be obtained by an additional rotation. Since $\mathbf{L}^{-1}(\beta) = \mathbf{L}(-\beta)$, we have from *Goldstein* eq. (7.11) and the fact that $\mathbf{F}$ must transform as $\mathbf{C}$ in problem 73 above ($v = c\beta$):

$$\mathbf{F}' = \frac{1}{c}\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix}\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{c}\begin{pmatrix} 0 & \gamma^2(1-\beta^2)E_x & \gamma(E_y - c\beta B_z) & \gamma(E_z + c\beta B_y) \\ \gamma^2(1-\beta^2)E_x & 0 & \gamma(-\beta E_y + cB_z) & \gamma(-\beta E_z - cB_y) \\ \gamma(E_y - c\beta B_z) & \gamma(\beta E_y - cB_z) & 0 & cB_x \\ \gamma(E_z + c\beta B_y) & \gamma(\beta E_z + cB_y) & -cB_x & 0 \end{pmatrix}.$$

We can then read off the transformed fields as:

$$E'_x = \gamma^2(1-\beta^2)E_x = E_x\,, \qquad E'_y = \gamma(E_y - vB_z)\,, \qquad E'z = \gamma(E_z + vB_y)\,,$$

$$B'_x = B_x\,, \qquad B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)\,, \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)\,.$$

By taking the parallel and perpendicular components of the results given on the problem sheet, on finds identical results. The identity $\gamma^2\beta^2 = \gamma^2 - 1$ may be useful in proving this.

b) Using the identity proven in problem 74b, one finds:

$$\mathbf{E}'\cdot\mathbf{B}' = \gamma^2\left[\mathbf{E}\cdot\mathbf{B} - \left(\frac{2\gamma}{\gamma+1} - \frac{\gamma^2\beta^2}{(\gamma+1)^2}\right)(\boldsymbol{\beta}\cdot\mathbf{E})(\boldsymbol{\beta}\cdot\mathbf{B}) - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\cdot\mathbf{v}\times\mathbf{B}\right]$$

$$= \gamma^2\left[\mathbf{E}\cdot\mathbf{B} - \left(\frac{2\gamma}{\gamma+1} - \frac{\gamma^2-1}{(\gamma+1)^2}\right)(\boldsymbol{\beta}\cdot\mathbf{E})(\boldsymbol{\beta}\cdot\mathbf{B}) - \beta^2\mathbf{E}\cdot\mathbf{B} + (\boldsymbol{\beta}\cdot\mathbf{E})(\boldsymbol{\beta}\cdot\mathbf{B})\right] = \mathbf{E}\cdot\mathbf{B}\,.$$

It would have sufficed to prove this for a boost in a specific direction, as the result is obviously rotation invariant. Note that this result means that if $\mathbf{E}$ and $\mathbf{B}$ are perpendicular in some inertial coordinate system, as they are for electromagnetic radiation, they are perpendicular in any frame.