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Set 12. Exercises for 10. November 2017

Problem 70: Find the equation of motion for the relativistic one-dimensional harmonic oscillator, *i.e.* relativistic motion in the potential $V(x) = \frac{1}{2}kx^2$, where k is the force constant: [Hint: See *Goldstein*, p. 316-7.]

Problem 71: *Goldstein*, exercise 1.9. [There is a printing error, the gauge transformation for ϕ shall read: $\phi \to \phi - \partial\psi/\partial t$].

Problem 72: Exam problem 2, 2013 fall.

Problem 73: Exam problem 1, 2015 spring.

Problem 74: Assume that two 4-vectors $\mathbf{a} = (a^\mu)$ and $\mathbf{b} = (b^\mu)$ are proportional, $\mathbf{a} = \mathbf{Cb}$ or $a^\mu = C^\mu{}_\nu b^\nu$, where $\mathbf{C} = (C^\mu{}_\nu)$ is a matrix.

a) Show that for this relation to be of the same form in all inertial coordinate systems, $\mathbf{a} \to \mathbf{a}' = \mathbf{La}$ etc., $\mathbf{C}$ must transform as:
$$\mathbf{C}' = \mathbf{LCL}^{-1}\,.$$

b) Show that the trace of $\mathbf{C}$, Tr $\mathbf{C} = C^\mu{}_\mu$, is a scalar.

c) Use this result to show that $\mathbf{E}^2 - c^2\mathbf{B}^2$ is invariant under a general Lorentz transformation.

d) Use the above to show that if there is a coordinate frame where $|\mathbf{E}| > c|\mathbf{B}|$, then $\mathbf{E}$ does not vanish in any coordinate system.

Problem 75: Prove the vector identities:

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})] = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\,.$$

[Hint:
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})\,.]$$

Problem 76:

a) Use the transformation formula in problem 72a for the field tensor, $F^\mu{}_\nu$ (see *Goldstein* eq. 7.71), to find the transformation law for electric and magnetic fields:

$$\mathbf{E}' = \gamma\left[\mathbf{E} + \mathbf{v}\times\mathbf{B} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})\right]$$
$$\mathbf{B}' = \gamma\left[\mathbf{B} - \frac{1}{c}\boldsymbol{\beta}\times\mathbf{E} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})\right]$$

[Hint: Explain why it suffices to check these formulas for a boost in, say, the x-direction. You can then use *Goldstein* eq. (7.16) for $\mathbf{L}$.]

b) Use the above result to show that $\mathbf{E}\cdot\mathbf{B}$ is a scalar under Lorentz transformations. [Hint: Why does it suffice to show that this is the case for a standard boost?].