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Additions and comments to *Classical Mechanics* by H. Goldstein & al. (3 ed. 2002).

Tensors

A *tensor* is an algebraic object which is defined by its transformation properties under a *linear* coordinate transformation. In general such a transformation transforms a vector $\mathbf{v}$ in a D dimensional real vector space according to:

$$\mathbf{v}' = A\,\mathbf{v} \qquad \Longleftrightarrow \qquad v'_i = A_{ij} v_j\,, \tag{0.43}$$

for some *invertible* matrix $A = (A_{ij})$. Any D-component object transforming like this is called a *vector* (under the transformation A). A *scalar*, ρ, under the same transformation is simply a quantity that does not change:

$$\rho' = \rho\,. \tag{0.44}$$

Furthermore, assume that we have some homogenous linear relation between two vectors $\mathbf{v}$ and $\mathbf{w}$, *i.e.* a relation of the form:

$$\mathbf{w} = M\,\mathbf{v} \qquad \Longleftrightarrow \qquad w_i = M_{ij} v_j\,, \tag{0.45}$$

In order for this relation to be retain its form under the transformation, so $\mathbf{w}' = M'\mathbf{v}'$ for any vector $\mathbf{v}$, we must have:

$$\mathbf{w}' = A\,\mathbf{w} = AM\mathbf{v} = AMA^{-1}\,\mathbf{v} = M'\mathbf{v}'\,.$$

This is achieved if M transforms as (cf. *Goldstein* p. 149):

$$M' = AMA^{-1} \qquad \Longleftrightarrow \qquad M'_{ij} = A_{ik} M_{kl} A^{-1}_{lj} = A_{ik} \widetilde{A^{-1}}_{jl} M_{kl}\,. \tag{4.41}$$

Any quantity that transforms like this is called a (mixed) *tensor of rank two* under the transformation A.

Normally one is interested in some set of linear transformations, such that two successive transformations, obtained by matrix multiplication, the inverse transformations and the trivial transformation with $A = \mathbb{1}$ are members of the set. This means that the set of transformations constitutes a *group*. In order to completely specify the tensor nature of a matrix, one has to specify the group of allowed transformations.

In Classical Mechanics, and many other applications, we are mostly interested in the group of *proper orthogonal* transformations, the rotations, with $A^{-1} = \widetilde{A}$ and $|A| = 1$, in $D = 3$ dimensions. This leaves eqs. (0.43) and (0.44) unchanged, while eq. (4.41) takes the particularly simple form:

$$M'_{ij} = A_{ik} A_{jl} M_{kl}\,. \tag{4.41$'$}$$

M is then called an Euclidean tensor, or simply just a tensor, of rank 2.

This equation has a generalization which takes us beyond standard linear algebra. We consider objects with N indices with transformations involving N orthogonal matrices, A, according to:

$$T'_{i_1 i_2 \ldots i_N} = A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_N j_N} T_{j_1 j_2 \ldots j_N}\,. \tag{5.10}$$

Such objects, taken collectively, are called *tensors of rank N* (under the rotation group), or are said to transform as tensors under the group. Such a tensor clearly has D^N components in D dimensions. In particular a scalar is a tensor of rank 0, a vector one of rank 1 and a matrix a tensor of rank 2. An abstract tensor, $\mathbf{T}$ is defined by the set of all its components, just like a matrix: $\mathbf{T} = \{T_{i_1 i_2 \ldots i_N}\}$. Two tensors $\mathbf{T}$ and $\mathbf{U}$ are equal if and only if they have the same components, $T_{i_1 i_2 \ldots i_N} = U_{i_1 i_2 \ldots i_N}$.

The simplest example of such a tensor of rank N is the *direct product*, or the *exterior product*, of N vectors, $\mathbf{a}, \mathbf{b}, \cdots \mathbf{z}$. This is simply defined as:

$$T_{i_1 \dots i_N} = a_{i_1} b_{i_2} \dots z_{i_N} \,. \tag{0.46}$$

In a rotated coordinate system it has components:

$$T'_{i_1 \dots i_N} = a'_{i_1} b'_{i_2} \cdots z'_{i_N} = A_{i_1 j_1} a_{j_1} A_{i_2 j_2} b_{j_2} \cdots A_{i_N j_N} z_{j_N} = A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_N j_N} T_{j_1 j_2 \dots j_N} \,, \tag{0.47}$$

so eq. (5.10) is satisfied. If we want a coordinate free notation for such a tensor, we use the symbol $\otimes$:

$$\mathbf{T} = \mathbf{a} \otimes \mathbf{b} \cdots \otimes \mathbf{z} \,. \tag{0.47$'$}$$

where $\mathbf{T}$ has the components $T_{i_1 \dots i_N}$ from eq. (0.46).

We note in passing that the direct product of 2 vectors is easily handled in matrix notation. For D=3:

$$\mathbf{a} \otimes \mathbf{b} = (a_i b_j) = \mathbf{a}\widetilde{\mathbf{b}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} (b_1, b_2, b_3) = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix} \neq \mathbf{b} \otimes \mathbf{a} = \widetilde{\mathbf{a} \otimes \mathbf{b}} \,. \tag{0.48}$$

The trace of this direct product is just the scalar product:

$$\mathrm{Tr}\ \mathbf{a} \otimes \mathbf{b} = a_i b_i = \widetilde{\mathbf{a}}\mathbf{b} = \mathbf{a} \cdot \mathbf{b} \,. \tag{0.49}$$

It is left as an exercise to find that the eigenvalues of $\mathbf{a} \otimes \mathbf{b}$ are $\mathbf{a} \cdot \mathbf{b}$ with eigenvector $\mathbf{a}$, and 0, which is doubly degenerate.

Eq. (0.49) has a very useful generalization. A *contraction*, a kind of a generalized trace, of a tensor is defined as the object obtained by summing the components over any pair of indices. Thus if $\mathbf{T}$ is a tensor of rank N, $\mathbf{U}$ defined by:

$$U_{i_3 \dots i_N} = T_{j j i_3 \dots i_N} \,, \tag{0.50}$$

is a contraction of $\mathbf{T}$, as is $T_{j i_2 \dots i_{N-1} j}$ and so on. Since any pair of indices can be contracted, there are altogether $\frac{1}{2}N(N-1)$ possible contractions of a tensor of rank N, which in general are different. They have the useful property that a contraction of a tensor of rank N is a tensor of rank $N-2$:

$$\begin{aligned} U'_{i_1 \dots i_{N-2}} &= T'_{j j i_1 \dots i_{N-2}} = A_{j k_1} A_{j k_2} A_{i_1 k_3} \cdots A_{i_{N-2} k_N} T_{k_1 k_2 k_3 \dots k_N} = \delta_{k_1 k_2} A_{i_1 k_3} \cdots A_{i_{N-2} k_N} T_{k_1 k_2 k_3 \dots k_N} \\ &= A_{i_1 k_3} \cdots A_{i_{N-2} k_N} T_{k_1 k_1 k_3 \dots k_N} = A_{i_1 k_3} \cdots A_{i_{N-2} k_N} U_{k_3 \dots k_N} \,, \end{aligned} \tag{0.51}$$

where we have used the orthogonality of A: $A_{ij} A_{ik} = \delta_{jk}$. This is indeed the correct transformation law for a tensor of rank $N-2$.

Tensors often occur in applications when one generalizes the concept of a linear relation to matrices. As an example, in continuum mechanics the general formulation og Hooke's law of elasticity, stating that forces are linearly proportional to deformations, but not necessarily acting in the same direction in an anisotropic medium. This can be written as a tensor equation:

$$\sigma_{ij} = C_{ijkl} D_{kl} \,,$$

where σ_{ij} is called the stress tensor, describing the forces on an element of the material, while D_{ij} is the shear tensor, describing the deformation (for further details, take the course in Petroleum Physics). Because σ_{ij} and D_{ij} by construction transforms as matrices under rotations, *i.e.* they are tensor of rank 2, a straightforward generalization of the argument used to derive eq. (4.41) yields that C_{ijkl}, which is called the stiffness tensor, has to be a tensor of rank 4 under rotations. In the same manner one proves the useful general result that if $\mathbf{A}$ is a tensor of rank M and B is a tensor of rank N which are proportional in the above sense, *i.e.* there is a set of numbers $C_{i_1 \dots \subset_{M+N}}$ such that:

$$B_{i_1 \dots i_N} = C_{i_1 \dots i_N j_1 \dots j_M} A_{j_1 \dots j_M} \,, \tag{0.52}$$

then the $C_{i_1 \dots i_{M+N}}$ transforms as a tensor of rank $M+N$.

A particularly useful tensor, of rank 3 for $D = 3$, is the *Levi–Civita tensor*, ϵ_{ijk} (see the lecture notes for 25.09 2017). We already know that the triple product between three vectors **a**, **b** and **c** can be written:

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \epsilon_{ijk} a_i b_j c_k \,, \tag{0.53}$$

But $a_i b_j c_k = (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c})_{ijk}$ are the components of a tensor of rank M=3, while the triple product is a scalar, *i.e.* a tensor of rank $N = 0$, so by the above result, it follows that ϵ_{ijk} is a tensor of rank $M + N = 3$. It has the peculiar property that the components are the same in all coordinate system.

To see why the Levi–Civita tensor is useful, we note that the formula for a determinant for $D = 3$ can be written:

$$\begin{aligned} |M| &= M_{11}M_{22}M_{33} - M_{12}M_{21}M_{33} - M_{13}M_{22}M_{31} - M_{11}M_{23}M_{32} + M_{13}M_{21}M_{32} + M_{12}M_{23}M_{31} \\ &= \epsilon_{ijk} M_{1i} M_{2j} M_{3k} = \epsilon_{ijk} M_{i1} M_{j2} M_{k3} \,. \end{aligned} \tag{0.54}$$

To interchange a pair of rows or columns of $|M|$ is equivalent to interchanging two of the indices $1, 2, 3$ on the right hand side of this equation. A series of such interchanges, which can generate an arbitrary permutation $1, 2, 3 \to l, m, n$, will therefore generate a sign which is simply ϵ_{ijk}. We can therefore write eq. (0.54) as:

$$\epsilon_{lmn} |M| = \epsilon_{ijk} M_{il} M_{jm} M_{kn} = \epsilon_{ijk} M_{li} M_{mj} M_{nk} \,. \tag{0.54$'$}$$

Since ϵ_{ijk} is a tensor of rank 3, we therefore have:

$$\epsilon'_{ijk} = A_{il} A_{jm} A_{kn} \epsilon_{lmn} = \epsilon_{ijk} |A| = \epsilon_{ijk} \,, \tag{0.55}$$

Since we have assumed $|A| = 1$, this confirms the invariance property of the Levi–Civita tensor.

However, if we enlarge our group to include *improper* orthogonal transformations, *i.e.* transformations with $|A| = -1$, we see that the Levi–Civita tensor is no longer a tensor. To handle this case, a new concept is introduced: The components $T_{i_1 \ldots i_n}$ form a *pseudotensor* if they have the transformation property;

$$T'_{i_1 i_2 \ldots i_N} = |A| A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_N j_N} T_{j_1 j_2 \ldots j_N} \qquad \text{Pseudotensor} \,. \tag{5.10a}$$

It is evident that the product of a tensor and a pseudotensor, possibly combined with one or more contractions, is a pseudotensor, since the transformation formula will contain only a single $|A|$, while the product of two pseudotensors yields a tensor, since $|A|^2 = 1$. Because of the appearance the pseudotensor ϵ_{ijk} in the definition, the vector product forms a pseudovector from two vectors, while the scalar triple product, eq. (0.53), forms a pseudoscalar, with the transformation law $\rho' = |A|\rho$, from three scalars, etc.

Finally we note that in D dimensions one defines a Levi–Civita tensor of rank D, with D indices, in complete analogy with eqs. (0.26) and (0.27). The obvious analogues of (0.54') and (0.55) remain valid. We also retain the definition of a pseudotensor.