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Additions and comments to *Classical Mechanics* by H. Goldstein & al. (3 ed. 2002).

Hamilton's principle with constraints

In *Goldstein* sect. 1.3 it was shown how problems with holonomic constraints can be solved by choosing generalized coordinates in such a way that the constraints are automatically obeyed. However, this is not always practical. In this note, which is an abbreviated version of section 2.4 of *Goldstein*, we shall discuss an alternative approach, based on Hamilton's principle. This method can also be applied to certain types of non-holonomic constraints. It works for constrained variational problems in general, but we shall restrict ourselves to the case of Hamilton's principle.

Let us assume that we have a problem were we know the Lagrangian $L = L(q_1, q_2, \dots q_n, \dot{q}_1, \dot{q}_2, \dots \dot{q}_n, t)$ in terms of a set of n coordinates, $\{q_i\}_{i=1}^n$, which are *not* independent, but subject to m constraints. These may depend not only on $\{q_i\}$, but also on the generalized velocities $\{\dot{q}_i\}$, but we shall exclude cases in which they depend on higher time derivatives of the coordinates. Thus we assume that we have m constraint relations:

$$f_\alpha(q_1, q_2, \dots q_n, \dot{q}_1, \dot{q}_2, \dots \dot{q}_n, t) = 0 \qquad \alpha = 1 \dots m\,. \tag{2.24}$$

Such constraints which also contain the velocities are called *semi-holonomic*.

We now use a trick which may be known from elementary analysis, namely the introduction of *Lagrangian multipliers*. We introduce a modified Lagrangian, which is a function of not only the q_i's, but also of m new variables, $\{\lambda_\alpha(t)\}_{\alpha=1}^m$, as follows†:

$$\widehat{L}(q_1, q_2, \dots q_n, \dot{q}_1, \dot{q}_2, \dots \dot{q}_n, \lambda_1, \lambda_2, \dots \lambda_m, t) = L + \sum_{\alpha=1}^{m} \lambda_\alpha(t) f_\alpha\,, \tag{2.20}$$

We now consider Hamilton's principle for this modified Lagrangian, assuming that *all* the $m+n$ variables $\{q_i\}$ and $\{\lambda_\alpha\}$ are independent:

$$\delta \widehat{I} = \delta \int_1^2 \widehat{L}(\{q_i\}, \{\dot{q}_i\}, \{\lambda_\alpha\}, t)\, \mathrm{d}t = 0\,. \tag{2.26}$$

If we first vary with respect to λ_α, noting that $\dot{\lambda}_\alpha$ does not appear in $\widehat{L}$, we find the Euler–Lagrange equations:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \widehat{L}}{\partial \dot{\lambda}_\alpha} + \frac{\partial \widehat{L}}{\partial \lambda_\alpha} = \frac{\partial \widehat{L}}{\partial \lambda_\alpha} = f_\alpha = 0\,,$$

so we just recover the constraints of eq. (2.24). But this means that we can vary over *all* q_i's in the variation in (2.26), because the constraints are automatically enforced by the variation over the λ_α's (this is where we use that f_α should only depend on $\dot{q}_i$, and not higher time derivatives). Thus we find the equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \widehat{L}}{\partial \dot{q}_i} - \frac{\partial \widehat{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} - Q_i = 0 \qquad i = 1 \dots n\,, \tag{2.27}$$

where Q_i contains all the terms involving the f_α's and the λ_α's:

$$Q_i = \sum_\alpha \left[\lambda_\alpha(t) \frac{\partial f_\alpha}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_\alpha \frac{\partial f_\alpha}{\partial \dot{q}_i} \right) \right]\,. \tag{2.27$'$}$$

† There is a consistency problem with the signs of the λ_α's in *Goldstein*. Note also that the book uses μ_α as symbol of a Lagrangian multiplier in the semi-holonomic case.

We see that we have obtained an expression for the *constraining forces* as the forces not derivable from the potential energy $V^{\ddagger}$. Note that we actually do not need to calculate the constraining generalized force Q_i acting on to q_i separately. We just write eq. (2.27) as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

where the left hand side put equal to zero is the Euler–Lagrange equation for q_i for the corresponding unconstrained problem. The constraining force Q_i on the right hand side is then the sum of all terms containing the constraints and the Lagrangian multipliers.

In the *holonomic case*, when the f_α's do not contain the $\dot{q}_i$'s, the price of this method is that we must solve n simultaneous differential equations for q_i, in addition to the m constraint equations. In contrast, if we are able to eliminate the constraints before deriving the Euler–Lagrange equations, we only have to solve $n - m$ equations. We also note that in the simplest cases, when the constraints simply hold some of the generalized coordinates fixed, the constraint equations can be written $f_\alpha = q_\alpha - c_\alpha = 0$ for some constants c_α. Hence $\partial f_\alpha / \partial q_i = \delta_{\alpha i}$, and the corresponding generalized force is simply given by the Lagrangian multiplier itself: $Q_\alpha = \lambda_\alpha$.

For *semi-holonomic* constraints the constraint equations depend on the $\dot{q}_i$'s. From eq. (2.27') we then see that if the f_α's have a more complex dependence on $\dot{q}_i$ than linear and/or quadratic terms, the resulting equations of motion, eq. (2.27), will become quite ugly nonlinear equations. Hence the method of Lagrangian multipliers in practice works only for simple semi-holonomic constraints.

Example

As a simple example of how to calculate the constraining force of a simple pendulum both using Newtons 2. law and Hamilton's principle with constraints.

We consider small oscillations of a plane pendulum of mass m swinging in the gravitational field. We choose coordinates such that x is horizontal and y vertical upwards, so the pendulum swings in the xy-plane. The Lagrangian is then, with g as the acceleration of gravity:

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy\,.$$

Choosing plane polar coordinates r, θ, with θ measured from the *negative* y-axis, we have $\mathbf{r} = [x, y] = r[\sin\theta, -\cos\theta]$, the Lagrangian becomes:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + mgr\cos\theta\,.$$

If the string is nor elastic, the constraint is $r = a$, the length of the pendulum, so $\dot{r} = 0$. With θ as generalized coordinate, we have:

$$L = \frac{1}{2}mra^2\dot{\phi}^2 + mga\cos\theta\,.$$

The Euler–Lagrange equation for θ is then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = ma^2\ddot{\theta} + mg\sin\theta = 0\,.$$

For small oscillations, $\sin\theta \approx \theta$, and the equation of motion reduces to:

$$\ddot{\theta} + \omega^2\theta = 0\,,$$

with $\omega^2 = g/a$. The general solution of this equation can be written:

$$\theta = A\cos(\omega t + \delta)\,, \qquad \dot{\theta} = -A\omega\sin(\omega t + \delta)\,,$$

‡ There is a serious printing error in *Goldstein* eq. (2.27).

where A and δ are constants of integrations. If the pendulum is released from rest at $t = 0$ at an angle of $\theta(0) = \theta_0$, one has $\dot{\theta}(0) = -A \sin\delta = 0$, so $\delta = 0$ (or $\delta = \pi$, which gives the same result with $A \to -A$), and therefore $\theta(0) = A \cos 0 = A = \theta_0$. Hence the solution is $\theta(t) = \theta_0 \cos\omega t$.

To find the constraining force in the string, we remember from eq. (0.41) in the lecture notes for 01.09 that the *centripetal acceleration*, which must be provided by the string, for constant $r = a$ is $F_c = -ma\dot{\theta}^2$. In addition this force must compensate for the r-component of the gravitational force. Thus the total force to be provided by the string is:

$$
\begin{aligned}
F_r &= -mg\cos\theta - ma\dot{\theta}^2 = -m[g\cos(\theta_0\cos\omega t) + a\theta_0^2\omega^2\sin^2\omega t] \\
&\approx -mg[(1 - \frac{1}{2}\theta_0^2\cos^2\omega t) + \theta_0^2\sin^2\omega t] = -mg[1 + \frac{1}{2}\theta_0^2(3\sin^2\omega t - 1)]\,,
\end{aligned}
$$

where we have used $g = a\omega^2$, $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ for $\phi \ll 1$ and $\cos^2\phi + \sin^2\phi = 1$.

If we instead want to calculate F_r from Hamilton's principle with constraints, we add the constraint condition, $f = r - a$ to the Lagrangian multiplied with a Lagrangian multiplier $\lambda(t)$, according to eq. (2.20):

$$
\widehat{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta + \lambda(r - a)\,.
$$

We then have to solve three simultaneous equations:

$$
\begin{aligned}
\lambda: &\quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widehat{L}}{\partial\dot{\lambda}} - \frac{\partial\widehat{L}}{\partial\lambda} = -(r - a) = 0\,, \\
r: &\quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widehat{L}}{\partial\dot{r}} - \frac{\partial\widehat{L}}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta - \lambda = 0\,, \\
\theta: &\quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widehat{L}}{\partial\dot{\theta}} - \frac{\partial\widehat{L}}{\partial\theta} = mr^2\ddot{\theta} + 2m\dot{r}\dot{\theta} + mg\sin\theta = 0\,.
\end{aligned}
$$

The first of these equations reproduces the constraint. Inserting this in the last equation leads to the previous equation of motion. The middle equation yields the constraining force, *i.e.* the string tension:

$$
Q = \lambda = -mg\cos\theta - ma\dot{\theta}^2 = F_r\,,
$$

as before.