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Additions and comments to *Classical Mechanics* by H. Goldstein & al. (3 ed. 2002).

Vector, or cross, product

In contrast to the scalar product, which can be defined (in many ways) in any number of dimensions, the *vector product*, or *cross product*, is only defined as such in 3 dimensional space. There are several possible definitions. In terms of vector components, it can be defined as:

$$\mathbf{r} \times \mathbf{s} = [r_2 s_3 - r_3 s_2, r_3 s_1 - r_1 s_3, r_1 s_2 - r_2 s_1] = -\mathbf{s} \times \mathbf{r}\,. \tag{0.21a}$$

The last relation is obtained by just interchanging the components of $\mathbf{r}$ and $\mathbf{s}$ in the definition. It can equivalently be written as a determinant:

$$\mathbf{r} \times \mathbf{s} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix} = (r_2 s_3 - r_3 s_2)\mathbf{i} + (r_3 s_1 - r_1 s_3)\mathbf{j} + (r_1 s_2 - r_2 s_1)\mathbf{k}\,, \tag{0.21b}$$

where we have expanded the determinant by rows. It immediately follows that the vector product of a vector with itself vanishes:

$$\mathbf{r} \times \mathbf{r} = -\mathbf{r} \times \mathbf{r} = 0\,. \tag{0.21c}$$

By a careful choice of coordinates, it is not difficult to show that (see Problem 3):

$$|\mathbf{r} \times \mathbf{s}| = rs\,|\sin\theta|\,, \tag{0.22}$$

where θ is the angle between the vectors $\mathbf{r}$ and $\mathbf{s}$ (cf. eq. (0.7)). Geometrically this is the area of the trapezoid, or twice the area of the triangle, spanned by the vectors $\mathbf{r}$ and $\mathbf{s}$, and hence a scalar quantity, independent of the choice of coordinate system. In particular, it follows from eq. (0.22) that if $\mathbf{r} \times \mathbf{s} = 0$ and $rs \neq 0$, then $\theta = 0$ or π, so $\mathbf{r}$ and $\mathbf{s}$ are parallel or antiparallel.

From eqs. (0.7), (0.22) and the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ finds:

$$(\mathbf{r} \cdot \mathbf{s})^2 + |\mathbf{r} \times \mathbf{s}|^2 = r^2 s^2\,. \tag{0.23}$$

By explicit calculations, we find that the *triple product* $\mathbf{r} \cdot (\mathbf{s} \times \mathbf{t})$ obeys:

$$\begin{aligned} \mathbf{r} \cdot (\mathbf{s} \times \mathbf{t}) &= (\mathbf{r} \times \mathbf{s}) \cdot \mathbf{t} = \mathbf{s} \cdot (\mathbf{t} \times \mathbf{r}) = \mathbf{t} \cdot (\mathbf{r} \times \mathbf{s}) = -\mathbf{s} \cdot (\mathbf{r} \times \mathbf{t}) = \ldots \\ &= r_1 s_2 t_3 - r_1 s_3 t_2 + r_2 s_3 t_1 - r_2 s_1 t_3 + r_3 s_1 t_2 - r_3 s_3 t_2\,, \end{aligned} \tag{0.24a}$$

Evidently the triple product is unchanged by a cyclical permutation of the vectors or if the scalar and vector multiplication signs are switched. If two vectors are interchanged, the sign of the answer is switched. Also note that since the expression $(\mathbf{r} \cdot \mathbf{s}) \times \mathbf{t}$ makes no sense, as we cannot take a cross product of the *scalar* $\mathbf{r} \cdot \mathbf{s}$ with the *vector* $\mathbf{t}$, we can drop the parentheses, and just write: $\mathbf{r} \cdot \mathbf{s} \times \mathbf{t} = \mathbf{r} \times \mathbf{s} \cdot \mathbf{t}$.

From eqs (0.24) and (0.21c) it follows that if two of the vectors in a triple product are equal, the result is zero:

$$\mathbf{r} \cdot \mathbf{r} \times \mathbf{s} = \mathbf{r} \times \mathbf{r} \cdot \mathbf{s} = 0\,. \tag{0.24b}$$

Hence $\mathbf{r} \times \mathbf{s}$ is perpendicular to both $\mathbf{r}$ and $\mathbf{s}$.

By differentiating the definition in eq. (0.21a) one finds that the rule for differentiating a vector product is the same as for a scalar product:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}\times\mathbf{s} = \cdots = \dot{\mathbf{r}}\times\mathbf{s} + \mathbf{r}\times\dot{\mathbf{s}}. \tag{0.25}$$

It should be noted that if one restricts oneself to a two-dimensional space, simply by putting $r_3 = s_3 = 0$, one finds:

$$\mathbf{r}\times\mathbf{s} = [0, 0, r_1 s_2 - r_2 s_1] = (r_1 s_2 - r_2 s_1)\mathbf{k}\,, \tag{0.21d}$$

but this is a vector pointing in the z-direction, and so not a vector in the two-dimensional space spanned by $\mathbf{r}$ and $\mathbf{s}$. In two dimensions the cross product can instead be defined as a scalar.

In 3 dimensions the vector product gives us a convenient way to construct a right handed coordinate system with two of its axes pointing along two given orthogonal vectors, say $\mathbf{a}$ and $\mathbf{b}$, with $\mathbf{a}\cdot\mathbf{b} = 0$. If $\mathbf{c} = \mathbf{a}\times\mathbf{b}$, then $\mathbf{c}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, and the corresponding unit vectors $\widehat{\mathbf{a}}$, $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{c}}$, in that order, forms the basis of a *right-handed* (check!) coordinate system, as do $(\widehat{\mathbf{b}}, \widehat{\mathbf{c}}, \widehat{\mathbf{a}})$ and $(\widehat{\mathbf{c}}, \widehat{\mathbf{a}}, \widehat{\mathbf{b}})$.

The Levi–Civita tensor

In order to simplify manipulations with cross products, it is useful to introduce the *permutation symbol* ϵ_{ijk} $(i, j, k = 1.2, 3)$, better known to physicists as the **Levi–Civita Tensor** (we shall briefly come back to the concept of a tensor later in this course). It is defined as 1 if $\{i, j, k\}$ is an even permutation of $\{1, 2, 3\}$, -1 if it is an odd permutation, and 0 if two indices are equal. Thus:

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1\,; \qquad \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1\,: \quad \epsilon_{ijk} = 0 \quad \text{otherwise}\,. \tag{0.26}$$

We see that we have:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{ikj} = -\epsilon_{kji} = -\epsilon_{jik}\,. \tag{0.27}$$

In other words, the component of the vector product are given by: $(\mathbf{a}\times\mathbf{b})_k = \epsilon_{ijk} a_i b_j$.

From this it follows that (with the summation convention):

$$\epsilon_{ijk}\, a_i b_j \mathbf{e}_k = a_1 b_2 \mathbf{e}_3 + a_2 b_3 \mathbf{e}_1 + a_3 b_1 \mathbf{e}_2 - a_1 b_3 \mathbf{e}_2 - a_2 b_1 \mathbf{e}_3 - a_3 b_2 \mathbf{e}_1 = \mathbf{a}\times\mathbf{b}\,. \tag{0.28}$$

The usefulness of this relation is due to the formula:

$$\epsilon_{ijk}\,\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\,. \tag{0.28}$$

Here δ_{ij} is **Kronecker's delta**:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & i \neq j, \end{cases} \tag{0.29}$$

which ar nothing but the matrix elements of the unit matrix. The usefulness of this symbol follows from the observation that for any function $f(i)$ we trivially have $\sum_{j=1}^{N} f(j)\,\delta_{ij} = f(i)$. We also have, using the summation convention:

$$\delta_{ii} = \sum_{i=1}^{N} \delta_{ii} = N\,. \tag{0.30}$$

Eq. (0.28) can be proven by noting that for each value of k in the sum either $i = l$ and $j = m$, in which case $\epsilon_{ijk}\,\epsilon_{lmk} = \epsilon_{ijk}\,\epsilon_{ijk} = 1$ or $i = m$ and $j = l$, in which case $\epsilon_{ijk}\,\epsilon_{lmk} = \epsilon_{ijk}\,\epsilon_{jik} = -1$, and each such term only occur once.

From this result it further follows:

$$\begin{aligned} \epsilon_{ijk}\,\epsilon_{ljk} &= \delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl} = 3\delta_{il} - \delta_{il} = 2\delta_{il}\,, \\ \epsilon_{ijk}\,\epsilon_{ijk} &= 2\delta_{ii} = 6 = 3!\,. \end{aligned} \tag{0.31}$$

The last result is actually trivial, because in the sum $\sum_{ijk} \epsilon_{ijk}^2$, according to (0.26) each nonvanishing term is equal to 1, and there are 6 such terms.

These formulas can be used to prove useful identities like:

$$\begin{aligned} \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= \epsilon_{ijk}\, a_i (\mathbf{b}\times\mathbf{c})_j\, \mathbf{e}_k = \epsilon_{ijk}\, a_i\, \epsilon_{lmj}\, b_l c_m\, \mathbf{e}_k = -\epsilon_{ikj}\,\epsilon_{lmj}\, a_i b_l c_m\, \mathbf{e}_k \\ &= -(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl})\, a_i b_l c_m\, \mathbf{e}_k = -(a_i b_i)(c_k \mathbf{e}_k) + (a_i c_i)(b_k \mathbf{e}_k) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})\,. \end{aligned} \tag{0.32}$$

This is sometimes called the "bac-cab" rule.