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Additions and comments to *Classical Mechanics* by H. Goldstein & al. (3 ed. 2002).

Kinematics

In Classical Mechanics, we are analyzing the motion of objects through space. In *kinematics*, we just concentrate on *describing* the motion mathematically, without asking questions about *why* objects move as they do.

The simplest object we can consider is a *point particle*. This is an idealization of a massive object moving through space under the idealized assumptions that the volume, shape and possible internal motions, like rotation, of the object do not influence its motion. We can then consider it in the limit that the volume goes to zero with the mass remaining constant. Such an idealized object we call a *point particle*. It might be noted that there are not so many interesting situations where this is a good approximation on Earth, although the situation is somewhat better in space. But it turns out that one can handle many more realistic situations by modelling extended bodies, and even fluids (gases and liquids), as assemblies of such point particles with certain well specified interactions.

A point particle has just two properties, the position, which we shall describe as a vector, $\mathbf{r} = [x, y, z]$, and the mass, m. The latter, however, does not enter into a kinematical description. We first assume that we express the position vector in a *fixed* coordinate system, so the basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, or $\mathbf{e}_i$, are the same at all times. If the particle moves, the coordinates change and become functions of time, t. But at any moment in time we can still write, from eqs. (0.4):

$$\mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} = r_1(t)\mathbf{e}_1 + r_2(t)\mathbf{e}_2 + r_3(t)\mathbf{e}_3 = \sum_i r_i(t)\mathbf{e}_i \equiv r_i(t)\mathbf{e}_i\,. \quad (0.13)$$

Thus, in kinematics we regard the coordinate vector as a *vector function* of time, $\mathbf{r} = \mathbf{r}(t)$. This function is also called the *trajectory* or the *orbit* of the particle.

As we remember, the *velocity* of a particle is its instantaneous rate of change of position. But it is an empirical fact that the coordinates of any object must be continuous functions of time, no real particle can move a *finite* distance instantaneously. Furthermore we shall see when we come to dynamics that Newton's laws of motion actually require that the coordinate functions are not only continuous, but even *smooth*, in the sense that their time derivatives much also exist.[†]

If a particle is at the point $\mathbf{r}(t) = [x(t), y(t), z(t)]$ at time t, and at $\mathbf{r}(t+\delta t) = [x(t+\delta t), y(t+\delta t), z(t+\delta t)]$ at a slightly different time $t + \delta t$, we calculate the instantaneous velocity at t as the limit:

$$\begin{aligned}\mathbf{v}(t) &= \lim_{\delta t \to 0} \frac{\mathbf{r}(t+\delta t) - \mathbf{r}(t)}{\delta t} = \lim_{\delta t \to 0} \left[\frac{x(t+\delta t) - x(t)}{\delta t}, \frac{y(t+\delta t) - y(t)}{\delta t}, \frac{z(t+\delta t) - z(t)}{\delta t} \right] \\ &= \left[\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right]\,.\end{aligned} \quad (1.1a)$$

All the steps in this computation are valid by the vector space rules, eqs. (0.2) and the definition of the derivative of scalar functions. Thus a *vector function* is differentiated by taking the derivative of the

[†] Actually, this requirement is not strictly necessary mathematically. After you have learned about *distributions* (generalized functions) in the course of mathematical modelling, you will know how to handle discontinuous velocities.

components, provided the basis vectors are constant. In Classical Mechanics we often use Newton's dot notation for *time derivatives*, $v_x = \dot{x}(t) = \mathrm{d}x/\mathrm{d}t$ etc. We thus write: We thus write:

$$\mathbf{v}(t) = [v_x(t), v_y(t), v_z(t)] = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \equiv \dot{\mathbf{r}}(t) = [\dot{x}(t), \dot{y}(t), \dot{z}(t)] = \sum_i \dot{r}_i(t)\,\mathbf{e}_i = \dot{r}_i(t)\,\mathbf{e}_i\,. \tag{1.1b}$$

The component-wise differentiation rule for vector functions is, of course, not restricted to coordinate vectors, it applies to any time dependent vector function $\mathbf{f}(\mathrm{t})$. In particular, we shall often need to calculate the acceleration, $\mathbf{a}(t)$, defined as the rate of change of the velocity with time, *i.e.* it is the time derivative of $\mathbf{v}(t)$. Repeating the above arguments and using the notation $\ddot{f}(t) = \mathrm{d}\dot{f}/\mathrm{d}t = \mathrm{d}^2 f/\mathrm{d}t^2$ for any function $f(t)$, we have in our various notations for vector functions:

$$\mathbf{a}(t) = [a_x, a_y, a_z] = \dot{\mathbf{v}}(t) = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = [\dot{v}_x(t), \dot{v}_y(t), \dot{v}_z(t)] = [\ddot{x}(t), \ddot{y}(t), \ddot{z}(t)] = \ddot{\mathbf{r}}(t) = \sum_i \ddot{r}_i(t)\,\mathbf{e}_i = \ddot{r}_i(t)\,\mathbf{e}_i\,. \tag{0.16}$$

The rule for differentiating a scalar product is the same as that for a normal product, as is easily verified:

$$\begin{aligned}\frac{\mathrm{d}(\mathbf{r}(t)\cdot\mathbf{s}(t))}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i r_i(t)s_i(t) = \sum_i \left(\frac{\mathrm{d}r_i(t)}{\mathrm{d}t}s_i(t) + r_i(t)\frac{\mathrm{d}s_i(t)}{\mathrm{d}t}\right)\\ &= \sum_i \left(\dot{r}_i(t)s_i(t) + r_i(t)\dot{s}_i(t)\right) = \dot{\mathbf{r}}(t)\cdot\mathbf{s}(t) + \mathbf{r}(t)\cdot\dot{\mathbf{s}}(t)\,.\end{aligned} \tag{0.17}$$

We proceed to consider how to generalize these formulas to the case where we allow also the coordinate system to change with time. This becomes particularly useful when we investigate the motion of rigid bodies in chapter 5. We shall still restrict ourselves to orthonormal coordinate systems, so we assume that we have three (or N) time dependent basis vectors, $\mathbf{e}_i(t)$, which are all unit vectors, $|\mathbf{e}_i(t)| = 1$, and which remain orthogonal at all times, $\mathbf{e}_i(t)\cdot\mathbf{e}_j(t) = 0$ $(i \neq j)$. At any fixed time, t, we can then expand $\mathbf{r}(t)$ as:

$$\mathbf{r}(t) = \sum_i r_i(t)\,\mathbf{e}_i(t)\,.$$

But this is nothing but a sum of products, and by carrying out a slightly lengthier calculation similar to the one above, we are not surprised to find that the velocity is given by the usual rule for differentiating sums of products:

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \sum_i \left(\frac{\mathrm{d}r_i}{\mathrm{d}t}\,\mathbf{e}_i(t) + r_i(t)\,\frac{\mathrm{d}\mathbf{e}_i}{\mathrm{d}t}\right) = \sum_i \left[\dot{r}_i\,\mathbf{e}_i(t) + r_i(t)\,\dot{\mathbf{e}}_i(t)\right]. \tag{0.18}$$

We see that this reduces to eq. (0.15b) if e_i is time independent so $\dot{\mathbf{e}}_i = 0$.

We note that eq. (0.17) tells us that if $\dot{\mathbf{e}}_i \neq 0$, we cannot find $v_i(t)$ from the time derivatives of the coordinates, $\dot{r}_i(t)$, alone. But we can say something general about the vector $\dot{\mathbf{e}}_i$, namely that it has no component along $\mathbf{e}_i$ itself: $\dot{\mathbf{e}}_i \perp \mathbf{e}_i$ or $\dot{\mathbf{e}}_i \cdot \mathbf{e}_i = 0$ (no sum over i). This is a special case of a more general result. If $\mathbf{n}(t)$ is a vector of constant length, $|\mathbf{n}(t)| = n$, a constant in time, then $\dot{\mathbf{n}}(t)\cdot\mathbf{n}(t) = 0$ or $\dot{\mathbf{n}}(t) \perp \mathbf{n}(t)$. The proof follows from eq. (0.17):

$$n^2 = \mathbf{n}(t)^2 \qquad \Longleftrightarrow \qquad 0 = \frac{\mathrm{d}n^2}{\mathrm{d}t} = \frac{\mathrm{d}(\mathbf{n}(t))^2}{\mathrm{d}t} = 2\mathbf{n}(t)\cdot\frac{\mathrm{d}\mathbf{n}(t)}{\mathrm{d}t} = 2\mathbf{n}(t)\cdot\dot{\mathbf{n}}(t)\,. \qquad \text{Q.E.D.}$$

Thus the tip of a moving vector of constant length always moves perpendicular to the vector itself.

Dynamics

We now start with **Sect. 1.1** of *Goldstein.* The basic principles of Classical Mechanics are summarized by Isaac Newton's three laws of motion, formulated more than 300 years ago. But before we discuss them, you are reminded about a basic theorem of *non-relativistic* physics:

> **Conservation of mass:** *The mass within a closed physical system remains constant in time for all processes.*

A closed system in this connection means that no matter crosses the boundary of the system. If m is the mass of such a system, or an *object* for simplicity, conservation of mass can simply be written:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \dot{m} = 0\,. \tag{0.19}$$

This is a very important example of a *conservation law.* In general, a *conserved quantity* is a function of the variables characterizing a physical system that does not change with time. Thus the quantity $C(t)$ is conserved if:

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \dot{C}(t) = 0\,. \tag{0.20}$$

For a closed system we have:

> **Newton's 1. law:** *For any object, or more generally any closed system, which is uninfluenced by the surroundings, there exist coordinate systems where the object is at rest, or move with constant velocity.*

Such coordinate systems are called *inertial*, or *Galilean*†. It is easy to convince oneself that if a point particle has coordinate vectors $\mathbf{r}(t)$ and $\mathbf{r}'(t)$, respectively, in two inertial coordinate systems, these are connected by a coordinate transformation of the form:

$$\mathbf{r}'(t) = \mathbf{r}(t) + \mathbf{r}_0 + \mathbf{v}_0 t\,, \tag{1.0}$$

where $\mathbf{r}_0$ and $\mathbf{v_0}$ are *constant vectors*, *i.e.* vectors with constant components. Note that Newton's first law should be interpreted to state that we in principle always can move an object away from all external influences, and therefore guarantees the existence of inertial systems. This is really an idealization. In practice it is actually impossible to find *exactly* inertial coordinate systems: A body on the Earth is influenced by the gravity of the Earth, the Sun, the Moon (tidal forces!), even the planets, the stars of the Milky Way, the members of the Local Group of galaxies, galaxy clusters, ... But in practice we can often find sufficiently accurate approximate inertial systems.

Before we introduce forces, we shall define the **momentum** vector of an object of mass m and velocity $\mathbf{v}$:‡

$$\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}\,. \tag{1.2}$$

For a composite object, we shall, for now, assume that all parts of it move with the same velocity.

In terms of the momentum, Newton's second law can be expressed as:

> **Newton's 2. law:** *In an inertial coordinate system, the acceleration of a particle is given by the equation of motion:*
>
> $$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \dot{\mathbf{p}}\,, \tag{1.3}$$
>
> *where* $\mathbf{F}$ *is the* **force** *vector.*

† After Galileo Galilei.

‡ In Norwegian, the correct translation of "momentum" is *bevegelsesmengde*, although the word *impuls*, which does not really cover the same concept, is often used informally. It should *never* be translated with *moment*, which, as you should know, is something else, and which we shall return to.

For closed systems the mass is conserved, so $\dot{m} = 0$ according to eq. (0.19), and the equation of motion can be written in a better known form, using eq. (0.16):

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = m\dot{\mathbf{v}} + \dot{m}\mathbf{v} = m\dot{\mathbf{v}} = m\mathbf{a} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = m\ddot{\mathbf{r}}\,. \tag{1.5}$$

Note that this equation is only valid in *inertial coordinate systems.* Also note that the first and second law together tells us that what needs to be explained in dynamics is the causes of *acceleration*, which are forces. This is in contrast to Aristotelian physics, which tried to explain *velocities* directly in terms of forces.

The expressions for the forces are not really a part of dynamics itself. Forces may be fundamental, like gravitational or electromagnetic forces, or *effective* forces, in principle derivable from the fundamental forces. Examples are friction, reaction forces and viscous forces. When studying dynamics, we take the force law as given, ultimately deduced from observations and experiments.

It should be stressed that eq. (1.5) is a *vector* equation, and comprises 3 scalar equations, one for each vector component. Thus, with $\mathbf{F} = [F_x, F_y, F_z]$:

$$\begin{aligned} F_x &= m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = m\ddot{x}(t) = m\dot{v}_x(t)\,, \\ F_y &= m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = m\ddot{y}(t) = m\dot{v}_y(t)\,, \\ F_z &= m\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = m\ddot{z}(t) = m\dot{v}_z(t)\,, \end{aligned} \tag{1.5a}$$

If we are able to use our freedom of choosing coordinates to find a coordinate system such that one of the components of the force vanishes (for all times) in that system, the solution of the equations simplifies significantly. Thus, if $F_z = 0$, we have $m\ddot{z}(t) = 0$. One integration yields $\dot{z}(t) = v_z^0$, for some constant velocity v_z^0, and another integration $z(t) = z^0 + v_z^0 t$, for another constant of integration z^0. Hence one of the three equations of motion is trivially solved. This can also be reformulated as $\dot{p}_z = 0$, which is of the form of eq. (0.20), and is a conservation law for the z-component of the momentum.

If all the three component of the force vanishes, $\mathbf{F} = \mathbf{0}$, the equation of motion reduces to:

$$m\mathbf{a} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = m\ddot{\mathbf{r}} = \dot{\mathbf{p}} = 0\,. \tag{1.5b}$$

We see that this is equivalent to a conservation law for each component of the momentum, *i.e. vector* conservation law. Each component of this equation can be solved as above, and we find the trajectory:

$$\mathbf{r}(t) = \mathbf{r}^0 + \mathbf{v}^0 t\,, \tag{1.5c}$$

for two vector constants of integration, $\mathbf{v}^0$ and $\mathbf{r}^0$, or 6 scalar constants of integration, $\mathbf{v}^0 = [v_x^0, v_y^0, v_z^0]$ and $\mathbf{r}^0 = [r_x^0, r_y^0, r_z^0]$. We see that this is just a restatement of Newton's 1. law. It is therefore indeed often claimed that the first law is just a special case of the second. But the latter only applies in inertial systems, and the first law is needed to establish that such systems exist.

Newton's third law considers the mutual forces between pairs of objects:

> **Newton's 3. law**, *or the* **law of action and reaction:** *If one object acts on another object with a force* $\mathbf{F}$*, the second object will act back on the first with an oppositely equal force,* $-\mathbf{F}$.

If we call the mass and position vector of the first object for m_A and $\mathbf{r}_A$ and those of the second m_B and $\mathbf{r}_B$, and we assume that no other forces act on either body, we can write this as:

$$m_B\ddot{\mathbf{r}}_B = -\mathbf{F} \qquad \Longleftrightarrow \qquad m_A\ddot{\mathbf{r}}_A = \mathbf{F}\,. \tag{1.5d}$$

Adding the two equations we find:

$$m_A\ddot{\mathbf{r}}_A + m_B\ddot{\mathbf{r}}_B = \frac{\mathrm{d}}{\mathrm{d}t}\left(m_A\dot{\mathbf{r}}_A + m_B\dot{\mathbf{r}}_B\right) = \dot{\mathbf{p}}_A + \dot{\mathbf{p}}_B = 0 \quad \Longleftrightarrow \quad \mathbf{p}_A + \mathbf{p}_B = \mathbf{P}\,, \tag{1.5e}$$

where $\mathbf{P}^0$ is a constant vector. This vector equation expresses the *conservation of the total momentum* (all three components!) for two mutually interacting particles.

It should be noted that Newton's 3. law in the form stated here is not as generally valid as the first and second laws. Thus it is not applicable to magnetic forces.