UNIVERSITY OF STAVANGER
Department of Mathematics and Natural Science

Examination MAF 500 Classical Mechanics and Field Theory, 2017 spring. Suggested solutions.

Problem 1

a) From the coordinate vector:

$$\mathbf{r} = [\rho\cos\phi, \rho\sin\phi, z]\,,$$

with $\rho = \sqrt{x^2+y^2}$ we find

$$\dot{\mathbf{r}} = \dot\rho\,[\cos\phi, \sin\phi, 0] + \rho\dot\phi\,[-\sin\phi, \cos\phi, 0] + \dot z\,[0,0,1] = \dot\rho\,\mathbf{e}_\rho + \rho\dot\phi\,\mathbf{e}_\phi + \dot z\,\mathbf{k}\,.$$

One easily checks that $\{\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{k}\}$ form an orthonormal set of vectors, so:

$$\dot{\mathbf{r}}^2 = \dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2\,.$$

b) Since the particle is constrained to move on a cylindrical surface with $\rho = a$, one has $\dot\rho = 0$. If the angle ϕ is measured from the horizontal plane, we have $\psi = \frac{\pi}{2} - \phi$ and $\dot\psi = -\dot\phi$. The height of a point on the cylinder above the horizontal plane is $h = a\cos\psi$, so the gravitational potential is $V = mgh = mga\cos\psi$. Thus the Lagrangian is:

$$L = T - V = \frac{1}{2}m\left(a^2\dot\psi^2 + \dot z^2\right) - mga\cos\psi\,.$$

The conjugate momenta to ψ and z are:

$$p_\psi = \ell = \frac{\partial L}{\partial\dot\psi} = ma^2\dot\psi\,, \qquad p_z = \frac{\partial L}{\partial \dot z} = m\dot z\,.$$

The Hamiltonian then is:

$$H = p_\psi\dot\psi + p_z\dot z - L = \frac{1}{2}m\left(a^2\dot\psi^2 + \dot z^2\right) + mga\cos\psi\,.$$

The Lagrangian does not contain z, so this is a cyclical coordinate, and therefore p_z is conserved. Furthermore, L is time independent, so $H = E$ is also conserved. [This needs not to be shown.]

c) Since p_z is conserved, we have, with $\dot z(0) = v_0$, $z(0) = z_0$:

$$p_z = m\dot z(t) = m\dot z(0) = mv_0 \qquad \Longleftrightarrow \qquad z = v_0 t + z_0\,.$$

The equation of motion for ψ is the Euler–Lagrange equation:

$$\frac{\partial L}{\partial\dot\psi} = \dot p_\psi = ma^2\ddot\psi = \frac{\partial L}{\partial\psi} = mga\,\sin\psi\,.$$

Multiplying this equation with $\dot\psi$, we see that it can be written:

$$\dot\psi\left(ma^2\ddot\psi - mga\,\sin\psi\right) = \frac{ma^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot\psi^2 + \frac{2g}{a}\cos\psi\right) = 0\,,$$

and the stated result follows. Equivalently, and even simpler, one finds from energy conservation:

$$\dot\psi^2 + \frac{2g}{a}\cos\psi = \frac{2}{ma^2}\left(E - \frac{1}{2}m\dot z^2\right) = \frac{2}{ma^2}\left(E - \frac{1}{2}mv_0^2\right) = C\,,$$

a constant.

d) We have two alternative approaches. One is to start with Newton's second law of motion in the radial direction for the unconstrained problem. This reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\rho}} - \frac{\partial L}{\partial \rho} = m\left(\ddot{\rho} + \rho\dot{\psi}^2 - g\cos\psi\right) = F_r\,,$$

since $m\rho\dot{\psi}^2$ is the centrifugal acceleration.

Alternatively, one can use the method of Lagrangian multipliers. With $\lambda(t)$ as the multiplier introduced to enforce the constraint $\rho = a$, the modified Lagrangian becomes:

$$\widetilde{L} = T - V - \lambda(\rho - a) = \frac{1}{2}m\left(\dot{\rho}^2 + a^2\dot{\psi}^2 + \dot{z}^2\right) - mga\cos\psi + \lambda(\rho - a)\,.$$

Variations with respect to λ just reproduces the constraint, the equation for ψ remains the same as before, while ρ-equation reproduces the previous equation, with $F_r = \lambda$, in accordance with the lectures. [Only one of these approaches is required.]

As long as the particle remains at the cylinder surface, $\rho = a$, so $\dot{\rho} = 0$, the constraining force is:

$$F_r(\psi) = m\left(a\dot{\psi}^2 - g\cos\psi\right)\,.$$

e) The condition for the particle to remain on the cylinder is $F_r(\psi) \geq 0$, since the reaction force from the cylinder can only push the particle outward. Thus the particle slides off the cylinder when:

$$F_r(\psi_c) = 0 \qquad \Longleftrightarrow \qquad \dot{\psi}^2|_{\psi=\psi_c} = \frac{g}{a}\cos\psi_c\,.$$

Using the equation of motion for ψ_c and using that from the initial conditions one finds C as:

$$C = \dot{\psi}(0)^2 + \frac{2g}{a}\cos\psi_0 = \frac{2g}{a}\cos\psi_0 \quad \rightarrow \quad \frac{2g}{a}\,.$$

Using this in the equation for ψ_c, we find:

$$\dot{\psi}^2|_{\psi=\psi_c} = C - \frac{2g}{a}\cos\psi_c = \frac{2g}{a}(1 - \cos\psi_c) = \frac{g}{a}\cos\psi_c\,,$$

$$\cos\psi_c = \frac{2}{3}\,, \qquad \psi_c = \arccos\frac{2}{3} \approx 0.841 = 48.2^\circ\,.$$

The angular velocity when $\psi = \psi_C$ is

$$\omega_c = \dot{\psi}_c = \sqrt{\frac{g}{a}\cos\psi_c} = \sqrt{\frac{2g}{3a}}\,,$$

where the positive root has been chosen, since the particle is sliding *down* the cylinder.

f) After leaving the cylinder the particle moves in a parabolic orbit in the gravitational field, with an initial position at $\rho = a$, $\psi = \psi_c$ and initial velocity components $\dot{\rho} = 0$, $\dot{\psi} = \omega_c$. From energy conservation, in the limit $\psi_0 \to 0$, the velocity v_f reaching the horizontal plane is:

$$\frac{1}{2}mv_f^2 = mga \qquad \Rightarrow \qquad v_f = \sqrt{2ga}\,.$$

g) After leaving the cylinder the particle moves with constant horizontal velocity. The moment it leaves it the speed is $a\omega_c$ in direction $\mathbf{e}_\psi = -\mathbf{e}_\phi = -[-\sin\phi, \cos\phi, 0] = [\cos\psi, -\sin\psi, 0]$ for $\psi = \psi_c$ (see part a above). The horizontal velocity is thus:

$$v_x = a\omega_c\cos\psi_c = \sqrt{\frac{8ga}{27}}\,.$$

It hits the ground with at an angle given by:

$$\cos\phi_f = \frac{v_x}{v_f} = \sqrt{\frac{2}{27}} \qquad \rightarrow \qquad \phi_f = \arccos\frac{2}{27} \approx 1.50 = 85.8^\circ\,.$$

Problem 2

a) By Newton's second law momentum is conserved for any system if there are no external forces acting on it. Therefore the combined lump must continue to move in the x- direction, which is the direction of the conserved total momentum, $\mathbf{P}$. The conservation of $P = |\mathbf{P}|$ then yields:

$$P = m_1 v = (m_1 + m_2)v' \qquad \Longrightarrow \qquad v' = \frac{m_1}{m_1 + m_2} v\,.$$

With E and E' as the energies before and after the collision, respectively, we find the energy loss, Q, as:

$$E = \frac{1}{2}m_1 v^2\,, \qquad E' = \frac{1}{2}(m_1 + m_2)v'^2 = \frac{1}{2}\frac{m_1^2}{m_1 + m_2}v^2\,, \qquad Q = E - E' = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}v^2 > 0\,.$$

We see that some of the kinetic energy of the incoming particle has been lost in the collision. It has been converted into heat.

b) The new coordinates are:

$$\mathbf{R} = \frac{1}{m_1 + m_2}\left(m_1\mathbf{r_1} + m_2\mathbf{r}_2\right)\,, \qquad \mathbf{r} = \mathbf{r_2} - \mathbf{r_1}\,.$$

The center of mass velocity is $\mathbf{V} = \dot{\mathbf{R}}$. After the collision $\mathbf{r_1} = \mathbf{r_2}$ and $\dot{\mathbf{r}}_1 = \dot{\mathbf{r}}_2 = v'\mathbf{i}$, so:

$$\mathbf{V} = \dot{\mathbf{R}} = \frac{1}{m_1 + m_2}\left(m_1 v' + m_2 v'\right)\mathbf{i} = v'\mathbf{i} = \dot{X}\mathbf{i}\,.$$

[This can be done in several ways.] After the collision the lump is at rest in the center of mass, so $E'_{\mathrm{CM}} = 0$.

c) The mass of a sphere of constant density d is $m = \frac{4\pi}{3}da^3$. Introducing standard cylindrical coordinates ρ, ϕ, z with origin at the center o fthe sphere, $\mathbf{r} = [\rho\cos\phi, \rho\sin\phi, z]$, and taking the axis of rotation to be the z-axis, the interior of the sphere is the region $\rho^2 + z^2 < a^2$. Thus the moment of inertia about any axis is:

$$\begin{aligned} I_0 &= \int_m \left(x^2 + y^2\right)\mathrm{d}m = d\int_{-a}^{a}\mathrm{d}z\int_0^{\sqrt{a^2-z^2}}\rho^2\rho\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\phi = 2\frac{2\pi}{4}d\int_0^a\left(a^2 - z^2\right)^2\mathrm{d}z \\ &= \pi d\left(a^4 z - \frac{2}{3}a^2 z^3 + \frac{1}{5}z^5\right)\Bigg|_0^a = \frac{8\pi}{15}da^5 = \frac{2}{5}ma^2\,. \end{aligned}$$

[This can also be done in standard spherical coordinates, using $x^2 + y^2 = r^2\sin^2\theta$ and substituting $\xi = \cos\theta$ so $\mathrm{d}\xi = -\sin\theta\mathrm{d}\theta$:

$$I_0 = 2\pi d\int_0^a r^4\,\mathrm{d}r\int_0^\pi \sin^3\theta\,\mathrm{d}\,\theta = \frac{2\pi}{5}a^5\int_{-1}^{1}\left(1 - \xi^2\right)\mathrm{d}\xi = \frac{8\pi}{15}da^5 = \frac{2}{5}ma^2\,.]$$

d) From Steiner's rule one finds the moment of inertia about an axis tangent to the sphere as:

$$I' = I_0 + ma^2 = \left(\frac{2}{5} + 1\right)ma^2 = \frac{7}{5}ma^2\,.$$

Since the two identical lumps have a common tangent at their point of contact, which becomes the common rotation axis after the collision, we find:

$$I = \int_{m_1+m_2} (y^2 + z^2)\,\mathrm{d}m = 2I' = \frac{14}{5} ma^2 \,.$$

e) In the center of mass system, before the collision the lumps approach each other with oppositely equal momenta, so the total momentum, which is conserved, is zero. Hence the compound lump is at rest. Each of the incoming particles approach the collision with velocity $v/2$ and impact parameter a, but on opposite sides of the center of mass. Angular momentum conservation then yields:

$$L = 2m\frac{v}{2}a = mav = I\omega \qquad \Longrightarrow \qquad \omega = \frac{mav}{I} = \frac{5v}{14a} \,.$$

The final rotation energy is:

$$E'_{\mathrm{R}} = \frac{1}{2} I\omega^2 = \frac{5}{28} mv^2 \,.$$

The initial energy for any impact parameter, with $m_1 = m_2 = m$, is:

$$E_{\mathrm{CM}} = 2\frac{1}{2}m\left(\frac{v}{2}\right)^2 = \frac{1}{4}mv^2 \,.$$

For a central collision, the energy loss is thus:

$$Q_0 = E_{\mathrm{CM}} - E'_{\mathrm{CM}} = \frac{1}{4}mv^2 \,,$$

while for the glancing collision it is:

$$Q_R = E_{\mathrm{CM}} - E'_{\mathrm{R}} = \frac{1}{14}mv^2 \,.$$

so $Q_R < Q_0$. This has to be so, because in the central collision, there is no motion after the merger, so the kinetic energy loss is maximal, while in the glancing case, there is still rotational energy in the final state.

f) The four-momentum is conserved, because there are no external forces acting on the system, and hence in particular no time-dependent forces. With

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} \,, \qquad \gamma' = \frac{1}{\sqrt{1-(v'/c)^2}} \,,$$

we can write the equations for momentum conservation in the x=direction and energy conservation as:

$$\gamma m_1 v = \gamma' m' v' \,, \qquad \gamma m_1 c^2 + m_2 c^2 = \gamma' m' c^2 \,,$$

where m' is the rest mass of the single lump after the collision. Introducing $\mu' = m'/m_1$ and $\mu_2 = m_2/m_1$, energy conservation can be written $\mu'\gamma' = \gamma + \mu_2$, which inserted in the momentum equation yields:

$$\gamma v = \mu'\gamma' v' = (\gamma + \mu_2) v' \qquad \Longrightarrow \qquad v' = \frac{\gamma}{\gamma + \mu_2} v \,.$$

From this follows:

$$\gamma' = \frac{1}{\sqrt{1 - v'/c^2}} = \frac{\gamma + \mu_2}{\sqrt{(\gamma+\mu_2)^2 - \gamma^2 v^2/c^2}} = \frac{\gamma + \mu_2}{\sqrt{1 + 2\gamma\mu_2 + {\mu_2}^2}} \,,$$

so

$$m' = \mu' m_1 = \frac{\gamma + \mu_2}{\gamma'} m_1 = \sqrt{1 + 2\gamma\mu_2 + {\mu_2}^2}\, m_1 \,.$$

[One sees that $m' = \sqrt{1 + 2\gamma\mu_2 + {\mu_2}^2}\, m_1 > \sqrt{1 + 2\mu_2 + {\mu_2}^2}\, m_1 = (1+\mu_2) m_1 = m_1 + m_2$, so the lost kinetic energy in the non-relativistic calculation is reflected by an increased rest mass in the energy-conserving relativistic case].