FACULTY OF SCIENCE AND TECHNOLOGY

COURSE: FYS500 — Classical Mechanics and Field Theory
DURATION: 09.00 – 13.00 (4 hours).
DATE: 23/11 2017
RESOURCES: Approved pocket calculator
SET: 2 problems on 2 pages

Universitetet i Stavanger

Each part of a problem counts equally, and most of them can be solved at least partly independently of the other parts. **The candidate is referred to the formulas at the end!**

Problem 1

a) Show that in two-dimensional polar coordinates, $x = r\cos\phi, y = r\sin\phi$, the velocity of a particle satisfies:

$$\dot{\mathbf{r}}^2 = \dot{r}^2 + r^2\dot{\phi}^2\,.$$

We shall consider a thin uniform rigid rod of length a and mass M swinging about its upper end like as a plane pendulum in the gravitational field, $\mathbf{g} = -g\,\mathbf{j}$.

b) Find the moment of the inertia, I_0, of the rod about its center of mass in a direction perpendicular to its length, and show that the moment for a parallel axis about an endpoint is:

$$I = \frac{1}{3}Ma^2$$

c) Write down the kinetic energy, T, and the Lagrangian, L, of the pendulum. Use the angle, ψ, that the rod is making with the negative y-axis as the generalized coordinate. Choose the zero-point of the potential energy so that it vanishes when the rod hangs motionless straight down. Identify the constants of motion of the system.

d) Find the equation of motion for ψ and solve it for $\psi(t) \ll 1$. Use the boundary conditions $\psi(0) = 0$, $\dot{\psi}(0) = \Omega$. What is the resulting amplitude of oscillations, ψ_0?

e) Find the total energy of the pendulum, and express it in terms of known quantities.

f) Find the forces acting on the suspension of the rod in the x- and y-directions when it is swinging with amplitude ψ_0.

g) We now assume that the suspension of the pendulum is moving, so its position is some known function $X(t)$, with $X(0) = 0$ and $\dot{X}(0) = v_0$. Keeping ψ as the coordinate, find the center of mass coordinates and use Chasle's (or Euler's) theorem to write down the Lagrangian.

h) Find the Euler-Lagrange equation for $\psi \ll 1$.

PROBLEM 2

Two particles of masses m_1 and m_2 with coordinate vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ interact with each other through a central potential $V(|\mathbf{r}_2 - \mathbf{r}_1|)$.

a) Write down the Lagrangian of the system, both expressed in terms of $\mathbf{r}_1$ and $\mathbf{r}_2$ and in terms of the center of mass coordinate $\mathbf{R}$ and the relative coordinate $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. Introduce the reduced mass, μ, and find the canonical momenta conjugate to $\mathbf{R}$ and $\mathbf{r}$.

b) What are the conserved quantities of the system? Why is the relative motion necessarily in a fixed plane? Also write down the Hamiltonian, H, of the system. Why is it conserved?

We shall now consider the harmonic potential:

$$V(\mathbf{r}) = \frac{1}{2} k\, \mathbf{r}^2 ,$$

with $k > 0$. Furthermore, let the motion take place in the xy-plane in the center of mass system, $\mathbf{R} = 0$.

c) Find the equations of motion for the relative motion, and solve them with the initial conditions $\mathbf{r}(0) = [x_0, 0, 0]$ and velocity $\dot{\mathbf{r}}(0) = [0, v_0, 0]$. Show that the resulting orbit is an ellipse, and find the period of the motion.

d) Express the angular momentum, $\boldsymbol{\ell}$ and energy, E, in terms of known quantities. Show that $\overline{T} = \overline{V} = \frac{1}{2}E$, where the bar means the average over a period. This is a special case of a well-known theorem of Classical Mechanics. Which?

e) Consider a two-dimensional one-body problem of a particle with mass m moving in a potential of the form:

$$V'(\mathbf{r}) = V_0 + \sum_{i=1}^{2} E_i x_i + \frac{1}{2} \sum_{i,j=1}^{2} k_{ij} x_i x_j .$$

Explain why we may assume that k_{ij} is a symmetric matrix, and why it therefore suffices to consider the case where the matrix (k_{ij}) is diagonal. Write down the Lagrangian in this case. Is angular momentum and energy conserved for a general k_{ij}?

f) Find and solve the equations of motion for the particle in the potential V' with the same initial conditions as in part c) above. What is the condition for the motion to be periodic?

SOME FORMULAS THAT MAY PROVE USEFUL:

For small angles $\phi \ll 1$ one has:

$$\sin\phi \approx \phi ; \qquad \cos\phi \approx 1 - \frac{1}{2}\phi^2 .$$

A pair of integrals:

$$\int_0^{2\pi} \sin^2(x)\, \mathrm{d}x = \int_0^{2\pi} \cos^2(x)\, \mathrm{d}x = \pi .$$