FACULTY OF SCIENCE AND TECHNOLOGY

COURSE: FYS500 — Classical Mechanics and Field Theory
DURATION: 09.00 – 13.00 (4 hours).
DATE: 9/3 2016
RESOURCES: Approved pocket calculator
SET: 2 problems on 2 pages

Universitetet i Stavanger

Each part of a problem counts equally, and most of them can be solved at least partly independently of the other parts. **The candidate is referred to the formulas at the end!**

Problem 1:

Two particles of masses m_1 and m_2 with coordinate vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ interact with each other through a central potential $V(|\mathbf{r}_2 - \mathbf{r}_1|)$.

a) Write down the Lagrangian of the system, both expressed in terms of $\mathbf{r}_1$ and $\mathbf{r}_2$ and in terms of the center of mass coordinate $\mathbf{R}$ and the relative coordinate $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. Introduce the reduced mass, μ, and find the momenta conjugate to $\mathbf{R}$ and $\mathbf{r}$.

b) What are the conserved quantities of the system? Why is the relative motion necessarily in a fixed plane? Also write down the Hamiltonian, H, of the system. Why is it conserved?

Introduce spherical polar coordinates, r, θ, ϕ, for the relative motion and let the motion take place in the xy-plane ($\theta = \pi/2$) in the center of mass system, $\mathbf{R} = 0$.

c) Write down the Lagrangian in these coordinates and explain why angular momentum, $\ell = \mu r^2 \dot{\phi}$, is conserved. Find the equation of motion for r.

We now consider the scattering of two protons on each other, so $m_1 = m_2 = m$, with a potential:

$$V(r) = \frac{q^2}{4\pi\epsilon_0 r} ,$$

where q is the proton charge, ϵ_0 the vacuum permittivity.

d) Show that the effective potential for fixed ℓ can be written:

$$V_\ell = V(r) + \frac{\ell^2}{2\mu r^2} .$$

Make a qualitative sketch of $V_\ell(r)$ and show the range of radial motion for fixed E. What is the possible range of E? Explain why there is only one turning point.

e) Show that the equation of motion for r leads to the equation for the orbit:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = -K .$$

where $u = 1/r$ for a constant K which shall be determined. [Hint: From the expression for ℓ we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\ell u^2}{\mu} \frac{\mathrm{d}}{\mathrm{d}\phi} . \Big]$$

f) Show, *e.g.* by direct substitution, that the orbit equation has a solution which leads to the following expression for r:

$$r(\phi) = \frac{1}{u(\phi)} = \frac{1}{K}\,\frac{1}{e\cos(\phi - \phi_0) - 1}\,,$$

where e and ϕ_0 are constants of integration. Determine ϕ_0 so that the turning point is at $\phi = 0$, and determine its radial distance in terms of K and e. [You need not find an expression for e.]

g) Two protons approach each other from infinity. Find the scattering angle, Θ, in the center of mass system, expressed in terms of the eccentricity, e.

Problem 2:

We have a circular disk of uniform mass density, diameter 0.5 m and negligible thickness.

a) Show that the moment of inertia about an axis in the plane of the disk through the center is:

$$I_0 = \frac{1}{4} M a^2\,,$$

where M is the disk mass, a the radius.

b) Find the moment of inertia, I, of the disk about an axis tangential to its rim. You may use the result of the previous part.

We place the disk in an inclined position with the edge on a horizontal surface. The point of contact is fixed, so the disk can only fall sideways, without rolling or slipping.

c) Write down the Lagrangian for the system, using the angle between the plane of the disk and the surface, θ, as coordinate. The acceleration of gravity is $g = 9.8\,\mathrm{m/s^2}$.

d) What is the energy of the falling disk. Why is it conserved? It is released from rest at an inclination of $\theta_0 = \pi/4$. What is the angular speed (numerical value), $\dot{\theta}$, as it hits the surface? [Do not try to solve the equation of motion].

We then let the disk roll on the horizontal surface in an inclined position.

e) It is possible for the disk to roll with a constant angular velocity, ω, about its center with a fixed inclination θ. The point of contact with the surface will then describe a circle of radius R ($R > a$). Make a sketch of the motion and show that the angular velocity of the contact point around the center of the circle is:

$$\omega_R = \omega \frac{a}{R}\,.$$

f) Find the centrifugal acceleration of the center of the disk in this motion. Also find an expression for ω in terms of θ, R, a and g in this case. [Hint: Use torque balance directly, do not try the Lagrangian approach].

A formula that may prove useful:

$$\int_0^{\pi} \sin^2\phi \, \mathrm{d}\phi = \frac{\pi}{2}$$