UNIVERSITY OF STAVANGER
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Examination MAF 500 Classical Mechanics and Field Theory, 2016 fall. Suggested solutions.

Problem 1

a) The mass of a sphere of constant density d is $M = \frac{4\pi}{3} d a^3$. Using standard cylindrical coordinates ρ, ϕ, z centered at the center of the ball, so $\mathbf{r} = [\rho\cos\phi, \rho\sin\phi, z]$, and taking the axis of rotation to be the z-axis, the interior of the sphere is the region $\rho^2 + z^2 < a^2$. Thus:

$$I = \int_M \left(x^2+y^2\right) \mathrm{d}M = d\int_{-a}^{a} \mathrm{d}z \int_0^{\sqrt{a^2-z^2}} \rho^2 \rho\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi = 2\frac{2\pi}{4} d \int_0^a \left(a^2-z^2\right)^2 \mathrm{d}z$$
$$= \pi d \left(a^4 z - \frac{2}{3}a^2 z^3 + \frac{1}{5}z^5\right)\Bigg|_0^a = \frac{8\pi}{15} d a^5 = \frac{2}{5} M a^2 \,.$$

[This can also be done in standard spherical coordinates, using $x^2+y^2 = r^2\sin^2\theta$ and substituting $\xi = \cos\theta$ so $\mathrm{d}\xi = -\sin\theta \mathrm{d}\theta$:

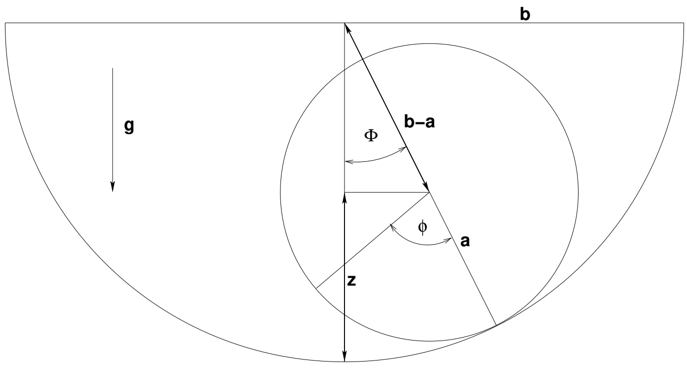
$$I = 2\pi d \int_0^a r^4\,\mathrm{d}r \int_0^\pi \sin^3\theta \mathrm{d}\theta = \frac{2\pi}{5} a^5 \int_{-1}^{1} \left(1-\xi^2\right) \mathrm{d}\xi = \frac{8\pi}{15} d a^5 = \frac{2}{5} M a^2 \,.]$$

b) The rolling condition is that the point of contact between ball and bowl is at rest with respect to both at all times. It must thus move at the same rate relative to both, so:

$$\mathrm{d}s = a\,\mathrm{d}\phi = b\,\mathrm{d}\Phi \qquad \Longleftrightarrow \qquad a\,\omega = b\,\Omega \,.$$

c) Since the radii of both ball and bowl have a common tangent at the point of contact, this and the centers of ball and bowl lie on a straight line. The distance between the centers is then $b-a$, and from the figure right we immediately read off:

$$z = b - (b-a)\cos\Phi \,.$$



d) Since the center of mass of a sphere is at its center, Chasle's (Euler's) theorem says that the instantaneous motion of any point of the ball is the sum of the motion of the center of mass, with velocity $(b-a)\Omega$, and the rotation of the ball about the center, with angular velocity ω. The Lagrangian can then be written, using the previous three parts of the problem:

$$L = T - V = \frac{1}{2} M (b-a)^2 \Omega^2 - Mgz + \frac{1}{2} I \omega^2$$
$$= \frac{1}{2} M \left(b^2 - 2ab + a^2 + \frac{2}{5} a^2 \frac{b^2}{a^2}\right) \dot{\Phi}^2 + Mg(b-a)\cos\Phi - Mgb$$
$$= \frac{1}{2} M \left(\frac{7}{5} b^2 - 2ab + a^2\right) \dot{\Phi}^2 + Mg(b-a)\cos\Phi - Mgb \,.$$

Since L only contains a quadratic term in $\dot{\Phi}$, the Hamiltonian follows immediately as:

$$H = T + V = \frac{1}{2}M\left(\frac{7}{5}b^2 - 2ab + a^2\right)\dot{\Phi}^2 - Mg(b-a)\cos\Phi + Mgb\,.$$

Alternatively, H can be calculated from the canonical momentum,

$$p_\Phi = \partial L/\partial\dot{\Phi} = M\left(\frac{7}{5}b^2 - 2ab + a^2\right)\dot{\Phi}\,,$$

and $H = p_\Phi\dot{\Phi} - L$.

e) From the Euler–Lagrange equations one finds the equation of motion as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\Phi}} = \dot{p}_\Phi = M\left(\frac{7}{5}b^2 - 2ab + a^2\right)\ddot{\Phi} = \frac{\partial L}{\partial\Phi} = -Mg(b-a)\sin\Phi\,.$$

For small oscillations $\sin\Phi \approx \Phi$, so the equation simplifies to:

$$\ddot{\Phi} + \Omega_s^2\Phi = 0\,; \qquad \Omega_s = \sqrt{\frac{g(b-a)}{\frac{7}{5}b^2 - 2ab + a^2}}\,.$$

Here Ω_s is the frequency of oscillation. [We see that $\Omega_s \to 0$ when $a \to b$, as it must.] The general solution of the equation is:

$$\Phi(t) = A\cos(\Omega_s t) + B\sin(\Omega_s t)\,,$$

where A and B are constants of integration. From the boundary conditions, $\Phi(0) = \Phi_0 = A$ and $\dot{\Phi}(0) = \Omega_0 = B\Omega_s$, so $B = \Omega_0/\Omega_s$. [The solution can also be written $\Phi(t) = C\cos(\Omega_s t - \delta)$, with

$$C = \sqrt{A^2 + B^2} = \sqrt{\Phi_0^2 + \left(\frac{\Omega_0}{\Omega_s}\right)^2}\,, \qquad \delta = \arctan\frac{B}{A} = \arctan\left(\frac{\Omega_0}{\Omega_s\Phi_0}\right).]$$

Problem 2

a) The Lorentz force, $\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$, has the following Cartesian components for the given fields:

$$F_x = q\dot{y}B_0\,; \qquad F_y = q(E_0 - \dot{x}B_0)\,; \qquad F_z = 0\,.$$

The work done by the magnetic field during a small displacement $\mathrm{d}\mathbf{r} = \dot{\mathbf{r}}\,\mathrm{d}t$ is

$$\mathrm{d}W = \mathbf{F}\cdot\mathrm{d}\mathbf{r} = q\dot{\mathbf{r}} \times \mathbf{B}\cdot\dot{\mathbf{r}}\,\mathrm{d}t = 0\,,$$

by the properties of the vector triple product.

b) With the fields as given we find $\Phi = -E_0 y$, $\mathbf{A} = \frac{1}{2}[-B_0 y, B_0 x, 0]$. This is consistent, since:

$$-\boldsymbol{\nabla}\Phi - \frac{\partial\mathbf{A}}{\partial t} = [0, E_0, 0] = \mathbf{E}\,; \qquad \boldsymbol{\nabla}\times\mathbf{A} = [\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}] = [0, 0, B_0] = \mathbf{B}\,.$$

c) From the formula for the Lagrangian given, we find for $V(r) = 0$:

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + qyE_0 + \frac{1}{2}q(-\dot{x}\,y + \dot{y}\,x)B_0\,.$$

The equations of motion are then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{x}} = m\ddot{x} - \frac{1}{2}q\dot{y}B_0 = \frac{\partial L}{\partial x} = \frac{1}{2}q\dot{y}B_0 \quad \Longrightarrow \quad m\ddot{x} = q\dot{y}B_0 = F_x\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{y}} = m\ddot{y} + \frac{1}{2}q\dot{x}B_0 = \frac{\partial L}{\partial y} = qE_0 - \frac{1}{2}q\dot{y}B_0 \quad \Longrightarrow \quad m\ddot{y} = qE_0 - q\dot{x}B_0 = F_y\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{z}} = m\ddot{z} = \frac{\partial L}{\partial z} = 0 \quad \Longrightarrow \quad m\ddot{z} = 0 = F_z\,,$$

L does not depend on z, so z is cyclical and $p_z = \partial L/\partial\dot{z} = m\dot{z}$ is conserved. In addition, L does not depend on time, so the energy, $E = H$ is conserved. [Formula is not required.]

d) Differentiating the equation of motion for y and using the one for x, we find, writing $v_y = \dot{y}$:

$$m\ddot{v}_y = -q\ddot{x}B_0 = -\frac{q^2B_0^2}{m}v_y \qquad \Longrightarrow \qquad \ddot{v}_y + \omega^2 v_y = 0\,.$$

where the *cyclotron frequency* is $\omega = qB_0/m$. This is a harmonic equation with solution:

$$v_y = \dot{y} = V\cos(\omega t - \delta) \qquad \Longrightarrow \qquad y = \int \dot{y}\,\mathrm{d}t = \frac{V}{\omega}\sin(\omega t - \delta) + y_0'\,.$$

where V, δ and y_0' are integration constants. We now can find x from:

$$\dot{x} = -\frac{m}{qB_0}\ddot{y} = -\frac{\ddot{y}}{\omega} = V\sin(\omega t - \delta) \qquad \Longrightarrow \qquad x = \int \dot{x}\,\mathrm{d}t = -\frac{V}{\omega}\cos(\omega t - \delta) + x_0'\,.$$

where x_0' is another integration constant. From the initial conditions we find:

$$\dot{x}(0) = V\sin\delta = 0\,, \quad \dot{y}(0) = V\cos\delta = v_0 \qquad \Longrightarrow \qquad \delta = 0, \quad V = v_0\,.$$

Then with $x(0) = x_0 = x_0' - v_0/\omega$ or $x_0' = v_0/\omega + x_0$ and $y(0) = y_0' = y_0$, all boundary conditions are satisfied. The equation of motion in the z direction is $\ddot{z} = 0$ with solution $z = w_0 t$ satisfying $z(0) = 0$. Thus the complete solution is:

$$\mathbf{r}(t) = \left[-\frac{v_0}{\omega}\cos\omega t + x_0',\ \frac{v_0}{\omega}\sin\omega t + y_0,\ w_0 t\right]\,.$$

We see that the projection of the motion in xy-plane is circular, satisfying $(x - x_0')^2 + (y - y_0)^2 = (v_0/\omega)^2$. Thus the overall motion is a spiral winding around a line parallel to the magnetic field through the point $[x_0', y_0', 0] = [x_0 + v_0/\omega, y_0, 0]$. The total velocity, $\sqrt{v_0^2 + w_0^2}$, and hence the kinetic energy, is conserved, consistent with the fact that the magnetic field does no work.

e) Using the strategy and notation of the previous part, the equation for v_y is unchanged, so the solution is still:

$$v_y = \dot{y} = V\cos(\omega t - \delta) \qquad \Longrightarrow \qquad y = \frac{V}{\omega}\sin(\omega t - \delta) + y_0'\,,$$

with ω as before. The equation for $\dot{x}$ is modified to:

$$\dot{x} = -\frac{1}{\omega}\ddot{y} + \frac{E_0}{B_0} = V\sin(\omega t - \delta) + \frac{E_0}{B_0} \qquad \Longrightarrow \qquad x = -\frac{V}{\omega}\cos(\omega t - \delta) + \frac{E_0}{B_0}t + x_0'\,.$$

The initial conditions become:

$$\dot{x}(0) = -V\sin\delta + \frac{E_0}{B_0} = 0\,, \quad \dot{y}(0) = V\cos\delta = v_0 \qquad \Longrightarrow$$

$$V\sin\delta = \frac{E_0}{B_0}\,, \qquad V\cos\delta = v_0 \qquad \Longleftrightarrow \qquad V = \sqrt{v_0^2 + \left(\frac{E_0}{B_0}\right)^2}\,, \quad \delta = \arctan\left(\frac{E_0}{v_0 B_0}\right)\,.$$

$x(0)$ and $y(0)$,and hence x_0' and y_0', are unchanged from part e. The equation of motion for z is unchanged, but now $w_0 = 0$, so $z(t) = 0$. We see that the motion in the xy-plane is almost the same as before, except that the center of the circle moves in the x direction with constant speed, $v_t = E_0/B_0$. If $E_0 = 0$, we recover the solution in the previous part.

f) In the relativistic case the expression for the Lorentz force is unchanged, but we must write the equations of motion in terms of the relativistic momentum, $\mathbf{p}$:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})\,.$$

The components of the relativistic momentum are:

$$p_i = \gamma m \dot{x}_i\,, \qquad \gamma = \frac{1}{\sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}}}\,,$$

where m is the rest mass. The proof that the magnetic field does no work remains valid. Thus, in the absence of an electric field the kinetic energy, and hence γ, is conserved. This means that we can solve for the motion exactly as in part above in the case $E_0 = 0$. The only difference is that $m \to \gamma m$, so the cyclotron frequency now depends on the velocity, $\omega = qB_0/\gamma m$. But if $E_0 \neq 0$, even in the non-relativistic case we have

$$\dot{x}^2 + \dot{y}^2 = V^2 + 2\frac{E_0}{B_0}V\sin(\omega t - \delta) + \left(\frac{E_0}{V_0}\right)^2\,,$$

so γ is not even constant in the non-relativistic limit, and we must look for another method of solution.