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Examination MAF 500 Classical Mechanics and Field Theory, 2015 spring. Suggested solutions.

Problem 1

a) In Cartesian coordinates we have:

$$L = T - V = \frac{1}{2}m\dot{\mathbf{r}}^2 - \frac{1}{2}k\mathbf{r}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}k(x^2 + y^2 + z^2)$$

Since the potential is central, $V(\mathbf{r}) = V(r)$, with $r = \sqrt{x^2 + y^2 + z^2}$, we find the force, using $\partial r/\partial x_i = x_i/r = \widehat{r}_i$:

$$\mathbf{F} = -\frac{\partial V}{\partial \mathbf{r}} = -\frac{\mathrm{d}V}{\mathrm{d}r}\left[\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right] = -\frac{\mathrm{d}V}{\mathrm{d}r}\,\widehat{\mathbf{r}} = -\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}\mathbf{r}\,.$$

We then find from the equation of motion for angular momentum that $\mathbf{L}$ is conserved ($\mathbf{N}$ is the torque):

$$\dot{\mathbf{L}} = \mathbf{N} = \mathbf{r} \times \mathbf{F} = -\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}\,\mathbf{r} \times \mathbf{r} = 0\,.$$

Since $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$ is conserved and perpendicular to $\mathbf{r}$ and $\dot{\mathbf{r}}$, which are both in the plane of motion, this plane is fixed, and can be chosen as the xy-plane.

b) The initial conditions imply that the motion is constrained to the xy-plane. We thus have $z = 0$ and $p_z = m\dot{z} = 0$ at all times. The remaining equations of motions are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \quad \Longrightarrow \quad m\ddot{x} = -\frac{\partial V}{\partial x} = -kx\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y} \quad \Longrightarrow \quad m\ddot{y} = -\frac{\partial V}{\partial y} = -ky\,.$$

This is a pair of harmonic equations, with solutions ($\omega = \sqrt{k/m}$):

$$x(t) = A\cos\omega t + B\sin\omega t\,, \qquad y(t) = C\cos\omega t + D\sin\omega t\,.$$

The initial condition $x(0) = x_0$ yields $A = x_0$, $\dot{x}(0) = B\omega = 0$ yields $B = 0$, $y(0) = 0$ yields $C = 0$ and $\dot{y}(0) = D\omega = v_0$ yields $D = y_0 = v_0/\omega$. Hence the solution is:

$$x(t) = x_0\cos\omega t\,, \qquad y(t) = y_0\sin\omega t\,, \qquad \frac{x^2}{{x_0}^2} + \frac{y^2}{{y_0}^2} = 1\,,$$

which is the equation of an ellipse with axes along the coordinate axes. Thus we have closed periodic orbits, with period $\tau = 2\pi/\omega$.

c) With $p_x = \partial L/\partial \dot{x} = m\dot{x}$ ands similarly $p_y = m\dot{y}$ we have the Hamiltonian as:

$$H = p_x\dot{x} + p_y\dot{y} + p_x\dot{z} - L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{1}{2}k(x^2 + y^2 + z^2)\,.$$

Since L is independent of time, $H = E$ is the conserved energy.

d) We have, retaining the z dependence:

$$\mathbf{r} = [r\cos\phi, r\sin\phi, z]\,,$$

with $r = \sqrt{x^2 + y^2}$. We then find:

$$\dot{\mathbf{r}} = \dot{r}\,[\cos\phi, \sin\phi, 0] + r\dot{\phi}\,[-\sin\phi, \cos\phi, 0] + \dot{z}\,[0, 0, 1] = \dot{r}\,\mathbf{e}_r + r\dot{\phi}\,\mathbf{e}_\phi + \dot{z}\,\mathbf{k}\,.$$

We easily check that $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{k}\}$ form an orthonormal set of vectors, so:

$$\dot{\mathbf{r}}^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\,.$$

e) We have:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - \frac{1}{2}k(r^2 + z^2)\,.$$

Thus L does not depend on ϕ, which is thus a cyclical variable, and the conjugate momentum:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = L_z = \ell\,,$$

is conserved. The equations of motions are:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} &= \frac{\partial L}{\partial r} \qquad &\Longrightarrow \qquad & m\ddot{r} = mr\dot{\phi}^2 - \frac{\partial V}{\partial r} = mr\dot{\phi}^2 - kr \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} &= \frac{\partial L}{\partial \phi} \qquad &\Longrightarrow \qquad & m\dot{\ell} = 0,, \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}} &= \frac{\partial L}{\partial z} \qquad &\Longrightarrow \qquad & m\ddot{z} = -\frac{\partial V}{\partial z} - kz\,.
\end{aligned}$$

The value of ℓ is:

$$\ell = mr^2\dot{\phi}\big|_{t=0} = mx_0v_0 = mx_0y_0\omega\,.$$

f) The magnetic vector potential $\mathbf{A}$ by definition satisfies $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. To check that the given formula, we use the formula at the end of the problem sheet, together with $\boldsymbol{\nabla}\cdot\mathbf{B} = \partial B/\partial z = 0$, $\nabla\cdot\mathbf{r} = \partial x/\partial x + \partial y/\partial y + \partial z/\partial z = 3$, $(\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{B} = (x\,\partial B/\partial x + y\,\partial B/\partial y + z\,\partial B/\partial z)\mathbf{k} = 0$ and $(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{r} = B\,\partial\mathbf{r}/\partial z$ = B$\mathbf{k}$=$\mathbf{B}$:

$$\boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}\times\frac{1}{2}(\mathbf{B}\times\mathbf{r}) = \frac{1}{2}[\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{r}) - \mathbf{r}(\boldsymbol{\nabla}\cdot\mathbf{B}) + (\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{B} - (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{r}] = \frac{1}{2}[3\mathbf{B} - 0 + 0 - \mathbf{B}] = \mathbf{B}\,.$$

Furthermore, using the properties of the triple product we find:

$$\dot{\mathbf{r}}\cdot\mathbf{a} = \frac{1}{2}\dot{\mathbf{r}}\cdot(\mathbf{B}\times\mathbf{r}) = \frac{1}{2}\mathbf{B}\cdot(\mathbf{r}\times\dot{\mathbf{r}}) = \frac{1}{2m}\mathbf{B}\cdot\mathbf{L}$$

where $\mathbf{L} = \mathbf{r}\times\dot{\mathbf{r}}$ is the *mechanical* angular momentum. If the initial conditions are unchanged, $\dot{z} = 0$ and $z = 0$, the motion will still be in the xy-plane, perpendicular to $\mathbf{B}$, since the Lorentz force, $\mathbf{F} = q\dot{\mathbf{r}}\times\mathbf{B}$, has no component out of that plane. Then $\mathbf{B}$ and $\mathbf{L}$ are parallel, and we can drop all z-dependence. With $\ell = |\mathbf{L}|\mathbf{k}$ we have:

$$\dot{\mathbf{r}}\cdot\mathbf{a} = \frac{1}{2m}\mathbf{B}\cdot\mathbf{L} = \frac{B\ell}{2m} = \frac{1}{2}Br^2\dot{\phi}\,.$$

Hence the Lagrangian is:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{1}{2}kr^2 + \frac{1}{2}qBr^2\dot{\phi}\,.$$

g) ϕ remains a cyclical variable, so the canonical angular momentum, p_ϕ, is still conserved. It is given by:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} + \frac{1}{2}qBr^2 = \ell + \frac{1}{2}qBr^2\,.$$

h) Using $\psi = \phi + \omega_0 t$ as a new generalized coordinate, we have $\dot{\psi} = \dot{\phi} - \omega_0$, so:

$$\begin{aligned}
L &= \frac{1}{2}m\left[\dot{r}^2 + r^2(\dot{\psi}^2 - 2\omega_0\dot{\psi} + {\omega_0}^2)\right] - \frac{1}{2}kr^2 + \frac{1}{2}qBr^2(\dot{\psi} - \omega_0) \\
&= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\psi}^2\right) + \frac{1}{2}(qB - 2m\omega_0)r^2\dot{\psi} - \frac{1}{2}(k + qB\omega_0 - m{w_0}^2)r^2\,.
\end{aligned}$$

We see that if we set:

$$\omega_0 = \frac{qB}{2m} \qquad k' = k + qB\omega_0 - m\omega_0^2 = k + \frac{q^2B^2}{4m}\,,$$

the Lagrangian expressed in terms of ψ has the same form as it had in the absence of a magnetic field if expressed in terms of ϕ, but with a changed value for the constant of elasticity, $k \to k'$:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\psi}^2\right) - \frac{1}{2}k'r^2\,.$$

The boundary conditions for ψ are $\psi(0) = \phi(0) = 0$ and $\dot{\psi}(0) = v_0/x_0 + \omega_0$. The motion in ψ will still be ellipses, with a slightly shortened period $\tau' = 2\pi/\omega' = 2\pi\sqrt{\frac{m}{k'}}$, since $k' > k$. But now these ellipses themselves rotate (or *precess*) with angular frequency ω_0, called the *Larmor frequency.*

Problem 2

a) Choosing coordinates such that the cylinder axis is the z-axis, and letting ℓ be the length of the cylinder and ρ the density, so that $m = \pi \ell a^2 \rho$, we find with $r = \sqrt{x^2 + y^2}$:

$$I = I_{zz} = \iiint \rho(x^2 + y^2)\,\mathrm{d}m = \rho \int_0^\ell \mathrm{d}z \int_0^a r^2\, 2\pi r\,\mathrm{d}r = 2\pi\ell \frac{1}{4} a^4 = \frac{1}{2} m a^2 .$$

b) The rolling condition is that the line of contact between the cylinder and the surface is at rest at all times. Since a point on the cylinder surface moves with a velocity $v = a\omega$ relative to the axis, this is the rolling condition, with v as the velocity of the axis, and in particular of the center of mass. By Chasles's (or Euler's) theorem, the kinetic energy is:

$$T = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 = \frac{1}{2}\left[m a^2 \omega^2 + \frac{1}{2} m a^2 \omega^2 \right] = \frac{3}{4} m a^2 \omega^2 = \frac{3}{4} m v^2 .$$

If the length of the incline is x_0, the height of the line of contact above the horizontal base of the wedge is $z_w = (x_0 - x)\sin\alpha$ (see figure). The *vertical* distance from the line of contact to the cylinder axis is $z_c = a\cos\alpha$, so the potential energy is:

$$V = mgz = mg(z_w + z_c) = mg\left[(x_0 - x)\sin\alpha + a\cos\alpha\right] = -mgx\sin\alpha + V_0 ,$$

where V_0 is an irrelevant constant. Hence the Lagrangian is ($a\omega = v = \dot{x}$):

$$L = T - V = \frac{3}{4} m \dot{x}^2 + mgx\sin\alpha - V_0 .$$

c) The Euler-Lagrange equation yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \qquad \Longrightarrow \qquad \frac{3}{2} m \ddot{x} = mg\sin\alpha \qquad \Longleftrightarrow \qquad \ddot{x} = \frac{2}{3} g \sin\alpha .$$

d) The kinetic energy of the wedge is $T_w = \frac{1}{2} M \dot{s}$, while its potential energy is constant, and can be neglected. The height of the center of mass of the cylinder above the contact point with the wedge is also unchanged, so the potential energy of the cylinder is $mgx\sin\alpha$ plus some constant. But the velocity of the contact point, $\mathbf{x} = [x, 0, 0]$ in the coordinate system with x-axis along the wedge, is $\mathbf{v} = \dot{\mathbf{s}} + \dot{\mathbf{x}}$ in the inertial system of the plane surface. Hence:

$$v^2 = \mathbf{v}^2 = (\dot{\mathbf{s}} + \dot{\mathbf{x}})^2 = \dot{s}^2 + \dot{x}^2 + 2\dot{s}\dot{x}\cos\alpha .$$

Using I from part a above and the rolling condition $\dot{x} = \omega a$, the Lagrangian can be written:

$$\begin{aligned} L' &= \frac{1}{2} M \dot{s}^2 + \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 + mgx\sin\alpha - V_0 \\ &= \frac{1}{2}(M + m)\dot{s}^2 + m\dot{s}\dot{x}\cos\alpha + \frac{3}{4} m \dot{x}^2 + mgx\sin\alpha - V_0 . \end{aligned}$$

This yields the equations of motion:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L'}{\partial \dot{s}} &= \frac{\partial L'}{\partial s} \qquad \Longrightarrow \qquad \frac{\mathrm{d}p_s}{\mathrm{d}t} = (M + m)\ddot{s} + m\ddot{x}\cos\alpha = 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L'}{\partial \dot{x}} &= \frac{\partial L'}{\partial x} \qquad \Longrightarrow \qquad \frac{3}{2} m \ddot{x} + m\cos\alpha\,\ddot{s} = mg\sin\alpha , \end{aligned}$$

where $p_s = \partial L'/\partial \dot{s} = (M + m)\dot{s} + m\dot{x}\cos\alpha$ is the conserved canonical momentum conjugate to s.

e) Inserting for $\ddot{s}$ from the equation of motion for s in that for x yields:

$$\frac{3}{2}m\ddot{x} + m\cos\alpha\,\ddot{s} = \frac{3}{2}m\ddot{x} - \frac{m^2\cos^2\alpha}{M+m}\ddot{x} = \left[\frac{3}{2} - \frac{m\cos^2\alpha}{M+m}\right]m\ddot{x} = mg\sin\alpha\,.$$

$$a_x = \ddot{x} = \frac{g\sin\alpha}{\frac{3}{2} - \frac{m\cos^2\alpha}{M+m}}\,.$$

Thus the acceleration of the cylinder down the wedge, a_x, is constant. We see that for an infinitely heavy wedge, $M \to \infty$, so the result from part c above is recovered. Using the boundary conditions, $x(0) = 0$, $\dot{x}(0) = 0$, we find $x(t) = \frac{1}{2}a_x t^2$. The time to travel a distance D is thus given by $D = \frac{1}{2}a_x t_D^2$, or:

$$t_D = \sqrt{\frac{2D}{a_x}} = \sqrt{\frac{3D}{g\sin\alpha}\left(1 - \frac{2}{3}\frac{m\cos^2\alpha}{M+m}\right)}\,.$$

With the boundary condition $\dot{s}(0) = 0$, we find $p_s = 0$, so with $s(0) = s_0$ we have:

$$s(t) = s_0 - \frac{m\cos\alpha}{M+m}x(t) \qquad \Longrightarrow \qquad s(t_D) - s_0 = -\frac{m\cos\alpha}{M+m}D\,.$$

The wedge moves in the opposite direction of the cylinder, a consequence of -momentum conservation for the combination of wedge and cylinder.