FACULTY OF SCIENCE AND TECHNOLOGY

COURSE: FYS500 — Classical Mechanics and Field Theory
DURATION: 09.00 – 13.00 (4 hours).
DATE: 18/2 2015
RESOURCES: Approved pocket calculator
SET: 2 problems on 3 pages

Each part of a problem counts equally, and most of them can be solved at least partly independently of the other parts. **The candidate is referred to the formulas at the end!**

Problem 1:

A particle of mass m and coordinate vector $\mathbf{r}(t) = [x(t), y(t), z(t)]$ moves in a fixed force field derived from a potential $V(\mathbf{r}) = \frac{1}{2}k\mathbf{r}^2$, with constant k.

a) Write down the Lagrangian, L, for the particle. What are the conserved quantities of motion? Why can you assume, without loss of generality, that the motion takes place in the xy-plane, *i.e.* with $z = 0$.

b) Find the equations of motion in Cartesian coordinates, and solve them with the initial conditions $\mathbf{r}(0) = [x_0, 0, 0]$ and velocity $\dot{\mathbf{r}}(0) = [0, v_0, 0]$. Show that the orbit of the particle is an ellipse with axes along the coordinate axes.

c) Define the Hamiltonian, H from L, and show that it is conserved if L is time independent. Express the energy of the system in terms of given quantities.

d) Show that in standard cylindrical coordinates r, ϕ, z, the velocity satisfies:

$$\dot{\mathbf{r}}^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2 \,.$$

e) Use the result of the previous point to write down the Lagrangian for the system in cylindrical coordinates, and find the resulting equations of motion. Show explicitly that the angular momentum about the z-axis, ℓ, is conserved, and find its value.

The particle has charge q, and is placed in a constant homogeneous magnetic field in the x-direction, $\mathbf{B} = B\,\mathbf{k}$.

f) Show that the magnetic vector potential for this field can be written:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B} \times \mathbf{r} \,.$$

Furthermore, express $\dot{\mathbf{r}} \cdot \mathbf{A}$ in cylinder coordinates, and use this to write down the Lagrangian, L, which includes the magnetic field.

g) What are the constants of motion now? Why can you assume that the motion still takes place in the xy-plane, if the initial conditions are unchanged. Also give an explicit expression for the angular momentum about the z-axis, ℓ.

h) Show that by introducing a new angular coordinate, $\psi = \phi + \omega_0 t$ for some constant ω_0, which you shall determine, you obtain a Lagrangian with coordinates r and ψ

which has effectively the same form as the one valid in absence of a magnetic field. Use this result to describe qualitatively the motion of the particle in the presence of a magnetic field.

PROBLEM 2:

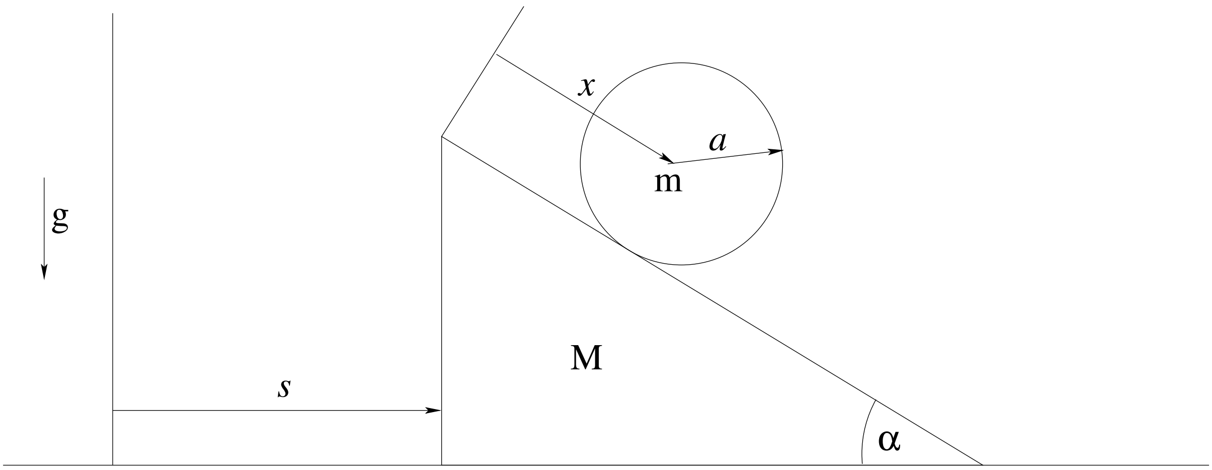
a) Show that a cylinder of mass m and radius a with uniform mass density has a moment of inertia, I, about its axis given by:

$$I = \frac{1}{2} m a^2 \,,$$

b) A cylinder rolls with angular frequency ω without slipping on a plane surface, The center of mass velocity is v. What is the rolling-condition? Furthermore, find the total kinetic energy and the Lagrangian for the cylinder when it rolling down an inclined plane in a gravitational field. The plane makes an angle α with the horizontal (see figure below), and the acceleration of gravity is g. Use the distance x along the plane ($\dot{x} = v$) as generalized coordinate.

c) Find the acceleration of the cylinder when rolling down the incline under the influence of gravity, without slipping.

We now assume that the cylinder is rolling down a wedge of mass M, which slides without friction on a horizontal plane:



d) With the distance along the horizontal plane, s, and x as generalized coordinates (see figure) write down the Lagrangian for the system and find the resulting equations of motion.

e) If the cylinder is released from rest at the top of the wedge ($x = 0$), find an expression for the time it takes before it has rolled a distance D down the wedge. How far, and in which direction, has the wedge moved then?

SOME FORMULAS THAT MAY PROVE USEFUL:

The equation for an ellipse with center at the origin and axes along the coordinate axes:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\,,$$

where a and b are the semimajor and the semiminor axis, respectively.

The Lagrangian for a particle of mass m and charge q in an electromagnetic field:

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - V(\mathbf{r}) - q\,\Phi(\mathbf{r}) + q\,\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r})\,,$$

where $V(\mathbf{r})$ is the non-electromagnetic potential, $\Phi(\mathbf{r})$ the electrostatic potential and $\mathbf{A}(\mathbf{r})$ the magnetic vector potential.

An identity from vector analysis:

$$\boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{W}) = \mathbf{V}(\boldsymbol{\nabla}\cdot\mathbf{W}) - \mathbf{W}(\boldsymbol{\nabla}\cdot\mathbf{V}) + (\mathbf{W}\cdot\boldsymbol{\nabla})\mathbf{V} - (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{W}\,,$$

where $\mathbf{V}$ and $\mathbf{W}$ are arbitrary vector functions.