FACULTY OF SCIENCE AND TECHNOLOGY

COURSE: FYS500 — Classical Mechanics and Field Theory
DURATION: 09.00 – 13.00 (4 hours).
DATE: 25/11 2015
RESOURCES: Approved pocket calculator
SET: 2 problems on 2 pages

Universitetet i Stavanger

Each part of a problem counts equally, and most of them can be solved at least partly independently of the other parts. **The candidate is referred to the formulas at the end!**

PROBLEM 1:

A mass M is fixed at one end of a massless rod of length D in the gravitational field. The other end is fixed so that the rod can move freely in all directions. Choose coordinates such that the fixed end of the rod is at the origin and the gravitational acceleration, g, is in the negative z-direction.

a) Show that in standard spherical polar coordinates r, ϕ, z, the velocity of the mass satisfies.

$$\dot{\mathbf{r}}^2 = D^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\,.$$

b) Using the angle $\psi = \pi - \theta$ between the negative z-axis and the rod an ϕ as generalized coordinates, write down expressions for the kinetic energy, T, the potential energy V and the Lagrangian, L. Also write down the energy, E, of the mass. Why is it conserved?

c) Use the Lagrangian to find the equations of motions. Show that there is one cyclic coordinate, and find the corresponding conjugate momentum, which we shall call ℓ. What is the physical interpretation of ℓ?

d) Show that the equations of motion have an equilibrium solution with constant $\omega = \dot{\phi}$ and equilibrium angle $\psi = \psi_0$. Find the tension in the rod, τ, in this equilibrium.

e) Find the moment of inertia, I, of a homogeneous sphere about a diameter through the center. The radius is a, the mass M.

f) Show that if the mass at the end of the rod is a homogeneous sphere of radius a, the expression for the kinetic energy T remains valid provided one replaces D^2 by $D'^2 = D^2 + \frac{2}{5}a^2$, while V is unchanged. What theorem is your argument based upon? [If you have not found I in the previous part, assume $I = ca^2$ for some constant c.]

g) Find the equilibrium configuration for constant ω in the case of a finite a. and show that this equilibrium is stable. Find the period of small oscillation about equilibrium for the same initial value of ℓ.

PROBLEM 2:

a) Write down the Lorentz force law for a particle of charge q that moves in both an electric and a magnetic field, and show that the magnetic field does no work on the particle.

b) Write down Newtons's equations for the particle if the electric and magnetic fields are parallel, and oriented along the z-axis: $\mathbf{E} = E_0\,\mathbf{k}$ and $\mathbf{B} = B_0\,\mathbf{k}$. Introducing the (mechanical) momentum vector $\mathbf{p}$, show that the equations can be written:

$$\dot{p}_x = \frac{qB_0}{m} p_y\,, \qquad \dot{p}_y = -\frac{qB_0}{m} p_x\,, \qquad \dot{p}_z = qE_0\,,$$

c) We observe that the equations of motion splits in two independent sets, one for $x(t)$ and $y(t)$ and one for $z(t)$. Solve the equation for the motion in the z-direction for a particle that starts from rest at $z = 0$ when $t = 0$.

d) Solve the equations of motion for $x(t)$ and $y(t)$ for a particle that starts at $t = 0$ from the point $[x_0, 0, 0]$ with the velocity $[0, v_0, 0]$. Describe the motion and check your solution by showing that the total kinetic energy is independent of B_0 for all times.

e) Write down the equations of motion for a *relativistic* particle moving in the same electromagnetic field in terms of the relativistic momentum in a form similar to that given in part b above. Why do these equations no longer split into two independent sets?

f) Solve the relativistic equations of motion for the case $B_0 = 0$, if the particle starts from rest at the origin. Check that your solution is physically acceptable at large t.

SOME FORMULAS THAT MAY PROVE USEFUL:

For small χ we have the approximations:

$$\sin(\psi + \chi) \approx \sin\psi + \chi\cos\psi\,, \qquad \cos(\psi\chi) \approx \cos\psi - \chi\sin\psi\,.$$

and for any number a:

$$(1+\chi)^a \approx 1 + a\chi$$

An integral that may prove useful:

$$\int \frac{t}{\sqrt{1+a^2t^2}}\,\mathrm{d}t = \frac{1}{a^2}\sqrt{1+a^2t^2}\,.$$