FACULTY OF SCIENCE AND TECHNOLOGY

COURSE: FYS500 — Classical Mechanics and Field Theory
DURATION: 09.00 – 13.00 (4 hours).
DATE: 21/02 2014
RESOURCES: Approved pocket calculator
SET: 2 problems on 3 pages

Universitetet i Stavanger

Each part of a problem counts equally, and most of them can be solved independently of the other parts.

Problem 1:

Two particles of masses m_1 and m_2 with coordinate vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ interact with each other through a central potential $V(|\mathbf{r}_2 - \mathbf{r}_1|)$.

a) Write down the Lagrangian of the system, and use it to find the canonical momenta conjugate to $\mathbf{r}_1$ and $\mathbf{r}_2$, and the Hamiltonian.

b) Introduce the center of mass coordinate $\mathbf{R}$ and the relative coordinate $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, express the Lagrangian in these coordinates and find the conjugate canonical momenta.

c) What are the conserved quantities of the system? Why is the relative motion necessarily in a fixed plane?

We now introduce spherical polar coordinates, r, θ, ϕ, for the relative motion and specialize to the case of a gravitational potential, $V(r) = -Gm_1m_2/r$. We take the plane of the motion to be the xy-plane ($\theta = \pi/2$) and consider the motion in the center of mass system, $\mathbf{R} = 0$. The Lagrangian for the relative motion then takes the form:

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - V(r)\,,$$

where $\ell = \mu r^2\dot{\phi}$ is the angular momentum and μ is the reduced mass of the two particles (this need not be shown).

d) Find the *effective* potential, $V_\ell(r)$, for fixed ℓ, and make a qualitative sketch of it. Show the possible range of radial motion for typical values of the energy, E. Mark in particular the turning point(s) of the orbit, and the radius of a circular orbit.

e) Derive Kepler's third law, $T^2 = ka^3$, in the case of a circular orbit of radius $r = a$. Express the constant k in terms of G, m_1 and m_2.

f) Find the equation of motion for $r(t)$ in the general case, and show that it leads to the equation for the orbit:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = K\,.$$

where $u = 1/r$ for a constant K which shall be determined. [Hint: From the expression for ℓ we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\ell}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\phi} = \frac{\ell u^2}{\mu}\frac{\mathrm{d}}{\mathrm{d}\phi}\,.\ ]$$

g) Show, *e.g.* by direct substitution, that the orbit equation has a solution which leads to the following expression for r:

$$r(\phi) = \frac{1}{u(\phi)} = \frac{1}{K}\frac{1}{1 + e\cos(\phi - \phi_0)},$$

where e and ϕ_0 are constants of integration. Determine ϕ_0 so that the (inner) turning point is at $\phi = 0$, and determine the radial distance(s) of the turning point(s) in terms of K and e.

h) Show that the eccentricity can be expressed as:

$$e = \sqrt{1 + \frac{2El^2}{\mu(Gm_1m_2)^2}}.$$

i) The two particles approach each other from infinity with asymptotic relative velocity v_0 and impact parameter s. Find the scattering angle in the center of mass, Θ, expressed in terms of v_0, s, m_1 and m_2.

PROBLEM 2:

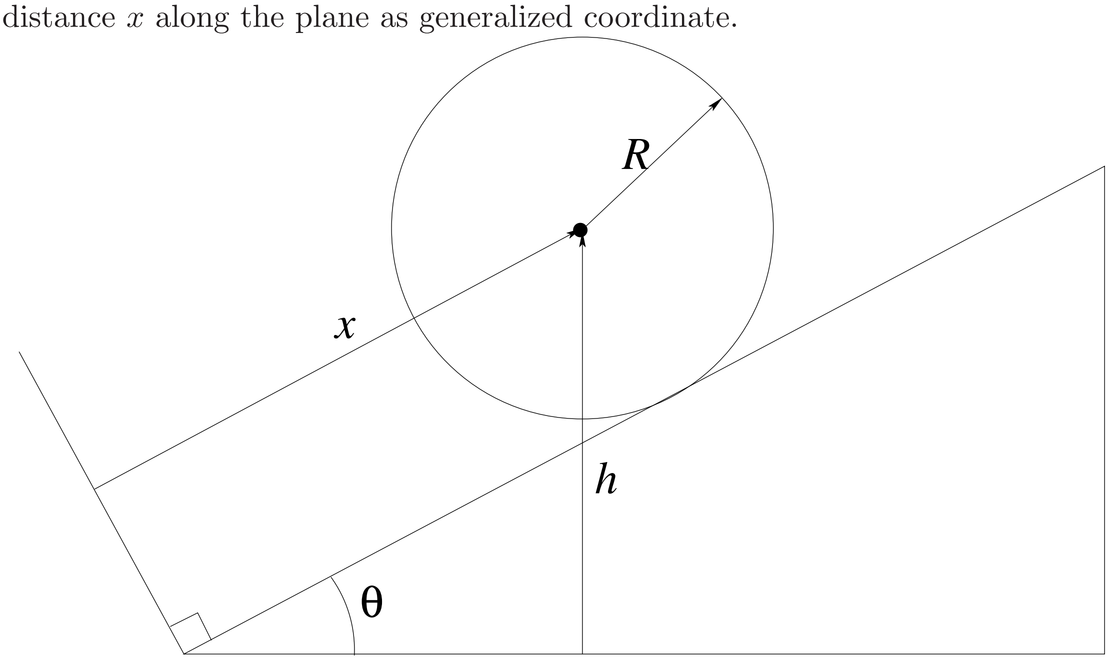
A cylinder of mass M and radius R of an arbitrary cylindrical symmetric internal mass distribution has a moment of inertia about its axis which can be written:

$$I = cMR^2,$$

where c is a constant that depends on the mass distribution.

a) Show that $c = 1$ for a hollow cylinder, with all its mass in the cylinder wall, and $c = 1/2$ for a cylinder of uniform density.

b) A cylinder rolls with angular frequency ω without slipping on a plane surface, The center of mass velocity is v. What is the rolling-condition? Also find the total kinetic energy and the Lagrangian for a cylinder with arbitrary c rolling down an inclined plane. The plane makes an angle θ with the horizontal (see figure below). Use the distance x along the plane as generalized coordinate.



c) Find the acceleration of each cylinder introduced in a) when rolling down the incline, without slipping.

We have to parallel cylinders with the same mass, M, and radius, R. One has uniform density, the other is hollow. The cylinders are connected by a massless rigid rod of length Λ, perpendicular to the two axes, in such a manner that they can rotate freely about their axes. The cylinders are placed on the incline so that they can roll straight down it, with the hollow one lowest down.

d) Set up the Lagrangian for the two connected cylinders, and find the acceleration of the system as it rolls down the incline, without slipping. [If you did not answer b), assume that the solution of that problem is that the Lagrangian can be written $L = A_c\dot{x}^2 + Bx$ for some constants A_c and B, which depends on the parameters of the problem.]

e) Find the tension in the connecting rod and the frictional forces acting at the two contact points between the cylinders and the incline as they roll down.