FACULTY OF SCIENCE AND TECHNOLOGY

COURSE: FYS500 — Classical Mechanics and Field Theory
DURATION: 09.00 – 13.00 (4 hours).
DATE: 26/11 2014
RESOURCES: Approved pocket calculator
SET: 2 problems on 2 pages

Universitetet i Stavanger

Each part of a problem counts equally, and most of them can be solved independently of the other parts. **The candidate is referred to the formula at the end!**

Problem 1:

Two particles with masses m_1 and m_2 have coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively.

a) Introduce the center of mass coordinate, $\mathbf{R}$ and relative coordinate $\mathbf{r}$, and express $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of $\mathbf{R}$ and $\mathbf{r}$. Also write down the kinetic energy, T, of the two particles in terms of these variables.

b) We introduce spherical polar coordinates for $\mathbf{r}$: $\mathbf{r} = r[\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]$. Show that the relative velocity satisfies:

$$\dot{\mathbf{r}}^2 = \dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\,.$$

c) The two particles are connected by a massless rigid rod of length a and are placed in the constant gravitational field $\mathbf{g} = -g\mathbf{k}$. Write down the Lagrangian for the system, and find all cyclical coordinates. Are there other, independent, conserved quantities?

d) Derive the equations of motion for the system and describe qualitatively, with explanation, the motion of the system if it is released with an initial center of mass velocity $\mathbf{V_0}$ and relative angular momentum $\mathbf{L}$.

We might alternatively wish to describe this system as a rigid body.

e) Show that the moment of inertia about an axis perpendicular to the rod can be written:

$$I = \mu a^2$$

where μ is the reduced mass of the system. Also explain why the moment of inertia about the axis of the rod vanishes.Use this to write down the corresponding Lagrangian, and show that it is the same as in c) above.

We use this system, with unchanged notation, to model of a satellite consisting of two massive modules connected by a rigid, practically massless, beam. The satellite is orbiting the Earth with the center of mass moving in a circular orbit of radius R. Neglect centrifugal and Coriolis terms in the following [they are not important].

f) Show that the fact that the gravitational forces on the two modules point in slightly different directions causes a torque on the satellite, which can be written ($r_i = |\mathbf{r}_i|$, $M = m_1 + m_2$ and G is Newton's constant):

$$\mathbf{N} = G\mu M_E\left(\frac{1}{r_2^3} - \frac{1}{r_1^3}\right)\mathbf{r}\times\mathbf{R}\,,$$

where M_E is the mass of the earth. By choosing the coordinate system such that ϕ is the angle between $\mathbf{R}$ and $\mathbf{r}$, one finds the excellent approximations:

$$r_{1,2} \approx R\left(1 \mp m_{2,1}\frac{a}{MR}\cos\phi\right) \qquad \frac{a}{R} \ll |\,,$$

(this needs not be shown). Use this to write the magnitude of the torque approximately as:

$$N = |\mathbf{N}| \approx \frac{3}{2}G\mu M_E\frac{a^2}{R^3}\sin\phi\cos\phi\,.$$

g) For which values of ϕ is the system in equilibrium? Investigate whether these equilibria are stable with respect to small perturbations. Also determine the angular frequency of small oscillation around the stable one(s), and show that it does not depend on a. What is the numerical value of the oscillation period close to the Earth's surface ($R \to R_E$), using $GM_E/R_E^2 = g = 9.8\,\mathrm{m/s^2}$.

PROBLEM 2:

In two space dimensions the potential energy both for electrostatics and Newtonian gravity has the form $V(r) = k\ln(r/a)$, where $k > 0$ and a are constants.

a) Write the Lagrangian in polar coordinates r, θ for a particle moving in such a potential. Find the angular momentum, ℓ, and the energy, E, and explain why both are conserved.

b) Find the *effective* potential, $V_\ell(r)$, for fixed ℓ, and make a qualitative sketch of it. Show the possible range of radial motion for typical values of the energy, E. Mark in particular the turning point(s) of the orbit, and the radius of a circular orbit. Does one have solutions where the particle can escape to infinity?.

c) Find the radius of a circular orbit, r_0, for fixed l, the corresponding angular velocity, and the relation between r_0 and the orbital period τ [analogous to Kepler's third law].

d) Obtain the equation of motion for $r(t)$ in the general case. Use this to find the equation of motion for a *nearly circular* orbit by writing $r(t) = r_0 + \rho(t)$ and making an expansion of $\frac{\partial V_\ell}{\partial r}(r_0 + \rho)$ to linear order in ρ. Show that the resulting motion is harmonic, and determine the angular frequency, ω.

e) Find $\theta(t)$ for a nearly circular orbit to lowest order in ρ. Also find how much the angular coordinate of a turning point changes in one revolution. [This is called the *advance of the apsides*]. If you have not found ω in the previous point, you can still explain how to do this for an arbitrary ω.

A FORMULA THAT MAY PROVE USEFUL:

From the binomial theorem:

$$(1+x)^a \approx 1 + xa$$

when $x \ll 1$ for any a.