FACULTY OF SCIENCE AND TECHNOLOGY

COURSE: FYS500 — Classical Mechanics and Field Theory
DURATION: 09.00 – 13.00 (4 hours).
DATE: 26/11 2013
RESOURCES: Approved pocket calculator
SET: 2 problems on 2 pages

Universitetet i Stavanger

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Each part of a problem counts equally, and most of them can be solved independently of the other parts. **The candidate is referred to the formulas at the end!**

## PROBLEM 1:

a) Show that if a point is constrained to move on a sphere of radius $a$, so that its position vector is $\mathbf{r} = a[\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]$ in standard spherical coordinate system, then its velocity satisfies:

$$\dot{\mathbf{r}}^2 = a^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\,.$$

A thin uniform rigid rod of constant mass density has mass $M$ and length $L$, and can swing freely about its end-point like a pendulum in the gravitational field. The acceleration of gravity is $g$.

b) Show that the moment of the inertia of the rod about its endpoints with respect to any direction perpendicular to the axis is given by:

$$I = \frac{1}{3}ML^2$$

c) Write down the kinetic energy, $T$, and the Lagrangian, $\mathcal{L}$, of this pendulum. Choose the zero-point of the potential energy so that it vanishes when the rod hangs straight down.

d) Show that there is one cyclic coordinate, and find the conjugate constant of motion, which we shall call $p$. What is the physical interpretation of $p$?

e) Show that the energy of this pendulum can be written:

$$E = \frac{1}{6}ML^2\left(\dot{\theta}^2 + \frac{K}{\sin^2\theta} + 2\omega_0^2(1-\cos\theta)\right)\,,$$

where $K$ $(>0)$ and $\omega_0$ shall be expressed in terms of known quantities, including $p$. Why is $E$ a constant of motion?

f) For small values of the angle $\theta$, the energy can be written approximately as:

$$\epsilon = \frac{E}{\frac{1}{6}ML^2} = \dot{\theta}^2 + \frac{K}{\theta^2} + \omega_0^2\theta^2\,.$$

Find the inner and outer turning point for $\theta$ as the pendulum swings around, expressed in terms of $\epsilon$, $K$ and $\omega_0$. What is the condition for a circular orbit?

g) Show that if $K = 0$ we have a plane pendulum. Determine its period of oscillation for small oscillations, and compare it with that of a plane pendulum consisting of a mass $M$ at the end of a massless rod or string of length $L$.

**PROBLEM 2:**

A point particle of charge $q$ is moving in a static Coulomb potential from a fixed point charge $Q$, with $Qq < 0$. We shall neglect radiation throughout this problem.

a) Explain why the motion of the particle is planar, and describe the qualitative nature of the orbits for different values of the energy of the particle, $E$. No calculations are required.

b) A constant magnetic field, $\mathbf{B}$, is switched on perpendicularly to the plane of the particle's trajectory. Write down the equation of motion in vector form for the particle, and explain why the motion is still planar, in the same plane as before.

c) Write down the equation of motion for the particle in a coordinate system which rotates with angular velocity $\boldsymbol{\omega}$ with respect to inertial space, where $\boldsymbol{\omega}$ is parallel to $\mathbf{B}$.

d) Find a value for $\boldsymbol{\omega}$, such that a constant magnetic field cancels to lowest (linear) order in $|\mathbf{B}|$. Interpret the result in terms of the motion of the particle as seen in a space-fixed coordinate system. [This value of $|\boldsymbol{\omega}|$ is called the Larmor frequency, $\omega_L$].

e) Assume that the particle is moving in a harmonic potential, $V(\mathbf{r}) = \frac{1}{2}k\mathbf{r}^2$ $(k > 0)$, instead of in the Coulomb potential. Show that the construction in the previous part can be used to find the particle's trajectory in the presence of a magnetic field $\mathbf{B}$ perpendicular to the trajectory exactly, if the trajectory for $\mathbf{B} = 0$ is known (for all $k$).

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**SOME FORMULAS THAT MAY PROVE USEFUL:**

Rate of change of an arbitrary vector, $\mathbf{V}$, in a rotating coordinate system:

$$\left(\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}\right)_r = \left(\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}\right)_s + \mathbf{V} \times \boldsymbol{\omega}\,,$$

where the subscript '$s$' refers to the stationary (inertial) system and '$r$' to the rotating one, while $\boldsymbol{\omega}$ is the angular velocity of the rotating system. If $\boldsymbol{\omega}$ is constant in time, the acceleration in the rotating frame is:

$$\mathbf{a}_r = \mathbf{a}_s + 2\mathbf{v}_r \times \boldsymbol{\omega} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\,,$$

where $v_r$ is the velocity in the rotating system.

A vector identity:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$