Fibrations on generalized Kummer varieties

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Introduction

In recent years, Matsushita has discovered the following remarkable property of symplectic varieties: Let X be a (projective, irreducible) symplectic $2n$-dimensional variety, and let $X \to S$ be a *fibration*, i.e. a map to an arbitrary variety S, such that a generic fibre X_s is connected and has positive dimension. Then a generic fibre is an n-dimensional abelian variety, and S is very close to being projective n-space. A precise statement is recalled in Section 1.2.

Throughout, we always assume a symplectic variety to be projective and irreducible, the latter basically meaning it cannot be written as a product of something smaller. Viewing symplectic varieties as higher dimensional analogues of K3 surfaces, Matsushita's result is a striking generalization of the fact that, if $X \to S$ is a K3 surface fibred over a curve S, then a generic fibre is an elliptic curve and S is the projective line. In other words, X is an elliptic K3 surface.

One might hope — although this is merely a hope at present — that a study of fibrations will lead to a better understanding of symplectic varieties. We remark that the classification problem for symplectic varieties is completely open. For a while they were even thought not to exist in dimension greater than two. The first examples appeared in the 80s: Fujiki [**10**] showed that the Hilbert scheme $S^{[2]}$, parametrizing pairs of points on a K3 surface, is symplectic. Shortly after, Beauville [**1**] showed that every Hilbert scheme $S^{[n]}$, parametrizing n points on a K3 surface, is symplectic. At the same time he showed that the (generalized) *Kummer variety* K^nA, parametrizing n points adding to zero on an abelian surface A, is symplectic. Thus there are two examples in every even dimension. To this date, only two further examples by O'Grady [**30, 31**] have been added to this list.

We study here basic existence and uniqueness problems for fibrations on Kummer varieties. Simultaneously, and independently (with the exception of Proposition 1.17, which is modelled upon a result from Markushevich's paper), Markushevich [**20**] and Sawon [**35**] have studied the case of Hilbert schemes of K3 surfaces. Finally, Rapagnetta [**34**] has described fibrations on O'Grady's examples.

An outline of this text, and its main results, goes as follows:

- In Chapter 1, we recall basic results from the theory of symplectic varieties, and in particular Matsushita's results on fibrations.

- In Chapter 2, we consider the Kummer varieties of a product $E \times E'$ of elliptic curves. These always admit fibrations, which in a certain sense are induced by the projections. As an application, we use the fibration found to provide an elementary proof of a formula due to Göttsche for the Euler characteristic of the Kummer varieties.
- Chapter 3 is the heart of the text. Given an effective divisor $C \subset A$ with self intersection $2n$, we construct a fibration on K^nA, which we call the *fundamental fibration*. Its existence is the main ingredient for the following criterion: If A is a generic, principally polarized abelian surface, then K^nA admits a (rational) fibration if and only if n is a perfect square.
- In Chapter 4 we study the fundamental fibration on the Kummer variety of 4 points, the first nontrivial perfect square. We find that its base locus is nonempty, and has the structure of a $\mathbb{P}^2$-bundle over A. Along the way we encounter two birational symplectic varieties which are not isomorphic. Such pairs play a special role in the theory of symplectic varieties, and are therefore of interest of their own.

It is a pleasure to express my heartfelt thanks to my advisor, prof. Geir Ellingsrud. His insight, patience and friendship have been of immense importance during these years.

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I also want to thank Anthony Maciocia, for sending me his unpublished notes on finite subschemes of abelian surfaces, and for allowing me to include the results that I need in Chapter 4.

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CHAPTER 1

Symplectic varieties and their fibrations

We begin by collecting some concepts from the theory of symplectic varieties, in particular the Beauville-Bogomolov form, which is a *pairing* of divisors, generalizing the intersection pairing on a surface. In Section 1.2, we quote Matsushita's results on fibrations on symplectic varieties, which form the background of the whole study undertaken here.

For background material on symplectic varieties, see Huybrechts' expositions [**15, 16**], which treat all the results we quote in Sections 1.1 and 1.2, and much more.

The reader already familiar with the basic properties of symplectic varieties might want to skip to Sections 1.3 and 1.4, where we give some weak results on existence and unicity of fibrations on symplectic varieties, and on Kummer varieties in particular.

1.1. Symplectic varieties

DEFINITION 1.1. A *symplectic variety* is a nonsingular complex variety X admitting an everywhere non-degenerate closed 2-form σ. It is *irreducible* if it is furthermore simply-connected and σ generates $\mathrm{H}^0(X,\Omega_X^2)$.

In *this text*, a symplectic variety will always be projective and irreducible, unless otherwise is stated.

REMARK 1.2. The word irreducible here refers to the Bogomolov decomposition theorem [**1**], which implies that every symplectic variety has a canonical factorization, up to an étale cover, where each "irreducible factor" is either an abelian variety or an irreducible symplectic variety.

REMARK 1.3. The study of symplectic varieties in the literature is often undertaken in the context of compact Kähler manifolds. The author finds it convenient to include the projectivity assumption from the beginning, mainly due to a preference for the language of projective algebraic geometry.

Note that the 2-form induces a skew-symmetric pairing

$$\sigma_x\colon T_X(x)\times T_X(x)\to\mathbb{C}$$

on the tangent space $T_X(x)$ at each point $x\in X$, and the condition of non-degeneracy is equivalent with each σ_x being perfect.

The first example of a symplectic variety is a K3 surface, and in fact, much of the study of symplectic varieties is modelled on the well-established theory of K3 surfaces.

Among the basic tools in the study of symplectic varieties is the Beauville-Bogomolov form q, which on a K3 surface is just the intersection pairing:

THEOREM 1.4 (Fujiki [**10**], Beauville [**1**]). *Let X be a symplectic variety of dimension $2n$. There exists an integral quadratic form q on $\mathrm{H}^2(X,\mathbb{Z})$, satisfying*

$$q(\alpha)^n = c\deg(\alpha^{2n})$$

for a real constant $c > 0$, and q is uniquely determined, up to a scalar, by this property.

REMARK 1.5. We will use the same symbol $q(-)$ for the quadratic form and the symmetric bilinear pairing $q(-,-)$ associated to it.

We will only need the characterization of q provided by the theorem, but let us mention how q can be constructed: Since σ generates $\mathrm{H}^0(X,\Omega_X^2)$, we can use Hodge decomposition to write

$$\alpha = r\sigma + \beta + s\overline{\sigma}$$

where β is a $(1,1)$-class and r,s are complex numbers. Then, up to a scalar, we have

$$q(\alpha) = rs + \frac{n}{2}\deg(\beta^2(\sigma\overline{\sigma})^{n-1}).$$

Here is the list of all currently known examples of symplectic varieties, up to deformation (in the compact Kähler setting, no further examples are known, except that the Hilbert schemes below may be replaced by Douady spaces, with underlying K3 or abelian surface possibly replaced by a nonprojective K3 or complex torus):

1.1.1. The Hilbert scheme of a K3 surface (Beauville [1]). Let S denote a (projective) K3 surface. It is well known that S itself satisfies the conditions of Definition 1.1. It turns out that the Hilbert scheme $S^{[n]}$, parametrizing finite subschemes $Z \subset S$ of length n, inherits these properties, and is thus a symplectic variety of dimension $2n$.

1.1.2. The generalized Kummer variety (Beauville [1]). Likewise, the Hilbert scheme $A^{[n]}$ of an abelian surface admits a non-degenerate 2-form, but it is not simply-connected, and hence not irreducible (in the symplectic sense). However, it turns out that the reducibility of $A^{[n]}$ is due to the surface A appearing as a factor in its Bogomolov decomposition, which can be "factored out" as follows: Composing the Hilbert-Chow morphism $A^{[n]} \to A^{(n)}$ with the n-fold addition map on A, we obtain a regular map

$$\sigma\colon A^{[n]} \to A$$

which is locally trivial (i.e. of product type) in the étale topology. A fibre K^nA of σ is in fact a (irreducible) symplectic variety, called the *generalized Kummer variety*. Since σ is locally trivial, all its fibres are isomorphic, but we will often choose K^nA to be the fibre over $0 \in A$. In this text, we will drop the word "generalized", and refer to K^nA simply as a Kummer variety. When $n = 2$, it is easy to verify that one recovers the classic Kummer surface associated to A. Thus, K^2A is in fact a K3 surface, but for $n > 2$, K^nA is not isomorphic to the Hilbert scheme of a K3. In fact, their second Betti numbers are distinct, so they are not even homeomorphic.

1.1.3. Interlude on moduli spaces of sheaves (Mukai [27]). Let S be either a K3 surface or an abelian surface. The Hilbert schemes $S^{[n]}$ occuring in the previous examples can be viewed as moduli spaces of sheaves, by associating to a subscheme $Z \subset S$ its ideal sheaf. The existence of a symplectic form on $S^{[n]}$ can then be explained, and generalized, as follows: Consider any moduli space M of stable sheaves on a surface S with trivial canonical sheaf. The tangent space to M at a sheaf $\mathscr{E} \in M$ can be identified with the vector space

$$T_M(\mathscr{E}) \cong \mathrm{Ext}^1_S(\mathscr{E}, \mathscr{E}).$$

Thus, the Yoneda product gives a pairing

$$T_M(\mathscr{E}) \times T_M(\mathscr{E}) \to \mathrm{Ext}^2_S(\mathscr{E}, \mathscr{E}).$$

Since the canonical sheaf on S is trivial, the vector space $\mathrm{Ext}^2_S(\mathscr{E}, \mathscr{E})$ is Serre dual to $\mathrm{Hom}_S(\mathscr{E}, \mathscr{E})$, and the latter is just $\mathbb{C}$, since a stable sheaf $\mathscr{E}$ is simple. Thus we have a natural pairing on each tangent space, and Mukai shows that they fit together to give a symplectic structure σ on M, and that M is non-singular. So M is symplectic, and is known to be projective in many cases (whenever semi-stability implies stability). The problem that remains is to find examples where M is *irreducible* symplectic, or has an irreducible symplectic variety as a Bogomolov factor. In all known examples, this has been proved by either deforming M to the Hilbert scheme of a K3 surface, or by deforming a Bogomolov factor of M to a Kummer variety. Thus, viewing each example considered so far as a family, the generalization of Mukai has not given any new families.

1.1.4. O'Grady's examples (O'Grady [30, 31]). This time, take a moduli space $\overline{M}$ of semi-stable sheaves on a K3 or an abelian surface, that is projective, but singular. By the results of Mukai, there exists a symplectic form on the (open) stable locus, and one might try to extend this to a non-degenerate form on a resolution of singularities of $\overline{M}$. This is by no means easy, but O'Grady found two examples — a 10-dimensional example obtained from a certain moduli space of rank 2 sheaves on a K3, and a 6-dimensional example obtained from a moduli space of rank 2 sheaves on an abelian surface (except that the abelian surface has to be "factored out" as in the case of a Kummer variety). Again it is shown that the examples produced are genuinely new by calculating their second Betti numbers, and noting that they do not appear among the previously known examples.

1.2. Regular and rational fibrations

DEFINITION 1.6. A map $X \to S$ of varieties is a *fibration* (of X over S) if a generic fibre is connected and has positive dimension.

EXAMPLE 1.7. If X is a K3 surface with a fibration $X \to S$, then S is the projective line $\mathbb{P}^1$, and a generic fibre X_s is an elliptic curve. In other words, X is an elliptic K3 surface.

1.2.1. Matsushita's results on fibrations. A seemingly over-optimistic generalization of Example 1.7 on elliptic K3 surfaces would be the following: If $X \to S$ is a fibration of a $2n$-dimensional symplectic variety, then S is projective n-space and a generic fibre X_s is an n-dimensional abelian variety. In fact, in a series of papers [**21, 22, 23, 24**], Matsushita showed that such a statement is very close to the truth. The strongest result so far in this direction is the following (due to Matsushita, but the form quoted here is taken from Huybrechts' exposition [**16**]):

THEOREM 1.8 (Matsushita). *Let $X \to S$ be a fibration of a $2n$-dimensional symplectic variety X over a projective nonsingular variety S.*

(1) *The base variety S is Fano, and its Hodge numbers equal those of $\mathbb{P}^n$.*
(2) *Every component of every fibre X_s is an n-dimensional Lagrangian subvariety of X, and every nonsingular fibre is an abelian variety.*

There are currently no known examples of fibrations of symplectic varieties with base different from $\mathbb{P}^n$, even when singular base varieties are allowed.

REMARK 1.9. It is possible that Matsushita's result may be the first step towards a classification of symplectic varieties: Every K3 surface can be deformed into an elliptic one — perhaps every symplectic variety can be deformed into one admitting a fibration? If so, to classify symplectic varieties up to deformation, and thus determine whether the list of examples in the previous section is exhaustive, it will be enough to understand fibred symplectic varieties. There is an expository paper by Sawon [**36**], where this line of thought is investigated further.

Although such a programme for classification is rather speculative at the moment, it motivates a study of fibrations on the known symplectic varieties.

1.2.2. Rational fibrations. It turns out to be fruitful to include certain fibrations with base points in our study.

DEFINITION 1.10. Let X be symplectic. A rational map $f\colon X \to S$ is a *rational fibration* if there exist another symplectic variety X' and a birational map $\phi\colon X' \widetilde{\dashrightarrow} X$ such that the composition $f \circ \phi$ is a (regular) fibration on X'.

Note that Matsushita's Theorem 1.8 applies to $f \circ \phi$, and in particular the conclusions on the base variety S hold also for rational fibrations.

There are several results, due to Huybrechts, expressing that birational symplectic varieties are *very similar*, to the extent that it is usually hard to show that birational symplectic varieties are not isomorphic. A first, rather easy instance of this principle is the following:

PROPOSITION 1.11 (Huybrechts [**16**, §21.3]). *Let $\phi\colon X' \widetilde{\dashrightarrow} X$ be a birational map between symplectic varieties. Then ϕ restricts to an isomorphism between the maximal open subsets $U' \subset X'$ and $U \subset X$ on which ϕ and ϕ^{-1} are defined.*

As the open subsets $U' \subset X'$ and $U \subset X$ have complements of dimension at least 2, a consequence is that the Picard groups, and the second cohomology groups, of X and X' are isomorphic.

In the theory of symplectic varieties, the second cohomology group, together with its Hodge structure and the Beauville-Bogomolov form, plays a fundamental role, analogous to the period of a K3 surface. However, as the next result shows, the period is not strong enough to distinguish between birational symplectic varieties.

PROPOSITION 1.12 (Huybrechts [**16**, §27.1]). *The isomorphism*

$$\mathrm{H}^2(X',\mathbb{Z}) \cong \mathrm{H}^2(X,\mathbb{Z}),$$

induced by a birational map $\phi\colon X' \widetilde{\dashrightarrow} X$ between symplectic varieties, is compatible with Hodge structures and preserves the Beauville-Bogomolov form.

For completeness, one should mention that Huybrechts has shown that any two birational symplectic varieties X and X' are deformation equivalent. In particular, they are diffeomorphic.

In conclusion, we might say that the resolution of base points provided by replacing X with X' in Definition 1.10 is a rather minor modification to X.

1.3. On existence and uniqueness of fibrations

Let $f\colon X \to \mathbb{P}^1$ be an elliptic K3 surface. Then the pullback $D = f^*(t)$ of a point $t \in \mathbb{P}^1$ has trivial self intersection. Conversely, if X is a K3 surface with a nontrivial (i.e. not linearly equivalent to 0) divisor D satisfying $D^2 = 0$, then there exists an elliptic fibration $f\colon X \to \mathbb{P}^1$.

An analogy for arbitrary symplectic varieties has been suggested by several people (I do not know who considered the question first, but see Section 21.4 of Huybrechts' exposition [**16**]):

CONJECTURE 1.13. *Let X be a symplectic variety of dimension $2n$. Then X admits a rational fibration over $\mathbb{P}^n$ if and only if there exists a nontrivial divisor D on X satisfying $q(D) = 0$.*

The necessity of the condition $q(D) = 0$ is easy to prove: If $f\colon X \to \mathbb{P}^n$ is a regular fibration, let $D = f^*H$ be the pullback of a hyperplane $H \subset \mathbb{P}^n$. Then the top self intersection D^{2n} vanishes. By Theorem 1.4, the Beauville-Bogomolov square $q(D)$ vanishes also. To extend the argument to rational fibrations, just apply Proposition 1.12.

Sawon [**35**] and Markushevich [**20**] independently proved that the conjecture holds when X is the Hilbert scheme of a sufficiently generic K3 surface. In Chapter 3, we prove that the conjecture holds also for the Kummer variety of a generic principally polarized abelian surface. We will also see an example in Chapter 4 of a Kummer variety admitting only a rational (and no regular) fibration.

In the rest of this chapter we will comment around the vanishing condition $q(D) = 0$, and in particular what the condition means for a Kummer variety.

1.3.1. The linear system defining a fibration. The following result is probably well known, but as we have found no reference, we include a proof. It shows that the linear system defining a fibration is necessarily complete. This holds for fibrations of any variety, not necessarily symplectic, and can be thought of as an analogue of projective normality for maps *onto* projective space rather than *into*.

PROPOSITION 1.14. *Let X be a variety over a field k of characteristic zero, and let $f\colon X \to \mathbb{P}^n_k$ be a proper fibration. Then the canonical map*

$$\mathrm{H}^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \to \mathrm{H}^0(X, f^*\mathscr{O}_{\mathbb{P}^n}(d))$$

is an isomorphism for every integer d. In particular, the linear system on X corresponding to f is complete.

PROOF. The hypothesis $\mathrm{char}(k) = 0$ implies that

$$f_*\mathscr{O}_X \cong \mathscr{O}_{\mathbb{P}^n} \tag{1.3.1}$$

by the following standard argument: As f is proper, it has a Stein factorization [**13**, III §4.3]

$$X \to \operatorname{Spec} f_*\mathscr{O}_X \xrightarrow{\alpha} \mathbb{P}^n.$$

Here, the map α is finite, and necessarily of degree one, since f has connected fibres. In other words α is bijective. Since the ground field has characteristic zero, generic smoothness [**14**, III Corollary 10.7] implies that α is birational. Moreover, the target variety $\mathbb{P}^n$ is nonsingular and hence normal, which implies that the bijective birational map α is an isomorphism. The isomorphism (1.3.1) follows.

We view $\mathrm{H}^0(X,-)$ as push forward to $\operatorname{Spec} k$, which obviously factors through push forward over f. Thus we identify

$$\mathrm{H}^0(X, f^*\mathscr{O}(d)) = \mathrm{H}^0(\mathbb{P}^n, f_*f^*\mathscr{O}(d)).$$

Using the projection formula and the isomorphism (1.3.1), we find

$$\begin{aligned} f_*f^*\mathscr{O}(d) &= f_*(\mathscr{O}_X \otimes f^*\mathscr{O}(d)) \\ &\cong f_*\mathscr{O}_X \otimes \mathscr{O}(d) \\ &\cong \mathscr{O}(d) \end{aligned}$$

and the first part of the proposition is established.

For the last sentence, let $P \subseteq |f^*\mathscr{O}(1)|$ denote the n-dimensional linear system corresponding to f. Now note that $|f^*\mathscr{O}(1)|$ has dimension n also, since $\mathrm{H}^0(X, f^*\mathscr{O}(1)) \cong \mathrm{H}^0(\mathbb{P}^n, \mathscr{O}(1))$. We conclude that $P = |f^*\mathscr{O}(1)|$, so P is complete. □

Now let $f\colon X \to \mathbb{P}^n$ be a fibration on a symplectic variety, and let $H \subset \mathbb{P}^n$ denote a hyperplane. By the proposition, the linear equivalence class of f^*H determines the fibration f.

Let $\mathrm{NS}(X)$ denote the Néron-Severi group of X, i.e. the image of the canonical (first Chern class) map

$$\mathrm{Pic}(X) = \mathrm{H}^1(X, \mathscr{O}_X^*) \to \mathrm{H}^2(X, \mathbb{Z})$$

induced by the exponential sequence. Since X is (irreducible) symplectic, we have $\mathrm{H}^1(X, \mathscr{O}_X) = 0$, hence this last map is injective, so $\mathrm{Pic}(X) \cong \mathrm{NS}(X)$. Thus, the fibration f is also determined by the class of f^*H in $\mathrm{NS}(X)$.

Temporarily denoting by $\mathrm{Fib}(X)$ the set of rational fibrations $f\colon X \dashrightarrow \mathbb{P}^n$, modulo automorphisms of $\mathbb{P}^n$, we have just shown that the map (of sets)

$$\mathrm{Fib}(X) \hookrightarrow \mathrm{NS}(X), \tag{1.3.2}$$

sending a fibration f to the class of f^*H, is an *injection*. Henceforth we will focus on finding the image of $\mathrm{Fib}(X)$ in $\mathrm{NS}(X)$.

1.3.2. A numerical criterion for the existence of a fibration. Let us investigate what the necessary condition $q(D) = 0$, for D to define a rational fibration, gives when the Picard number of the symplectic variety X is small.

If the Picard number of X is 1, then no rational fibration exists: Let D_0 be the ample generator of $\mathrm{NS}(X)$. Then $q(D_0)$ is nonzero, for instance by Theorem 1.4 again. But then any nontrivial D in $\mathrm{NS}(X)$ has $q(D) \neq 0$.

As we will see in Section 1.4.1, the Kummer varieties have Picard number at least 2 (and so do the Hilbert schemes of K3 surfaces), so there is still hope. In fact, as we will see in the next section, whenever the underlying abelian surface is sufficiently general, the Picard number of the Kummer variety is exactly 2. In this case we have the following:

PROPOSITION 1.15. *If X is a symplectic variety with Picard number* 2*, then there exist at most two non-equivalent rational fibrations on X over $\mathbb{P}^n$.*

PROOF. In general, if ρ is the Picard number of X, then the Beauville-Bogomolov form on $\mathrm{NS}(X)_{\mathbb{R}}$ has index $(1,\rho-1)$, i.e. it has one positive eigenvalue and the rest are negative [**1**, Théorème 5]. Thus, when $\rho=2$, we can find real classes A and B such that there is an orthogonal decomposition with respect to q,

$$\mathrm{NS}(X)_{\mathbb{R}} = \mathbb{R}A \oplus \mathbb{R}B,$$

with $q(A)=1$ and $q(B)=-1$. If $D=rA+sB$ is an arbitrary divisor class with $q(D)=0$, we have

$$0 = q(D) = r^2 - s^2$$

so $r=\pm s$ and hence D is a (real) multiple of $A\pm B$.

Next, we claim that there can exist at most one real number r (for each choice of sign) such that $r(A\pm B)$ is a divisor defining a fibration on X. This will prove the proposition.

Slightly more generally, we prove that if D_1 and D_2 are proportional divisor classes, both defining rational fibrations on X, then they are equal. For this, choose a "common multiple" D such that $D=d_1D_1=d_2D_2$ for integers d_1 and d_2, which we may assume are not both negative. By definition of rational fibrations, there exist symplectic varieties X_i and birational maps $X_i \overset{\sim}{\dashrightarrow} X$ such that D_i is base point free on X_i. By Proposition 1.11, birational symplectic varieties are isomorphic in codimension 1, so we have

$$\mathrm{H}^0(X_1,\mathscr{O}(D)) \cong \mathrm{H}^0(X,\mathscr{O}(D)) \cong \mathrm{H}^0(X_2,\mathscr{O}(D)).$$

Now apply Proposition 1.14 twice to obtain isomorphisms

$$\mathrm{H}^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(d_1)) \cong \mathrm{H}^0(X,\mathscr{O}(D)) \cong \mathrm{H}^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(d_2)).$$

Since at least one d_i is nonnegative, we conclude $d_1=d_2$, and thus $D_1=D_2$. □

In the next section we will see that, at least for the Kummer varieties, a rational fibration, if one exists, is in fact *unique*.

1.4. Fibrations on Kummer varieties

We first describe the Néron-Severi group of the Kummer varieties, and then strengthen Proposition 1.15 in the case of Kummer varieties.

1.4.1. The Beauville-Bogomolov form on Kummer varieties. We need to know the exact structure of the Néron-Severi group and the Beauville-Bogomolov form on a Kummer variety. To state the result, due to Beauville [**1**] and Britze [**4**], we let $E \subset K^nA$ be the locus of non-reduced subschemes $Z \in K^nA$.

Then E is a divisor on K^nA, divisible by 2 in $\mathrm{NS}(K^nA)$, and we define

$$\varepsilon = E/2 \in \mathrm{NS}(K^nA)$$

which is a primitive class.

Here, and throughout, we use the same symbol to denote a divisor and its class in the Néron-Severi group.

THEOREM 1.16. *There is a direct sum decomposition*

$$\mathrm{NS}(K^nA) \cong \mathrm{NS}(A) \oplus \mathbb{Z}\varepsilon$$

which is orthogonal with respect to the Beauville-Bogomolov form q. The restriction of q to $\mathrm{NS}(A)$ is the intersection form on A, whereas $q(\varepsilon) = -2n$.

Here is some explanation. We construct a map

$$i\colon\ \mathrm{NS}(A) \to \mathrm{NS}(K^nA)$$

as follows: Let p_i denote the i'th projection from the product A^n, and let

$$A^n \xrightarrow{\ p\ } A^{(n)} \xleftarrow{\ q\ } A^{[n]}$$

be the quotient map by the symmetric group and the Hilbert-Chow morphism. If D is a divisor on A, the divisor $\sum_i p_i^*D$ on A^n is invariant under the action of the symmetric group and descends to $A^{(n)}$. In other words, we have

$$p^*(D') = \sum_i p_i^*D$$

for a uniquely determined divisor D' on $A^{(n)}$. We define $i(D)$ to be the restriction of $q^*(D')$ to K^nA.

By exactly the same construction we may define a map

$$i\colon\ \mathrm{H}^2(A,\mathbb{C}) \to \mathrm{H}^2(K^nA,\mathbb{C}).$$

What Beauville [**1**] proves is that this map is *injective*, and that $\mathrm{H}^2(K^nA,\mathbb{C})$ is the direct sum of the image of i and the complex line spanned by ε, i.e.

$$\mathrm{H}^2(K^nA,\mathbb{C}) = \mathrm{H}^2(A,\mathbb{C}) \oplus \mathbb{C}\varepsilon.$$

The decomposition in Theorem 1.16 follows from the Lefschetz theorem on $(1,1)$-classes, since i is compatible with Hodge decomposition, and ε is a primitive $(1,1)$-class.

Finally, the calculation of the Beauville-Bogomolov form can be found in Britze's thesis [**4**].

1.4.2. Fibrations on Kummer varieties. Let A be an abelian surface with Picard number 1, and let H be the ample generator of its Néron-Severi group. By Theorem 1.16 we have

$$\mathrm{NS}(K^nA) \cong \mathbb{Z}H \oplus \mathbb{Z}\varepsilon. \tag{1.4.1}$$

In particular, the Kummer variety K^nA has Picard number 2, so we can apply Proposition 1.15 to conclude that K^nA admits at most two rational fibrations. But in fact, as we will show, a rational fibration, if one exists, is unique.

The whole point is the following: In the proof of Proposition 1.15, we found two possible candidates for a divisor defining a rational fibration. We did not, however, use the fact that such a divisor must be without fixed components. This condition rules out one of the divisors. The argument is essentially copied from Markushevich [**20**], who proved the analogous result for the Hilbert scheme of a K3 surface with Picard number 1.

PROPOSITION 1.17. *There exists at most one rational fibration on the Kummer variety K^nA of an abelian surface A with Picard number* 1.

In the following, we will write $\widetilde{H}$ for the divisor class in $\mathrm{NS}(K^nA)$ corresponding to H under the isomorphism (1.4.1).

PROOF. We claim that if a divisor $D = k\widetilde{H} + l\varepsilon$ is without fixed components, then k must be positive and l negative.

First of all, let us show that if k is negative, then $\widetilde{H}$ is a fixed component for D. Let $\sigma(\alpha) \in A$ denote the sum of a 0-cycle α on A, with respect to the group law. Fix an effective 0-cycle α of degree $n-2$, and consider the map

$$\gamma\colon H \to A^{(n)}$$

sending $x \in H$ to the 0-cycle obtained by adjoining the two points x and $-x-\sigma(\alpha)$ to α. For generic α, the image of γ is a curve avoiding the big diagonal in $A^{(n)}$, and can be considered as a curve on K^nA, which we denote by C. As C is disjoint from $E \subset A^{[n]}$, we have $\varepsilon \cdot C = 0$, and using the H is ample, one checks that

$$\widetilde{H} \cdot C > 0.$$

Hence, if k is negative, then C is contained in the base locus of $|D|$. Now, by construction of $\widetilde{H}$, the curve C is contained in $\widetilde{H}$. By varying the choice of the cycle α, we obtain a dense open subset of $\widetilde{H}$ consisting entirely of base points for $|D|$, and hence we conclude that $\widetilde{H}$ is a fixed component. Thus k must be positive.

Secondly, let C be the fibre of the Hilbert-Chow morphism (restricted to K^nA)

$$K^nA \to A^{(n)}$$

over a cycle $\alpha \in A^{(n)}$ with exactly one double point, adding to zero under the group law on A. Recall the following: If $A^{[n]}_*$ and $A^{(n)}_*$ denote the open subsets of the Hilbert scheme and the symmetric product, respectively, consisting of schemes or 0-cycles with at most one double point, then it is well known that the Hilbert-Chow morphism $A^{[n]}_* \to A^{(n)}_*$ is just the blowup of the big diagonal, i.e. the locus of 0-cycles *with* a double point. Since ε is one half the class of the exceptional divisor of this blowup, it follows that ε intersects the fibre C negatively. Thus we have

$$\varepsilon \cdot C < 0$$

and since $\widetilde{H}$ is a pullback from $A^{(n)}$, by construction, we have $\widetilde{H} \cdot C = 0$. We conclude that, if l is positive, then C is contained in the base locus of $|D|$. By varying the choice of the fibre C, we get a dense subset of the exceptional divisor E consisting of base points for $|D|$, so E is a fixed component for D unless l is negative.

Finally, apply Proposition 1.15 as follows: A diagonalization of the Beauville-Bogomolov form on $\mathrm{NS}(K^nA)_{\mathbb{R}}$ is provided by the basis $A = (1/\sqrt{H^2})H$ and $B = (1/\sqrt{2n})\varepsilon$. In the proof of Proposition 1.15, we found exactly two candidates for a divisor D to define a rational fibration on K^nA; one was proportional to $A+B$ and the other was proportional to $A-B$. But only the latter can be written $kH + l\varepsilon$ with k positive and l negative. □

We close this section by giving the criterion for the existence of a divisor D on K^nA with $q(D) = 0$, in the case where A admits a *principal* polarization and has Picard number 1. In the notation used in the proof of Proposition 1.15, such a divisor must lie on the lines in $\mathrm{NS}(K^nA)_{\mathbb{R}}$ spanned by $A \pm B$, thus the question is whether these lines contain integral points.

PROPOSITION 1.18. *Let A be a principally polarized abelian surface with Picard number* 1. *Then K^nA carries a divisor D satisfying $q(D) = 0$ if and only if n is a perfect square. Hence, if K^nA admits a fibration, then n is a perfect square.*

PROOF. Let H be the principal polarization. If $D = k\widetilde{H} + l\varepsilon$, then

$$q(k\widetilde{H} + l\varepsilon) = k^2H^2 - 2l^2n$$

and $H^2 = 2$ since H is principal. Thus, if q vanishes on D, we have $n = k^2/l^2$. Conversely, if $n = m^2$, then $D = m\widetilde{H} - \varepsilon$ satisfies $q(D) = 0$. □

In Chapter 3 we will complete this result by showing that if n *is* a perfect square, then K^nA does admit a rational fibration.

CHAPTER 2

Products of elliptic curves

Recall that every known example of a symplectic variety (irreducible and projective) is deformation equivalent to, and not isomorphic to, one of the examples of Beauville (Sections 1.1.1 and 1.1.2) and O'Grady (Section 1.1.4). In other words, each example should be considered as a family. With this in mind, a first question to ask about fibrations on symplectic varieties, would be the following:

> Is each of the examples of Beauville and O'Grady deformation equivalent to a symplectic variety admitting a fibration over a projective space?

The answer is yes: For the Hilbert scheme of a K3 surface, one can deform the K3 surface into an elliptic one $\pi\colon S \to \mathbb{P}^1$. The projection π induces in a natural way a fibration on $S^{[n]}$ over $(\mathbb{P}^1)^{(n)} \cong \mathbb{P}^n$. In this chapter, we will give a similar construction to provide fibrations of all Kummer varieties after deformation, by deforming the underlying abelian surface to a product $E \times E'$ of elliptic curves, and then describe a simple fibration on $K^n(E \times E')$. Finally, (rational) fibrations on O'Grady's examples have been described by Rapagnetta [**34**].

The fibration on $K^n(E \times E')$, that we are going to describe, may be known among experts, but it doesn't seem to have appeared in the literature. We also remark that the construction utilized by Debarre [**6**] for computing the Euler characteristic of the Kummer varieties, provides another example of fibrations of Kummer varieties after deformation, although less elementary. In fact, his fibration is a special case of the fundamental fibration, which we will construct in Chapter 3.

As an application, inspired by the cited paper of Debarre, we use our fibration on $K^n(E \times E')$ to give an elementary proof of the formula

$$\chi(K^n A) = n^3 \sigma(n)$$

for the Euler characteristic of the Kummer varieties, where $\sigma(n) = \sum_{d|n} d$ denotes the sum of divisors function.

2.1. Deformations of the abelian surface underlying a Kummer variety

We want to deform the Kummer variety $K^n A$ of an arbitrary abelian surface into the Kummer variety $K^n(E \times E')$ of a product of elliptic curves. In fact, we will show that this is possible for any choice of elliptic curves E and E'.

PROPOSITION 2.1. *Let A be an abelian surface and E, E' elliptic curves. Then there exists, for each n, a smooth family of varieties over a nonsingular curve having $K^n A$ and $K^n(E \times E')$ among its fibres.*

The proof consists of two steps: First we will explain why there exists a family deforming the surface A itself into $E \times E'$. Then we will apply the Kummer variety construction to the whole family to produce the wanted deformation.

2.1.1. Splitting abelian varieties by deformation. Let (A,H) be a polarized abelian variety of arbitrary dimension g. Denoting linear equivalence by $\sim$, consider the group

$$K(H) = \{a \in A \,|\, T_a^* H \sim H\},$$

which is finite when H is ample [**28**, §6]. Recall that there exists an isomorphism

$$\alpha\colon K(H) \xrightarrow{\sim} \bigoplus_{i=1}^{g} (\mathbb{Z}/d_i\mathbb{Z})^2$$

for nonnegative integers d_i with d_i diving d_{i+1}. The tuple $d = (d_1, d_2, \dots, d_g)$ is called the *type* of H, and the choice of an isomorphism α is called a *level structure* (although there exist other structures referred to by the same name).

There exists [**2**, §8] a moduli space $\mathscr{A}_{g,d}$ for polarized abelian varieties of dimension g and type d, together with a level structure. This space is an irreducible algebraic variety, carrying a universal family whenever $d_1 \geq 3$.

LEMMA 2.2. *Let A be an abelian variety of dimension g, and let $E_1, \dots, E_g$ be elliptic curves. Then there exists an abelian scheme*

$$X \to S$$

over a nonsingular curve S, having A and $\prod_{i=1}^{g} E_i$ among its fibres.

PROOF. Let us first observe that a product $\prod_i E_i$ of elliptic curves admits polarizations of any given type $d = (d_1, \dots, d_g)$. To see this, choose divisors D_i on E_i of degree d_i, and, denoting projection to the i'th factor by p_i, let

$$H' = \sum_{i=1}^{g} p_i^* D_i.$$

Then H' defines a polarization of type d.

The irreducibility of the moduli space $\mathscr{A}_{g,d}$ now implies the assertion of the lemma: Choose an ample polarization H on A of any type $d = (d_1, \dots, d_g)$. By ampleness we have $d_1 > 0$, and, after possibly replacing H with a multiple, we may assume $d_1 \geq 3$. Now equip $\prod_i E_i$ with a polarization H' of the same type d. Then the two polarized abelian varieties (A,H) and $(\prod_i E_i, H')$, together with arbitrarily chosen level structures, define two points in the same moduli space $\mathscr{A}_{g,d}$, which is irreducible and carries a universal family. Thus we may take S to be (the normalization of) any irreducible curve passing through these two points, and X to be the pullback of the universal family to S, and we are done. For the existence of an irreducible curve through any two points on any variety, see Mumford [**28**, §6, Lemma]. □

2.1.2. The Kummer variety of a family. If $X \to S$ is a projective morphism of schemes, we denote by $X_S^{[n]}$ and $X_S^{(n)}$ the relative Hilbert scheme and the relative symmetric product, respectively. There exists [**18**] a relative Hilbert-Chow morphism $X_S^{[n]} \to X_S^{(n)}$ over S. When $X \to S$ is an abelian scheme, the group law on X induces a morphism $X_S^{(n)} \to X$ over S, and, precomposing with the relative Hilbert-Chow morphism, we get a relative addition map

$$\sigma\colon X_S^{[n]} \to X \tag{2.1.1}$$

just as in the case of an abelian variety. We remark that the relative Hilbert-Chow morphism respects base change, and hence the addition map (2.1.1) does also.

DEFINITION 2.3. Let $X \to S$ be an abelian scheme and n a natural number. *The Kummer variety of X over S* is the fibre product K^nX in the diagram

$$\begin{array}{ccc} K^nX & \longrightarrow & X_S^{[n]} \\ \downarrow & & \downarrow \sigma \\ S & \xrightarrow{\;0\;} & X \end{array}$$

where σ is the addition map (2.1.1) and 0 is the zero section.

Clearly, when the base is $S = \operatorname{Spec}\mathbb{C}$, we recover the Kummer variety of Beauville. The construction of the relative Kummer variety commutes with base change:

PROPOSITION 2.4. *Let $X \to S$ be an abelian scheme and $T \to S$ a morphism. Then there is a canonical isomorphism*

$$(K^nX)_T \cong K^n(X_T)$$

where subscript T denotes the fibre product with T over S.

PROOF. This follows from the fact that formation of the Hilbert scheme, the symmetric product, the addition map (2.1.1) and the zero section all commute with base change. □

In particular, the fibre $K^nX \otimes k(s)$ over $s \in S$ is canonically identified with the Kummer variety of $X \otimes k(s)$.

PROOF OF PROPOSITION 2.1. By an application of Lemma 2.2, we can find an abelian scheme $X \to S$ of relative dimension 2 over a nonsingular curve

S, having A and $E \times E'$ among its fibres. Then, as we just saw, the relative Kummer variety $K^nX \to S$ has K^nA and $K^n(E \times E')$ among its fibres. It only remains to prove that K^nX is smooth over S. Recall that X itself is smooth over S; this is included in the definition of an abelian scheme.

We basically follow Beauville's proof [**1**] for the nonsingularity of the Kummer variety: It is straight forward to verify that there is a cartesian diagram

$$\begin{array}{ccc} K^nX \times_S X & \xrightarrow{\nu} & X_S^{[n]} \\ \downarrow q & & \downarrow \sigma \\ X & \xrightarrow{n_X} & X \end{array}$$

where ν is induced by the natural action of X on the Hilbert scheme, σ is the addition map, q is the second projection and n_X denotes multiplication by the natural number n. By Fogarty's result [**9**, Theorem 2.9], the Hilbert scheme $X_S^{[n]}$ is smooth over S; this is where we use the fact that S is a curve. As n_X is étale, so is ν. Thus the composition

$$K^nX \times_S X \xrightarrow{\nu} X_S^{[n]} \to S$$

is smooth also.

In particular, both X and $K^nX \times_S X$ are flat over S. It follows that $K^nX \times_S X$ is flat over X via second projection. By pulling back q over the zero section $S \to X$, we recover the structure map $K^nX \to S$, which thus is flat.

Since $K^nX \to S$ is flat, to prove it is smooth it is enough to prove that every geometric fibre is nonsingular. In other words, we may assume that S is the spectrum of an algebraically closed field k, in which case the fact that $K^nX \times_k X$ is nonsingular implies that K^nX is nonsingular. □

2.2. The fibration induced by projection

Let $A = E \times E'$ be the product of two elliptic curves. The first projection $p\colon A \to E$ induces a map $p^{(n)}\colon A^{(n)} \to E^{(n)}$ of symmetric products, which we may precompose with the Hilbert-Chow morphism to obtain a map

$$\Pi\colon A^{[n]} \to E^{(n)}. \tag{2.2.1}$$

Let π denote the restriction of Π to the Kummer variety K^nA, and let $P \subset E^{(n)}$ be the image of π. We will show that the regular map

$$\pi\colon K^nA \to P \subset E^{(n)} \tag{2.2.2}$$

has connected fibres — i.e. it is a fibration — and that the image P is a projective space.

EXAMPLE 2.5. In the case $n = 3$, the map π has a nice geometric interpretation: Let us embed the elliptic curve E as a cubic in $\mathbb{P}^2$, sending the origin to an inflection point, such that the usual geometric description of the group law applies. In particular, three points on E add to zero if and only if they are collinear.

Now, if Z is a point of the Kummer variety K^3A, its underlying cycle adds to zero under the group law on A. By definition of the group law on the product $A = E \times E'$, the “first coordinates” $\pi(Z)$ add to zero on E also, hence they are collinear. Thus $\pi(Z)$ spans a uniquely determined line $\ell \in \check{\mathbb{P}}^2$. Conversely, any such line intersects E in an effective zero cycle of degree 3, adding to zero under the group law.

Thus we see that the image P of π can be identified with the dual projective plane $\check{\mathbb{P}}^2$, and the map

$$\pi\colon K^3A \to \check{\mathbb{P}}^2,$$

sends $Z \in K^3A$ to the unique line spanned by the first coordinates of the 3 points underlying Z.

THEOREM 2.6. *Let $\pi\colon K^nA \to P$ be the map in equation* (2.2.2)*. Then P is isomorphic to $\mathbb{P}^{n-1}$, and a generic fibre of π is isomorphic to $(E')^{n-1}$.*

PROOF. The image P is the set of effective divisors of degree n on E, mapping to the zero element $0 \in E$ under the n-fold addition map

$$E^{(n)} \to E.$$

On an elliptic curve, the condition that the underlying points of a degree n divisor D add to zero, is equivalent to D being linearly equivalent to the divisor $n \cdot 0 = 0 + \cdots + 0$. Thus P is just the linear system $|n \cdot 0| \cong \mathbb{P}^{n-1}$.

For the second statement, consider a divisor $D = \sum x_i$ in P, where the supporting points x_i are all distinct. Then a point in $\pi^{-1}(D)$ is an n-tuple of points of the form $(x_i, y_i) \in E \times E'$, with the y_i's summing to zero on E'. By forgetting y_n and remembering $y_1, \ldots, y_{n-1}$ we can identify $\pi^{-1}(D)$ with $(E')^{n-1}$. □

COROLLARY 2.7. *For every abelian surface A and every n, the Kummer variety K^nA admits a fibration over $\mathbb{P}^{n-1}$ after (algebraic) deformation.*

PROOF. By Proposition 2.1, the Kummer variety K^nA of A can be deformed to the Kummer variety $K^n(E \times E')$ of a product of elliptic curves, which admits a fibration by Theorem 2.6. □

EXAMPLE 2.8. Continuing Example 2.5, let us describe the fibres of

$$\pi\colon K^3A \to \check{\mathbb{P}}^2.$$

Let $\check{E} \subset \check{\mathbb{P}}^2$ denote the dual curve of E, i.e. the locus of tangent lines to E. Then a point $\ell \in \check{\mathbb{P}}^2$ outside $\check{E}$ is a line in $\mathbb{P}^2$ intersecting E in 3 distinct points. Hence we have

$$\pi^{-1}(\ell) \cong E' \times E'$$

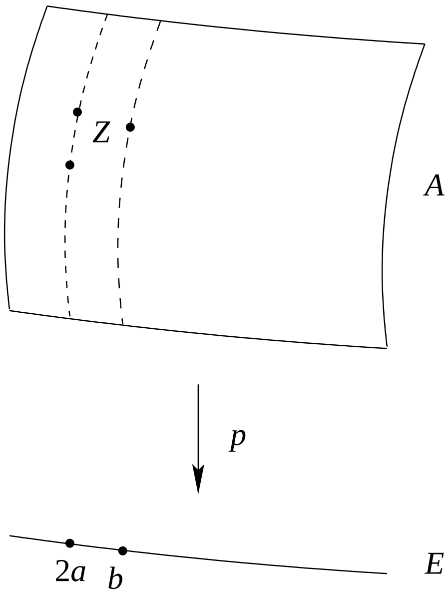

FIGURE 1. Three points on $A = E \times E'$

as in Theorem 2.6. On the other hand, a point $\ell \in \check{E}$ is a line in $\mathbb{P}^2$ with, as cycles on E,

$$\ell \cap E = 2a + b \qquad \text{if } \ell \text{ is a nonsingular point of } \check{E}$$

$$\ell \cap E = 3a \qquad \text{if } \ell \text{ is a singular point of } \check{E}$$

where a and b are distinct points. The first case is illustrated in Figure 1: An element Z of $\pi^{-1}(\ell)$ has two points in the fibre $\{a\} \times E'$, and one point in $\{b\} \times E'$, which is determined by the former two, since the sum of all three should be zero. Then there are two possibilities: Either the two points in $\{a\} \times E'$ are distinct, as in the figure, or they coincide. Each case is parametrized by a 2-dimensional locus in $\pi^{-1}(\ell)$, which thus has two components.

Similarly one shows that, when ℓ is a singular point of $\check{E}$, the fibre $\pi^{-1}(\ell)$ has eleven components. Nine of these correspond to subschemes $Z \subset A$ supported at a single point (a, a'): Then a' must be one of the nine 3-torsion points on E', each giving rise to a component of $\pi^{-1}(\ell)$. A generic point in the remaining two components corresponds to a subscheme Z having either a double point or being reduced.

2.3. Application: The Euler characteristic

Using the fibration of Theorem 2.6 we will give an elementary proof of the following theorem.

THEOREM 2.9 (Göttsche [**11**]). *The topological Euler characteristic of the Kummer variety is given by*

$$\chi(K^nA) = n^3\sigma(n)$$

where $\sigma(n)$ *denotes the sum of divisors function* $\sigma(n) = \sum_{d|n} d$.

This formula was first found by Göttsche [**11**, Corollary 2.4.13], as a corollary of his computation of the Betti numbers of K^nA. If one aims only at the Euler characteristic, however, a much simpler proof is possible, and by now there exist several proofs in the literature. In particular, as mentioned in the beginning of this chapter, Debarre [**6**] gave a proof utilizing a certain fibration on a deformation of the Kummer variety. Inspired by this, the author realized that a study of the fibres of the very simple fibration provided by Theorem 2.6, would provide a completely elementary proof of Theorem 2.9.

Let us first note that, by Proposition 2.1, we may assume that the underlying abelian surface is a product of elliptic curves: Recall that a smooth map between nonsingular varieties is submersive, and that all fibres of a submersive map of (real) smooth manifolds are diffeomorphic. Thus the smooth deformation in Proposition 2.1 shows that K^nA and $K^n(E\times E')$ are diffeomorphic (and hence, in fact, all Kummer varieties are diffeomorphic, as E and E' can be chosen independently of A). In particular, their Euler characteristics coincide. Throughout this section we will, therefore, assume that $A = E\times E'$ is already a product of elliptic curves.

2.3.1. Preparation. We will use the fact that the Euler characteristic is both additive and multiplicative: Additivity means that, whenever X is a disjoint union $\bigcup_i X_i$ of locally closed subsets $X_i \subset X$, we have

$$\chi(X) = \sum_i \chi(X_i).$$

By multiplicativity, we mean the following:

PROPOSITION 2.10. *Let* $f\colon X \to Y$ *be a regular map of algebraic varieties, such that the Euler characteristic of the fibres* $f^{-1}(y)$ *is constant, i.e. independent of* $y \in Y$. *Then the Euler characteristic of* X *is the product of the Euler characteristics of a fibre* $f^{-1}(y)$ *and the base* Y.

PROOF. There exists [**37**, Corollaire 5.1] a stratification

$$Y = \bigcup_i Y_i$$

where $Y_i \subset Y$ are (Zariski) locally closed, such that the restriction $f^{-1}(Y_i) \to Y_i$ of f to each stratum is locally trivial in the transcendent topology, i.e. locally isomorphic to the projection from a product $X = Y\times Z \to Y$. Thus, by additivity of the Euler characteristic, it is enough to prove the claim when f is the projection from a product, in which case the result is elementary. □

We will eventually reduce the computation of the Euler characteristic of the Kummer variety to the following well-established result: Let $H(n)$ denote the punctual Hilbert scheme, which parametrizes length n subschemes of a surface, supported at a fixed nonsingular point. The punctual Hilbert scheme can be realized as a closed subset of the Hilbert scheme $S^{[n]}$ of an arbitrary nonsingular surface S, and is given the reduced induced structure. We need its Euler characteristic:

THEOREM 2.11 (Ellingsrud and Strømme [**8**]). *The Euler characteristic of the punctual Hilbert scheme $H(n)$ equals the number $p(n)$ of partitions of n.*

2.3.2. Only degenerate fibres contribute. Consider again the fibration $\pi\colon K^nA \to P$ constructed in the previous section. We start by showing that most fibres have vanishing Euler characteristic, leaving only a finite number of fibres that contribute to the Euler characteristic of K^nA:

LEMMA 2.12. *Let D denote an effective n-cycle in the image $P \subset E^{(n)}$ of π.*

(1) *If the Euler characteristic of the fibre $\pi^{-1}(D)$ is nonzero, then $D = n\cdot a$ for some point $a \in E$.*
(2) *The Euler characteristic of K^nA satisfies*
$$\chi(K^nA) = n^2\chi(F)$$
where F is the fibre $\pi^{-1}(n\cdot 0)$ over $n\cdot 0$.

Before giving the proof, let us introduce a piece of notation: If $D = \sum k_i a_i$ is an effective divisor of degree $n = \sum k_i$ on E, we let $W(D) \subset A^{[n]}$ denote the (closed) subset consisting of length n subschemes $Z \subset A$ that can be expressed as a disjoint union
$$Z = \bigcup_i Z_i \tag{2.3.1}$$
where Z_i has length k_i and is supported in $\{a_i\} \times E'$. In other words, $W(D)$ is the fibre above D of the map $\Pi\colon A^{[n]} \to E^{(n)}$ in equation (2.2.1).

As the decomposition (2.3.1) is necessarily unique, we have an isomorphism
$$W(D) \cong \prod_i W(k_i a_i)$$
where $W(k_i a_i)$ is considered as a subset of $A^{[k_i]}$. Note that the fibre $\pi^{-1}(D)$ considered in the theorem is just the intersection $W(D) \cap K^nA$. We are only considering topological properties here, so we will not need to worry about the scheme structure of $W(D)$.

PROOF. (1) Suppose on the contrary that D has at least two distinct points in its support. Choosing a distinguished point $a \in \operatorname{Supp} D$, we have
$$D = ka + D'$$

where $k < n$ and D' is an effective divisor, not having a in its support. Consider a point $Z \in K^n A$ with $\pi(Z) = D$. The underlying zero cycle of Z has the form

$$[Z] = \sum_{i=1}^{k}(a, a_i') + \sum_{i=k+1}^{n}(a_i, a_i') \tag{2.3.2}$$

for some points $a_i' \in E'$ (and where $D' = \sum_{i=k+1}^{n} a_i$). The idea is that a labels the first k points, and we obtain a map

$$\nu\colon \pi^{-1}(D) \to E'$$

by sending Z to the sum $\sum_{i=1}^{k} a_i'$ under the group law on E'. We claim ν is regular. In fact, in the notation introduced above, ν is just the restriction to $\pi^{-1}(D) \subset W(D)$ of the composite map

$$\begin{array}{ccccccc} W(D) \cong W(ka) \times W(D') & & & & & & \\ \downarrow p & & & & & & \\ W(ka) & \subset & A^{[k]} & \xrightarrow{\Pi'} & E'^{(k)} & \xrightarrow{\sigma} & E' \end{array}$$

where Π' is induced by second projection as in equation (2.2.1), and σ is the summation map.

Next, we show that all fibres of ν are isomorphic. The fibre of ν above $p \in E'$ is the set of subschemes $Z \subset A$ having underlying zero-cycle of the above form (2.3.2), satisfying

$$ka + \sum_{i=k+1}^{n} a_i = 0 \qquad \sum_{i=1}^{n} a_i' = 0 \qquad \sum_{i=1}^{k} a_i' = p \tag{2.3.3}$$

under the group laws on E and E'. As E' is divisible as a group, we may choose $q, r \in E'$ such that

$$kq = p \quad \text{and} \quad (n-k)r = -p.$$

Then replacing a_i' with $a_i' - q$ when $i \leq k$ and with $a_i' - r$ when $i > k$ does not affect the first two equations of (2.3.3) , whereas the right hand side p of the last equation is replaced by 0. In other words, the translation map

$$W(D) \cong W(ka) \times W(D') \xrightarrow{T^*_{(0,q)} \times T^*_{(0,r)}} W(ka) \times W(D') \cong W(D)$$

restricts to an isomorphism between $\nu^{-1}(p)$ and $\nu^{-1}(0)$.

Now we can apply the multiplicative property of the Euler characteristic: The base space E' for ν has Euler characteristic zero, and all the fibres are isomorphic, so we conclude that the total space $\pi^{-1}(D)$ has Euler characteristic zero also. The first part of the lemma is established.

(2) If a divisor of the form $n \cdot a$ is in P, then a is an element of the group $E_n \subset E$ of n-division points. Let $U \subset P$ be the complement of the set of divisors

of this form. By additivity we have

$$\chi(K^nA) = \chi(\pi^{-1}(U)) + \sum_{a\in E_n} \chi(\pi^{-1}(n\cdot a)).$$

By the first part of the lemma, all fibres of π above U have Euler characteristic zero. Hence $\pi^{-1}(U)$ has Euler characteristic zero also. Furthermore, every fibre $\pi^{-1}(D)$ above a divisor $D = n\cdot a$ is isomorphic, via translation by $(a,0)$, to the fibre F above $n\cdot 0$. Finally, the group of n-division points on an elliptic curve is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$, which has order n^2, so

$$\sum_{a\in E_n} \chi(\pi^{-1}(n\cdot a)) = n^2\chi(F)$$

and we are done. □

2.3.3. The Euler characteristic of a degenerate fibre. We next study the fibre $F = \pi^{-1}(n\cdot 0)$ by means of the map

$$\pi'\colon K^nA \to P' \subset E'^{(n)}$$

induced by the *second* projection $A = E\times E' \to E'$.

Note that F consists of the subschemes $Z \in A^{[n]}$ supported in $\{0\}\times E'$, such that the sum of its underlying cycle under the group law on E' equals 0. This set can be stratified according to the multiplicities of the points in Z: Denote by $\alpha = (1^{\alpha_1}2^{\alpha_2}\cdots)$ the partition of $n = \sum i\alpha_i$ in which i occurs α_i times, and define a locally closed subset $U(\alpha) \subset F$ by

$$U(\alpha) = \{Z \in F \,|\, Z \text{ has } \alpha_i \text{ points of multiplicity } i\}.$$

We define a corresponding locally closed subset $V(\alpha) \subset P'$ by

$$V(\alpha) = \{D \in P' \,|\, D \text{ has } \alpha_i \text{ points of multiplicity } i\}.$$

The computation of the Euler characteristic of F can be reduced to finding the Euler characteristic of each $V(\alpha)$:

LEMMA 2.13. *With $V(\alpha)$ as above we have*

$$\chi(F) = \sum_{\alpha\vdash n}\prod_i p(i)^{\alpha_i}\chi(V(\alpha))$$

where $p(i)$ denotes the number of partitions of i, and the sum runs over all partitions α of n.

PROOF. Clearly, the map $\pi'\colon K^nA \to P'$ sends $U(\alpha)$ to $V(\alpha)$. Let

$$\pi'_\alpha\colon U(\alpha) \to V(\alpha)$$

denote the restricted map. A divisor $D \in V(\alpha)$ can be written

$$D = \sum_i \big(i\sum_{j=1}^{\alpha_i} p_{ij}\big)$$

where the $p_{ij} \in E'$ are distinct points. Hence the fibre of π'_α above D consists of subschemes of the form $Z = \cup_{ij} Z_{ij}$, where each component Z_{ij} has length i and is supported at $(0, p_{ij}) \in A$. Thus every fibre of π'_α is isomorphic to a product $\prod_i H(i)^{\alpha_i}$ of punctual Hilbert schemes, and hence has Euler characteristic

$$\chi(\prod_i H(i)^{\alpha_i}) = \prod_i p(i)^{\alpha_i}$$

by Theorem 2.11. Now apply Proposition 2.10 to π'_α to find

$$\chi(U(\alpha)) = \prod_i p(i)^{\alpha_i} \chi(V(\alpha)).$$

Finally, since $F = \bigcup_{\alpha \vdash n} U(\alpha)$ is a disjoint union of locally closed subsets, we obtain the result by summing the last formula over all partitions of n. □

2.3.4. A recurrence for the Euler characteristic of $V(\alpha)$. We will need an expression for the sum of divisors function $\sigma(n)$ in terms of the number of partitions function $p(n)$. Our starting point is the well known formula

$$p(n) = \frac{1}{n} \sum_{k=1}^{n} \sigma(k) p(n-k), \tag{2.3.4}$$

which may be proved either using Euler's generating function for $p(n)$ or by a counting argument [**12**, Theorem 6, Chapter 12].

Solving equation (2.3.4) for $\sigma(n)$ we find by induction the formula

$$\sigma(n) = \sum_{\alpha \vdash n} \prod_i p(i)^{\alpha_i} c(\alpha) \tag{2.3.5}$$

where $c(\alpha)$ are integers satisfying the recursion

$$c(\alpha) = \begin{cases} n & \text{if } \alpha = (n^1) \\ -\sum_i c(1^{\alpha_1} \cdots i^{\alpha_i - 1} \cdots) & \text{otherwise.} \end{cases} \tag{2.3.6}$$

In this formula, the partitions on the right hand side are obtained from $\alpha = (1^{\alpha_1} 2^{\alpha_2} \cdots)$ by lowering the i'th exponent by 1. If α_i is already zero, we interpret $c(1^{\alpha_1} \cdots i^{\alpha_i - 1} \cdots)$ as being zero.

Now compare Lemma 2.13 with equation (2.3.5): If we can show that $\frac{1}{n}\chi(V(\alpha))$ satisfies the recurrence relation (2.3.6), then it follows that $\chi(F) = n\sigma(n)$ and hence the formula $\chi(K^n A) = n^3 \sigma(n)$ follows from Lemma 2.12.

Since we have now left the surface A altogether, and are only working with the elliptic curve E', let us henceforth write E instead of E'. Recall that $V(\alpha)$ is defined as the set of effective divisors of degree n on E, adding to zero under the group law on E, and having α_i points of multiplicity i.

LEMMA 2.14. *We have $\chi(V(n^1)) = n^2$ for every n.*

PROOF. $V(n^1)$ consists of the divisors of the form $D = n \cdot a$, where a is an n-division point on E. Hence we can identify $V(n^1)$ with the set of n-division points $E_n \cong (\mathbb{Z}/n\mathbb{Z})^2$, which is a finite group of order n^2. □

LEMMA 2.15. *Let $\alpha = (1^{\alpha_1}2^{\alpha_2}\cdots)$ be a partition of n, not equal to (n^1), and let i be an index such that $\alpha_i \neq 0$. Let*

$$\alpha' = (1^{\alpha_1}\cdots i^{\alpha_i - 1}\cdots)$$

denote the partition of $n-i$ obtained from α by lowering the i'th exponent by one. Then

$$\chi(V(\alpha)) = -\frac{n^2(\sum_j \alpha_j - 1)}{\alpha_i(n-i)^2}\chi(V(\alpha')).$$

PROOF. Basically, we would like to compare $V(\alpha)$ and $V(\alpha')$ by means of the incidence variety

$$\{(a,D)\,|\,D \text{ has multiplicity } i \text{ at } a\} \subset E \times V(\alpha).$$

However, if we remove from D the component supported at a, we do get an effective divisor of degree $n-i$, but the sum of its points under the group law on E is no longer zero. Thus there is no natural map from the incidence variety to $V(\alpha')$.

Instead, we let

$$Y = \left\{(a,b,D)\,\middle|\,\begin{array}{l} D \text{ has multiplicity } i \text{ at } a\\ \text{and } (n-i)b = ia \text{ on } E\end{array}\right\} \subset E \times E \times V(\alpha).$$

It is clearly an algebraic subset. There are maps

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & V(\alpha) \\ {\scriptstyle\psi}\downarrow & & \\ V(\alpha') & & \end{array}$$

where ϕ is induced by projection to the third factor, and

$$\psi(a,b,D) = T_b(D - i\cdot a).$$

Here $D - i\cdot a$ denotes the effective divisor obtained from D by removing the component supported at a, and T_b is translation by b. Note that the sum of the supporting points of $T_b(D-i\cdot a)$, with multiplicities, is zero, so ψ is indeed a map to $V(\alpha')$.

We want to calculate the Euler characteristic $\chi(Y)$ twice, using each of the maps ϕ and ψ, and equate the results.

First, let $D \in V(\alpha)$ and consider the fibre

$$\phi^{-1}(D) \cong \left\{(a,b)\,\middle|\,\begin{array}{l} D \text{ has multiplicity } i \text{ at } a\\ \text{and } (n-i)b = ia \text{ on } E\end{array}\right\} \subset E \times E.$$

Now D has α_i points of multiplicity i. Let us denote them a_j with $j = 1,\ldots,\alpha_i$. Then $\phi^{-1}(D)$ is just the disjoint union of the α_i sets

$$\{b\,|\,(n-i)b = ia_j\} \subset E$$

and each of these consists of $(n-i)^2$ points. Thus every fibre of ϕ is a discrete set of $\alpha_i(n-i)^2$ points. By the multiplicativity of the Euler characteristic, we conclude

$$\chi(Y) = \alpha_i(n-i)^2\chi(V(\alpha)). \tag{2.3.7}$$

Next, the fibre of ψ above a point $D' \in V(\alpha')$ can be described as

$$\psi^{-1}(D') \cong \left\{ (a,b) \,\middle|\, \begin{array}{c} a+b \notin D' \text{ and} \\ (n-i)b = ia \end{array} \right\} \subset E \times E.$$

This identification comes about since, if $\psi(a,b,D) = D'$, then the divisor D is uniquely determined by the pair (a,b) as

$$D = T_b^{-1}(D') + i \cdot a,$$

and this divisor has multiplicity i at a if and only if $a \notin T_b^{-1}(D')$, or equivalently $a+b \notin D'$.

Now $\psi^{-1}(D')$ is contained in the slightly bigger set

$$B = \{(a,b) \,|\, (n-i)b = ia\} \subset E \times E, \tag{2.3.8}$$

which has Euler characteristic zero, as can be seen by projecting to e.g. the second factor, and noting that all fibres are isomorphic (in fact, they are discrete sets of i^2 points).

It remains to count the pairs $(a,b) \in B$ with $a+b \in D'$. For each point c in the support of D', the set

$$\{(a,b) \,|\, (n-i)b = ia \text{ and } a+b = c\} \cong \{b \,|\, nb = ic\} \subset E$$

consists of n^2 points. Since there are $\sum_j \alpha_j - 1$ distinct points $c \in D'$, we see that $\psi^{-1}(D')$ is the complement in B to $n^2(\sum_j \alpha_j - 1)$ points. Hence we have

$$\chi(Y) = -n^2(\sum_j \alpha_j - 1)\chi(V(\alpha'))$$

and equating with (2.3.7) gives the result. □

We can now finish the proof of Theorem 2.9 by verifying that $\frac{1}{n}\chi(V(\alpha))$ satisfies the relation (2.3.6), that is,

$$\frac{1}{n}\chi(V(\alpha)) = \begin{cases} n & \text{if } \alpha = (n^1) \\ -\sum_i \frac{1}{n-i}\chi(V(1^{\alpha_1}\cdots i^{\alpha_i-1}\cdots)) & \text{otherwise.} \end{cases}$$

In fact, the first equality is Lemma 2.14, and by Lemma 2.15 we have

$$\begin{aligned} -\sum_i \frac{1}{n-i}\chi(V(1^{\alpha_1}\cdots i^{\alpha_i-1}\cdots)) &= \sum_i \frac{\alpha_i(n-i)}{n^2(\sum_j \alpha_j - 1)}\chi(V(\alpha)) \\ &= \frac{1}{n}\chi(V(\alpha))\frac{\sum_i \alpha_i(n-i)}{n(\sum_j \alpha_j - 1)} \\ &= \frac{1}{n}\chi(V(\alpha)), \end{aligned}$$

where we in the last equality use that $n = \sum_i i\alpha_i$, since α is a partition of n.

REMARK 2.16. The recursion in Lemma 2.15 is easier to solve than (2.3.6). In fact, we find

$$c(\alpha) = \frac{1}{n}\chi(\alpha) = (-1)^{\sum_i \alpha_i - 1} n \frac{(\sum_i \alpha_i - 1)!}{\prod_i(\alpha_i!)}$$

giving a closed solution to (2.3.6).

CHAPTER 3

The fundamental fibration

In this chapter we will construct a rational fibration on the Kummer variety K^nA, whenever the abelian surface A admits an effective divisor C with self intersection $2n$.

Our main application is to the case when A is principally polarized with Picard number 1. In this case, we can verify Conjecture 1.13 by showing that K^nA admits a rational fibration if and only if n is a perfect square, which in turn is equivalent to the existence of a divisor D on K^nA with $q(D) = 0$.

Here is one viewpoint: In the previous chapter, we showed that the Kummer variety K^nA admits fibrations for every n, when the underlying surface A is *special*, namely a product of elliptic curves. The assumption that the Picard number of A equals 1, on the other hand, is *generic*: Recall [**2**, §8] that there exist irreducible moduli spaces $\mathscr{A}_{g,d}$ parametrizing polarized abelian varieties of given dimension g and type d. The locus in $\mathscr{A}_{g,d}$ consisting of abelian varieties with Picard number greater than 1 is at most a countable union of proper analytic subvarieties [**2**, Exercise 8.11.1]. Thus the assumption that the Picard number equals 1 is “generic”.

When A is non-principally polarized, we only obtain partial results. See Section 3.3.1 for further comments. As the type of a polarization is discrete, there is no such thing as a “generic type”. Without implying any meaning, or usefulness, at all, I am tempted to say that the principal type after all has some genericity taste to it: It is the one you most often encounter out in the wild; for Jacobian varieties are principally polarized.

In this chapter we will make freely use of the results (and notations) on Fourier-Mukai transforms and moduli spaces, that are summarized in Appendices A and B.

3.1. Overview

3.1.1. Statement. If $C \in \mathrm{Pic}(A)$ is the class of an effective divisor on A, then there is a canonically defined dual divisor class $\widehat{C} \in \mathrm{Pic}(\widehat{A})$ having the same self intersection as C. We postpone the construction of $\widehat{C}$ until Section 3.1.2.

Our main task is to prove the following:

THEOREM 3.1. *Let A be an abelian surface carrying an effective divisor C with self intersection $2n$, where $n > 2$. If a generic divisor in the linear system*

$|\widehat{C}|$ is a nonsingular, irreducible curve, then the Kummer variety K^nA admits a rational fibration

$$f\colon K^nA \dashrightarrow |\widehat{C}| \cong \mathbb{P}^{n-1}. \tag{3.1.1}$$

REMARK 3.2. The assumption that a generic element in $|\widehat{C}|$ is nonsingular and irreducible is only used for the verification that (a resolution of) f has connected fibres, and hence is a rational fibration according to Definitions 1.6 and 1.10. This assumption is almost always satisfied: In fact, it can only fail if both A is a product of elliptic curves and C is primitive, i.e. indivisible in $\mathrm{NS}(A)$. This follows from Bertini's theorem [**2**, Theorem 4.3.6], using that $|\widehat{C}|$ is base point free whenever C (and hence $\widehat{C}$) is divisible [**28**, §6, §16] or A (and hence $\widehat{A}$) is simple [**2**, §10.1], i.e. not a product.

The theorem is proved in Section 3.2. The proof is constructive, and we will refer to the rational fibration f as *the fundamental fibration* (associated to C).

We have the following corollary, which confirms Conjecture 1.13 for the Kummer varieties associated to a generic principally polarized abelian surface:

COROLLARY 3.3. *If the abelian surface A has Picard number one and admits a principal polarization, then the following are equivalent, for each $n > 2$:*

(1) *The Kummer variety K^nA admits a rational fibration over $\mathbb{P}^{n-1}$.*
(2) *K^nA carries a divisor with vanishing Beauville-Bogomolov square.*
(3) *n is a perfect square.*

PROOF. By Proposition 1.18, we only have to show that (3) implies (1). If H is the principal polarization, then $C = mH$ has self intersection $2m^2$, so by the theorem, the Kummer variety $K^{m^2}A$ admits a rational fibration. □

Also note that the implication from (3) to (1) holds whenever A admits a principal polarization, independent of its Picard number. Thus, for instance, when $A = E \times E'$ is a product of elliptic curves, the fundamental fibration on $K^n(E \times E')$ exists for every perfect square n. This fibration is different from the one considered in Chapter 2.

3.1.2. The dual divisor class. Let C be an effective divisor on an abelian surface A, and let $\mathscr{L} = \mathscr{O}_A(C)$ denote the corresponding invertible sheaf. By Mumford's index theorem [**28**, §16], we have

$$\mathrm{H}^p(A, \mathscr{L} \otimes \mathscr{P}_x) = 0, \quad \text{for all } p > 0 \text{ and all } x \in \widehat{A}.$$

Hence, $\mathscr{L}$ satisfies IT with index 0, and its Fourier-Mukai transform $\widehat{\mathscr{L}}$ is locally free on $\widehat{A}$.

Mukai's Theorem A.14 says, in this case, that the rank, first Chern class and Euler characteristic of the Fourier-Mukai transform of $\mathscr{L}$ are

$$r(\widehat{\mathscr{L}}) = \chi(\mathscr{L}) \qquad c_1(\widehat{\mathscr{L}}) = -c_1(\mathscr{L}) \qquad \chi(\widehat{\mathscr{L}}) = r(\mathscr{L})$$

where we, for the first Chern class, are suppressing the canonical isomorphism

$$\mathrm{H}^2(A,\mathbb{Z}) \cong \mathrm{H}^2(\widehat{A},\mathbb{Z}). \tag{3.1.2}$$

The following defines a divisor class $\widehat{C}$ on $\widehat{A}$, such that the classes of C and $\widehat{C}$ in $\mathrm{H}^2(-,\mathbb{Z})$ agree via the last isomorphism.

DEFINITION 3.4. The *dual divisor class* $\widehat{C} \in \mathrm{Pic}(\widehat{A})$ of an effective divisor C on A is the unique class satisfying

$$\mathscr{O}_{\widehat{A}}(-\widehat{C}) \cong \det \widehat{\mathscr{O}_A(C)}.$$

Let, as usual,

$$\phi_C \colon A \to \widehat{A}$$

denote the map sending $a \in A$ to the class of $T_a^*C - C$, where $\widehat{A}$ is considered as the group of homogeneous divisor classes on A. Then we have [**25**, Proposition 3.11]

$$\phi_C^*(\widehat{\mathscr{O}_A(C)}) = \mathscr{O}_A(-C)^{\oplus d}$$

where $d = C^2/2$, which is also the degree of ϕ_C. It follows that

$$\phi_C^*(\widehat{C}) = dC.$$

Consequently, $\widehat{C}$ is ample, and its self intersection is $\widehat{C}^2 = C^2$. Also note that, if C is a principal polarization, then $d = 1$, and hence C and $\widehat{C}$ agree via the isomorphism ϕ_C.

3.1.3. Sketch. Consider the setup of Theorem 3.1, i.e. we have an effective divisor C with self intersection $2n$ on the abelian surface A.

To construct the fibration in Theorem 3.1, we want to associate to a finite subscheme $Z \subset A$ of length n, a curve in a certain linear system. As a first try, one might ask whether there exists a curve in the linear system $|C|$ containing Z. This turns out to be too restrictive:

LEMMA 3.5. *A generic element $Z \in A^{[n]}$ is not contained in any curve in the linear system $|C|$.*

PROOF. By the index theorem of Mumford again, all higher cohomology groups of $\mathscr{O}_A(C)$ vanish, so by Riemann-Roch,

$$\dim \mathrm{H}^0(A, \mathscr{O}_A(C)) = \chi(\mathscr{O}_A(C)) = n.$$

Thus the complete linear system $|C|$ has dimension $n-1$. It follows that the set of subschemes $Z \in A^{[n]}$ contained in a curve in $|C|$ forms a family of dimension $2n-1$. On the other hand, $A^{[n]}$ has dimension $2n$. □

Let us, starting from the observation in the lemma, sketch our construction: Recall (see Appendix A) that each point $x \in \widehat{A}$ corresponds to a homogeneous invertible sheaf $\mathscr{P}_x$ on A. By allowing not only curves in $|C|$, but in the linear systems associated to $\mathscr{P}_x(C)$ for any $x \in \widehat{A}$, we "win" two more degrees of freedom: The set of length n subschemes contained in a curve in $|\mathscr{P}_x(C)|$, for some $x \in \widehat{A}$, forms a family of dimension $2n+1$. Since, again, $A^{[n]}$ has dimension $2n$, we expect the locus

$$D_Z = \{x \in \widehat{A} \mid \mathrm{H}^0(A, \mathscr{I}_Z \otimes \mathscr{P}_x(C)) \neq 0\} \tag{3.1.3}$$

to be a curve. We will see that this is indeed true for generic Z, and furthermore, when Z is a generic element of the Kummer variety K^nA, the curve D_Z belongs to the linear system $|\widehat{C}|$. The fibration f in Theorem 3.1 is given by sending Z to D_Z.

To "symplectically resolve" the map f, in the sense of Definition 1.10, we proceed as follows: Associated to $Z \in A^{[n]}$, we have not only the curve D_Z, but also the vector space $\mathrm{H}^0(A, \mathscr{I}_Z \otimes \mathscr{P}_x(C))$ attached to each point $x \in D_Z$. One might suspect that these vector spaces form a sheaf on D_Z, so that we may factor f via a suitable moduli space of sheaves supported on curves in $|\widehat{C}|$. This is almost what we will do, except that we should replace H^0 with H^1, basically since the latter is the one that is well behaved with respect to base change. Note that the Euler characteristic of $\mathscr{I}_Z \otimes \mathscr{P}_x(C)$ is zero, and its second cohomology vanishes, so

$$D_Z = \{x \in \widehat{A} \mid \mathrm{H}^1(A, \mathscr{I}_Z \otimes \mathscr{P}_x(C)) \neq 0\} \tag{3.1.4}$$

is the same curve as defined in equation (3.1.3).

More precisely, we prove that, for generic $Z \in K^nA$, the sheaf $\mathscr{I}_Z(C)$ satisfies WIT_1. The fibres of $\widehat{\mathscr{I}_Z(C)}$ are precisely the vector spaces

$$\widehat{\mathscr{I}_Z(C)} \otimes k(x) \cong \mathrm{H}^1(A, \mathscr{I}_Z \otimes \mathscr{P}_x(C))$$

so D_Z is the support of $\widehat{\mathscr{I}_Z(C)}$. Sending Z to the Fourier-Mukai transform $\widehat{\mathscr{I}_Z(C)}$ defines a birational map

$$\phi\colon K^nA \overset{\sim}{\dashrightarrow} K_{\widehat{A}}(0, \widehat{C}, -1) \tag{3.1.5}$$

where the target space is a Bogomolov irreducible factor of the moduli space $M_{\widehat{A}}(0, \widehat{C}, -1)$; see Appendix B. A generic element of $K_{\widehat{A}}(0, \widehat{C}, -1)$ is an invertible sheaf on a curve in the linear system $|\widehat{C}|$, and there is a natural map

$$f'\colon K_{\widehat{A}}(0, \widehat{C}, -1) \to |\widehat{C}| \tag{3.1.6}$$

essentially given by sending a sheaf to its support. We recover our map f as the composition of f' and ϕ. Finally, we show that f is a rational fibration by verifying that a generic fibre of f' is connected. By the assumption on the linear system $|\widehat{C}|$, this amounts to checking that the fibre of f' over a nonsingular, irreducible curve in $|\widehat{C}|$ is connected.

3.2. Construction

It turns out to be convenient to extend the setup as follows: We begin in Section 3.2.1 by identifying $A^{[n]} \times \widehat{A} \cong M_A(1,C,0)$ in such a way that the Kummer variety is recovered as the fibres of the map

$$\alpha\colon M_A(1,C,0) \to A \times \widehat{A}$$

of Yoshioka, described in Appendix B.

Then, in the remaining sections, we will construct a commutative diagram

$$\begin{array}{ccccc} M_A(1,C,0) & \overset{\psi}{\dashrightarrow} & M_{\widehat{A}}(0,\widehat{C},-1) & \xrightarrow{F} & P \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow \\ A \times \widehat{A} & \overset{\eta}{\xrightarrow{\sim}} & \widehat{A} \times A & \xrightarrow{q} & A \end{array} \qquad (3.2.1)$$

where ψ is a birational map induced by the Fourier-Mukai transform, η is an isomorphism, q is second projection and $P \to A$ is a projective bundle with the complete linear system $|\mathscr{P}_a(\widehat{C})|$ as the fibre over a. Choosing compatible base points for the varieties in the lower row, and restricting the upper row to the respective fibres, we recover the maps (3.1.5) and (3.1.6).

3.2.1. Rank one sheaves and the Hilbert scheme. As usual, the Hilbert scheme $A^{[n]}$ can be regarded as a moduli space of rank one sheaves on A. More precisely, there is an isomorphism

$$A^{[n]} \times \widehat{A} \cong M_A(1,0,-n) \qquad (3.2.2)$$

which, on the level of sets, is given by the map

$$(Z,x) \mapsto \mathscr{I}_Z \otimes \mathscr{P}_x.$$

By twisting with C, we get a canoncial isomorphism from $M_A(1,0,-n)$ to $M_A(1,C,0)$. Including the isomorphism (3.2.2), we can thus identify

$$A^{[n]} \times \widehat{A} \cong M_A(1,C,0).$$

We want to describe the composition

$$A^{[n]} \times \widehat{A} \cong M_A(1,C,0) \xrightarrow{\alpha} A \times \widehat{A}$$

where α is the map of Yoshioka. Recall that, to define α, we must choose representatives $\mathscr{L} \in \mathrm{Pic}(A)$ and $\mathscr{L}' \in \mathrm{Pic}(\widehat{A})$ for the class of C in $\mathrm{H}^2(A,\mathbb{Z})$ and the corresponding class in $\mathrm{H}^2(\widehat{A},\mathbb{Z})$ (via the canonical isomorphism (3.1.2)). With the natural choices

$$\mathscr{L} = \mathscr{O}_A(C) \qquad\qquad \mathscr{L}' = \mathscr{O}_{\widehat{A}}(\widehat{C}),$$

we have

$$\alpha = (\delta, \widehat{\delta}) \quad \text{where} \quad \begin{cases} \delta(\mathscr{E}) = \det(R\widehat{S}(\mathscr{E}))^{-1} \otimes \mathscr{O}_{\widehat{A}}(-\widehat{C}) \\ \widehat{\delta}(\mathscr{E}) = \det(\mathscr{E}) \otimes \mathscr{O}_A(-C). \end{cases}$$

LEMMA 3.6. *The diagram*

$$\begin{array}{ccc} A^{[n]} \times \widehat{A} & \cong & M_A(1,C,0) \\ \Big\downarrow{\scriptstyle \sigma \times 1_{\widehat{A}}} & & \Big\downarrow{\scriptstyle \alpha} \\ A \times \widehat{A} & \xrightarrow{\theta} & A \times \widehat{A} \end{array}$$

is commutative, where θ is the automorphism

$$\theta(a,x) = (a + \phi_{\widehat{C}}(x), x)$$

and σ is the addition map. In particular, the fibres K^nA on the left are taken isomorphically to the fibres $K_A(1,C,0)$ on the right.

PROOF. Let us, for the sake of readability, use additive notation in the Picard groups. Firstly, we have

$$\widehat{\delta}(\mathscr{I}_Z \otimes \mathscr{P}_x(C)) = \det(\mathscr{I}_Z \otimes \mathscr{P}_x(C)) + \mathscr{O}_A(-C) = \mathscr{P}_x.$$

Secondly, applying the Fourier-Mukai functor to the short exact sequence

$$0 \to \mathscr{I}_Z \otimes \mathscr{P}_x(C) \to \mathscr{P}_x(C) \to \mathscr{O}_Z \to 0 \tag{3.2.3}$$

we obtain an exact sequence

$$0 \to \widehat{S}(\mathscr{I}_Z \otimes \mathscr{P}_x(C)) \to \widehat{S}(\mathscr{P}_x(C)) \to \widehat{S}(\mathscr{O}_Z) \to R^1\widehat{S}(\mathscr{I}_Z \otimes \mathscr{P}_x(C)) \to 0,$$

since $\mathscr{P}_x(C)$ satisfies IT_0, by Mumford's index theorem (see Section 3.1.2). Thus we have

$$\delta(\mathscr{I}_Z \otimes \mathscr{P}_x(C)) = -\det\widehat{S}(\mathscr{P}_x(C)) + \det\widehat{S}(\mathscr{O}_Z) + \mathscr{O}_{\widehat{A}}(-\widehat{C}).$$

By direct computation, we find that the Fourier-Mukai transform $\widehat{S}(\mathscr{O}_Z)$ is the direct sum $\bigoplus_{a \in Z} \mathscr{P}_a$, where the points $a \in Z$ should be repeated according to their multiplicity. Hence

$$\det\widehat{S}(\mathscr{O}_Z) = \mathscr{P}_{\sigma(Z)}.$$

Furthermore, by Proposition A.10, tensoring with $\mathscr{P}_x$ before applying $\widehat{S}$ is the same thing as translating with x after applying $\widehat{S}$, and hence

$$\det\widehat{S}(\mathscr{P}_x(C)) = \mathscr{O}_A(-T_x^*\widehat{C})$$

by definition of $\widehat{C}$. Thus

$$\delta(\mathscr{I}_Z \otimes \mathscr{P}_{\sigma(Z)}(C)) = \mathscr{P}_{\sigma(Z)} + \mathscr{O}_A(T_x^*\widehat{C} - \widehat{C}).$$

More concisely, we may write this as

$$\alpha(\mathscr{I}_Z \otimes \mathscr{P}_x(C)) = (\sigma(Z) + \phi_{\widehat{C}}(x), x),$$

which is what we wanted to prove. □

3.2.2. The weak index property.

PROPOSITION 3.7. *Let $Z \in A^{[n]}$. The sheaf $\mathscr{I}_Z(C)$ fails WIT if and only if every translate of Z is contained in a divisor in the linear system $|C|$.*

PROOF. First of all, we have

$$\mathrm{H}^2(A, \mathscr{I}_Z(C) \otimes \mathscr{P}_x) = 0$$

for every $x \in \widehat{A}$, for instance by the short exact sequence (3.2.3). Thus, by the base change theorem in cohomology, we always have $R^2\widehat{S}(\mathscr{I}_Z(C)) = 0$, so WITness is equivalent to the vanishing of $\widehat{S}(\mathscr{I}_Z(C))$. By base change again, this last sheaf vanishes exactly when

$$\mathrm{H}^0(A, \mathscr{I}_Z(C) \otimes \mathscr{P}_x) = 0$$

for *some* $x \in \widehat{A}$, since $\widehat{S}(\mathscr{I}_Z(C))$ is torsion free. This says that $\mathscr{I}_Z(C)$ fails WIT if and only if there exists a divisor in $|\mathscr{P}_x(C)|$ containing Z, for every $x \in \widehat{A}$. Since C is ample, the map $\phi_C \colon A \to \widehat{A}$ is surjective. Hence, the collection of linear systems $|\mathscr{P}_x(C)|$, for varying $x \in \widehat{A}$, coincides with the collection of linear systems $|T_a^* C|$, for varying $a \in A$. This shows that $\mathscr{I}_Z(C)$ fails WIT if and only if Z is contained in a divisor in every translated linear system $|T_a^* C|$, which is clearly equivalent to the claim. □

LEMMA 3.8. *The (open) locus of sheaves $\mathscr{E} \in M_A(1, C, 0)$ satisfying WIT with index 1 is non empty. In fact, there exist WIT-sheaves in every fibre $K_A(1, C, 0)$ of α.*

PROOF. By Lemma 3.5, a generic element $Z \in A^{[n]}$ is not contained in any curve in $|C|$. This is a stronger statement than needed in Proposition 3.7, to conclude that $\mathscr{E} = \mathscr{I}_Z(C)$ satisfies WIT.

To prove the second statement, note that the sheaves obtained from a WIT-sheaf $\mathscr{E}$, by translating and twisting with a homogeneous invertible sheaf

$$\mathscr{E} \mapsto T_a^* \mathscr{E}, \quad \mathscr{E} \mapsto \mathscr{E} \otimes \mathscr{P}_x,$$

again satisfy WIT, by Proposition A.10. A straight forward calculation shows that

$$\alpha(\mathscr{P}_x \otimes T_a^* \mathscr{E}) = \alpha(\mathscr{E}) + (\phi_{\widehat{C}}(x), \phi_C(a) + x).$$

so that we can move $\mathscr{E}$ to any fibre of α by translating and twisting. This proves the lemma. □

3.2.3. Stability. If $\mathscr{E} \in M_A(1, C, 0)$ satisfies WIT with index 1, then the invariants of its Fourier-Mukai transform are

$$r(\widehat{\mathscr{E}}) = 0 \qquad c_1(\widehat{\mathscr{E}}) = \widehat{C} \qquad \chi(\widehat{\mathscr{E}}) = -1 \tag{3.2.4}$$

by Mukai's Theorem A.14. Thus, provided $\widehat{\mathscr{E}}$ is stable, it defines a point in $M_{\widehat{A}}(0,\widehat{C},-1)$. We next show that $\widehat{\mathscr{E}}$ is indeed always stable. Note that a semi-stable sheaf with the invariants (3.2.4) is automatically stable, as can be seen for instance by applying the stability criterion from Section B.1.

LEMMA 3.9. *Let $\mathscr{E}$ be a sheaf in $M_A(1,C,0)$ satisfying WIT with index* 1. *Then the Fourier-Mukai transform $\widehat{\mathscr{E}}$ is stable with respect to any polarization of $\widehat{A}$.*

PROOF. We first show that $\widehat{\mathscr{E}}$ is pure. Being the Fourier-Mukai transform of a WIT-sheaf with index 1, $\widehat{\mathscr{E}}$ itself satisfies WIT with index 1, by Corollary A.7. Since it has rank zero, to prove it is purely one-dimensional, it suffices to show that it contains no zero-dimensional subsheaf. Now, if $\mathscr{T}\subset\widehat{\mathscr{E}}$ were a zero-dimensional subsheaf, then $\mathscr{T}$ would satisfy WIT with index 0, but

$$S(\mathscr{T})\subseteq S(\widehat{\mathscr{E}})=0$$

and hence $\mathscr{T}=0$.

Next, suppose $\mathscr{F}\subset\widehat{\mathscr{E}}$ were a destabilizing subsheaf. Then $S(\mathscr{F})=0$ for the same reason as above, and $R^2S(\mathscr{F})=0$ since $\mathscr{F}$ is one-dimensional. Thus $\mathscr{F}$ also satisfies WIT with index 1.

As $\widehat{\mathscr{E}}/\mathscr{F}$ is torsion, its degree is nonnegative, so we have

$$\deg(\mathscr{F})\leq\deg(\widehat{\mathscr{E}})$$

with respect to any polarization. Now we apply the stability criterion from Section B.1: Since $\mathscr{F}$ is destabilizing, we have

$$\frac{\chi(\mathscr{F})}{\deg(\mathscr{F})}>\frac{\chi(\widehat{\mathscr{E}})}{\deg(\widehat{\mathscr{E}})}$$

and thus

$$\chi(\mathscr{F})>\chi(\widehat{\mathscr{E}})=-1.$$

Since the Fourier-Mukai transform $\widehat{\mathscr{F}}$ has rank $-\chi(\mathscr{F})<1$ by Theorem A.14, it must be a torsion sheaf. Now, applying the Fourier-Mukai functor to the exact sequence

$$0\to\mathscr{F}\to\widehat{\mathscr{E}}\to\widehat{\mathscr{E}}/\mathscr{F}\to 0$$

we obtain a left exact sequence

$$0\to S(\widehat{\mathscr{E}}/\mathscr{F})\to\widehat{\mathscr{F}}\to\widehat{\widehat{\mathscr{E}}}\cong(-1)^*\mathscr{E}$$

where Corollary A.7 is applied to obtain the isomorphism on the right. But both $S(\widehat{\mathscr{E}}/\mathscr{F})$ and $(-1)^*\mathscr{E}$ are torsion free, hence it is impossible for the middle term $\widehat{\mathscr{F}}$ to be torsion. Thus we have reached a contradiction. □

We are now ready to construct the leftmost square in diagram (3.2.1): Let $U\subset M_A(1,C,0)$ denote the set of sheaves satisfying WIT with index 1. Then U is open by Theorem A.8, and by Lemma 3.8, it is nonempty. Let $\mathscr{U}$ denote the restriction of the universal family on $M_A(1,C,0)$ to U. Applying Theorem A.8 again, we find that $\mathscr{U}$ satisfies WIT with index 1, and its Fourier-Mukai transform $\widehat{\mathscr{U}}$ is a flat family of sheaves on $\widehat{A}$ parametrized by U. The fibres of $\widehat{\mathscr{U}}$ are stable by Lemma 3.9, so there is an induced rational map

$$\psi\colon M_A(1,C,0) \dashrightarrow M_{\widehat{A}}(0,\widehat{C},-1)$$

which is regular on U. In fact, by theorem A.2, the restriction of ψ to U is an open immersion. As $M_{\widehat{A}}(0,\widehat{C},-1)$ is irreducible, by Theorem B.1, the map ψ is birational. Let us verify that ψ fits into diagram (3.2.1), i.e. we check the commutativity of the leftmost square. So let $\mathscr{E}$ be a sheaf in $M_A(1,C,0)$ satisfying WIT with index 1. Then

$$\delta(\mathscr{E}) = \det(\widehat{\mathscr{E}}) \otimes \mathscr{O}_{\widehat{A}}(-\widehat{C})$$
$$\widehat{\delta}(\mathscr{E}) = \det(\mathscr{E}) \otimes \mathscr{O}_A(-C)$$

whereas

$$\delta(\widehat{\mathscr{E}}) = \det(\widehat{\widehat{\mathscr{E}}}) \otimes \mathscr{O}_A(-C) = (-1)_A^* \det(\mathscr{E}) \otimes \mathscr{O}_A(-C)$$
$$\widehat{\delta}(\widehat{\mathscr{E}}) = \det(\widehat{\mathscr{E}}) \otimes \mathscr{O}_{\widehat{A}}(-\widehat{C}).$$

Thus we see that, if we define the map η in diagram (3.2.1) by

$$\eta(a,x) = (-x,a) + ((-1)^*C - C, 0),$$

then the left square in that diagram commutes. Since, by Lemma 3.8, no fibre $K_A(1,C,0)$ of α is contained in the base locus of ψ, we conclude that ψ restricts to a birational equivalence

$$\phi\colon K_A(1,C,0) \overset{\sim}{\dashrightarrow} K_{\widehat{A}}(0,\widehat{C},-1).$$

3.2.4. The fibration. Let $\mathscr{G}$ be the Fourier-Mukai transform of $\mathscr{O}_{\widehat{A}}(\widehat{C})$. By the base change theorem in cohomology, the fibre of $\mathscr{G}$ over $a \in A$ is canonically isomorphic to $\mathrm{H}^0(\widehat{A}, \mathscr{P}_a(\widehat{C}))$. Thus, the associated projective bundle

$$P = \mathbb{P}\left(\mathscr{G}^\vee\right) \to A \tag{3.2.5}$$

has the complete linear systems associated to $\mathscr{P}_a(\widehat{C})$ as fibres.

The Fitting ideal of a sheaf $\mathscr{F}$ in $M_{\widehat{A}}(0,\widehat{C},-1)$ defines a curve representing the first Chern class of $\mathscr{F}$, and hence a point in the bundle P. The map of sets

$$F\colon M_{\widehat{A}}(0,\widehat{C},-1) \to P \tag{3.2.6}$$

thus obtained is a (regular) map of varieties, since formation of the Fitting ideal commutes with base change.

We remark that Mumford [**29**, §5.3] has constructed an effective Cartier divisor $\mathrm{Div}(\mathscr{F})$ representing the first Chern class of a torsion sheaf $\mathscr{F}$ in greater generality. On a nonsingular surface the ideal defining $\mathrm{Div}(\mathscr{F})$ is precisely the Fitting ideal of $\mathscr{F}$.

Clearly, F fits into diagram (3.2.1), making its rightmost square commute. Thus, restricting F to the fibre $K_{\widehat{A}}(0,\widehat{C},-1)$ above zero in $\widehat{A}\times A$ yields a map

$$f\colon K_{\widehat{A}}(0,\widehat{C},-1)\to|\widehat{C}|. \tag{3.2.7}$$

It remains only to prove that f is a fibration, i.e. a generic fibre is connected. To show this, we need the following fact, which seems to be standard, but we include a proof.

PROPOSITION 3.10. *Let C be a nonsingular ample curve on an abelian surface A. Then the restriction map $\widehat{A}\to \mathrm{Jac}(C)$ is an embedding.*

PROOF. As the restriction map is a homomorphism of abelian groups, i.e. it is both regular and a group homomorphism, it suffices to check injectivity.

Let $x\in\widehat{A}$ and consider the exact sequence

$$0\to\mathscr{P}_x(-C)\to\mathscr{P}_x\to\mathscr{P}_x|_C\to 0.$$

As C is ample, the cohomology spaces $\mathrm{H}^i(A,\mathscr{P}_x(-C))$ vanish for $i<2$, and hence the induced long exact sequence in cohomology gives an isomorphism

$$\mathrm{H}^0(A,\mathscr{P}_x)\xrightarrow{\sim}\mathrm{H}^0(C,\mathscr{P}_x|_C).$$

Now apply the fact [**28**, §8] that $\mathscr{P}_x$ admits a global section if and only if $x=0$. It follows that the restriction map is injective. □

LEMMA 3.11. *Let $D\subset\widehat{A}$ be a nonsingular irreducible curve in the linear system $|\widehat{C}|$. Then the fibre $f^{-1}(D)$ of the map* (3.2.7) *can be identified with a fibre of the canonical map*

$$\mathrm{Jac}(D)\to\widehat{A}$$

induced by the Albanese property of the Jacobian $\mathrm{Jac}(D)$ *of D. In particular, the fibre $f^{-1}(D)$ is connected.*

PROOF. Viewing D as a point in the projective bundle P of equation (3.2.5), the fibre $F^{-1}(D)$ of the map (3.2.6) is just the Jacobian of D, parametrizing invertible sheaves of degree $n-1$, which we may identify with the Jacobian of degree 0 invertible sheaves. We claim there exists a commutative diagram

$$\begin{array}{ccc} F^{-1}(D) & \xrightarrow{\alpha} & \widehat{A}\\ \wr\| & & \wr\| \\ \mathrm{Jac}(D) & \longrightarrow & \widehat{A}\end{array}$$

where the automorphism of $\widehat{A}$ on the right is translation by a fixed point in $\widehat{A}$, and the map at the bottom is the canonical map in the lemma, which sends a divisor $\sum n_i x_i$ on D to the point $\sum n_i x_i\in\widehat{A}$, where the last sum denotes the group law on $\widehat{A}$.

To show that we have such a commutative diagram, we must verify that, if $\mathscr{L}$ corresponds to the divisor $\sum n_i x_i$, then

$$\det(RS(\mathscr{L})) \otimes \mathscr{O}_A(-C) = \sum n_i x_i, \tag{3.2.8}$$

up to translation by a fixed point in $\widehat{A}$.

To prove equation (3.2.8), note that the composite functor $\det(RS(-))$ is additive over short exact sequences. Thus, if $x \in D$ is a point and $\mathscr{L}$ is an invertible sheaf on D, the short exact sequence

$$0 \to \mathscr{L} \to \mathscr{L}(x) \to k(x) \to 0$$

gives

$$\det(RS(\mathscr{L}(x))) = \det(RS(\mathscr{L})) + x$$

since $RS(k(x)) = \mathscr{P}_x$ by Example A.13. Equation (3.2.8) follows.

It remains to check that the fibres of the map in the lemma are connected. A map of abelian varieties has connected fibres if and only if the dual map has. The dual map is the canonical map

$$A \to \mathrm{Jac}(D)$$

obtained by viewing A and $\mathrm{Jac}(D)$ as the Picard varieties of $\widehat{A}$ and D, respectively. Recall that, on any abelian variety, an effective divisor is ample if and only if it has positive self intersection [**28**, §8]. Thus we can apply Proposition 3.10 to conclude that the last map is injective. The lemma is established. □

The lemma concludes the proof of Theorem 3.1.

3.3. Comments

3.3.1. Non-principal polarizations. Assume A has Picard number 1, and let H again be the ample generator of $\mathrm{NS}(A)$. Let

$$d = H^2/2$$

which is greater than 1 if H is non-principal. In this situation, Theorem 3.1 only gives partial results. Namely, if we try to copy the argument in Proposition 1.18, we see that, if K^nA admits a rational fibration, and hence carries a divisor $D = k\widetilde{H} + l\varepsilon$ with $q(D) = 0$, then

$$k^2 d = l^2 n,$$

but here we are stuck. However, if we also assume that d divides n, then we can conclude that n/d is a perfect square, and $C = \sqrt{n/d}H$ has self intersection $2n$. Thus, the generalization of Corollary 3.3 we can prove is the following:

COROLLARY 3.12 (of Theorem 3.1). *If $d = H^2/2$ divides n then the following are equivalent.*

(1) *K^nA admits a rational fibration.*

(2) *K^nA carries a divisor D with vanishing Beauville-Bogomolov square.*
(3) *n/d is a perfect square.*

On the other hand, we cannot use Theorem 3.1 to construct fibrations on K^nA when n is not a multiple of d, even though K^nA may very well carry a divisor D with $q(D) = 0$. For instance, if $d = 2n$, then the divisor $\widetilde{H} - 2\varepsilon$ on K^nA satisfies

$$q(\widetilde{H} - 2\varepsilon) = 2d - 4n = 0,$$

but A carries no curve with self intersection $2n$.

We remark that Sawon [**35**] and Markushevich [**20**] have studied fibrations on Hilbert schemes on K3 surfaces, and they verified Conjecture 1.13 for *any* generic K3 surface. Their construction is similar to ours, but in place of our moduli space $M_{\widehat{A}}(0, \widehat{C}, -1)$, they utilize a certain moduli space of *twisted sheaves*. It might be possible to overcome the difficulties in the non-principally polarized case by adapting their construction to Kummer varieties.

Here is, however, a positive result: If the polarization H has type $(1,d)$, then we can apply Theorem 3.1 with $C = H$ to obtain a rational fibration on K^dA. The primitivity of H implies that the birational map ψ in diagram (3.2.1) is in fact an isomorphism: This is a special case of a result of Yoshioka [**38**, Prop. 3.5]. I thank Sawon for pointing this out to me. We conclude that the fundamental fibration on K^dA is regular.

In contrast to this, the next chapter provides an example were the fundamental fibration has nonempty base locus.

3.3.2. Fundamental fibrations in the literature. When A is a generic abelian surface with a polarization of type $(1,d)$, the fundamental fibration on K^dA has appeared a couple of places in the literature.

Debarre [**6**, §1] utilized a fibration on a certain relative Jacobian in order to compute the Euler characteristic of K^dA. In our language, this corresponds to the (regular) fibration (3.2.7) on $K_{\widehat{A}}(0, \widehat{H}, -1)$. As mentioned above, this variety is isomorphic to K^dA by Yoshioka's results, but Debarre's work predates Yoshioka's. Debarre instead related $K_A(0, H, -2)$ to K^dA through a series of deformations and birational constructions.

Furthermore, Sawon has informed me that Braden and Hollowood [**3**] have described the fundamental fibration on K^dA in the language of mathematical physics. In response, Sawon wrote a private letter to Braden, where he used the result of Yoshioka, as mentioned above, to get the fundamental fibration on K^dA.

CHAPTER 4

The Kummer variety of four points

In this chapter, we will make a detailed study of the fundamental fibration on the six-dimensional Kummer variety K^4A, in the case where the abelian surface A has Picard number 1 and admits a principal polarization. One motivation for such a study is to demonstrate that the fundamental fibration can have base points.

To understand the fundamental fibration, we need to understand the birational map

$$\phi\colon K^4A \overset{\sim}{\dashrightarrow} K_{\widehat{A}}(0,\widehat{C},-1)$$

constructed in Chapter 3. Birational maps between symplectic varieties are of interest on their own, as we indicated in Section 1.2.2. Our study of ϕ suggests rather strongly that ϕ is a *Mukai elementary transform*. In particular, we find that the base locus of ϕ is a codimension 2 subvariety $Q \subset K^4A$, having the structure of a $\mathbb{P}^2$-bundle over A. Even though we are not able to prove that ϕ is a Mukai elementary transform, that assumption — or belief — sets out the direction for our study of ϕ.

From our description of ϕ it follows, that the fundamental fibration

$$K^4A \dashrightarrow \mathbb{P}^3$$

also has nontrivial base locus Q. From this we deduce that the two varieties K^4A and $K_{\widehat{A}}(0,\widehat{C},-1)$ cannot be isomorphic at all, since only the latter admits a *regular* fibration. This provides an example, and I believe it is the first known, of two projective irreducible symplectic varieties that are birational, but not isomorphic. A nonprojective, compact Kähler example of such a pair has been described by Debarre [**5**].

This chapter is organized as follows: In Section 4.1, we formulate our description of the birational map ϕ, or rather the map ψ constructed in Section 3.2, from which ϕ is obtained by restriction. In Section 4.2, we recall the construction of Mukai's elementary transform. Sections 4.3, 4.4 and 4.5 contain the hard work, where we prove the statements from Section 4.1. Finally, in Section 4.6, we use our results to find the base loci of ψ, ϕ and the fundamental fibration.

4.1. Notation and statements

Throughout this chapter, we fix an abelian surface A with Picard number 1, admitting a principal polarization H. Define $C = 2H$ and

$$M = M_A(1,C,0) \qquad\qquad M' = M_{\widehat{A}}(0,\widehat{C},-1).$$

We will tend to denote sheaves in M and M' by $\mathscr{E}$ and $\mathscr{F}$, respectively.

We will analyse the birational map

$$\psi\colon M \dashrightarrow M'$$

from Chapter 3 as follows: Firstly, we determine the base loci $P \subset M$ and $P' \subset M'$ of ψ and ψ^{-1}, respectively. Secondly, given a curve (the assumptions will be made precise in a moment)

$$\gamma\colon T \to M$$

intersecting P transversally in a point $0 \in T$, we will describe the corresponding point

$$\lim_{t\to 0}(\psi\circ\gamma)(t) \in P' \tag{4.1.1}$$

in terms of $\gamma(0) \in P$ and the tangent vector $\gamma'(0)$.

The corresponding results for the birational map $\phi\colon K^4A \dashrightarrow K_{\widehat{A}}(0,\widehat{C},-1)$ follow by restriction, as we will see in Section 4.6.2

We now give the exact statements, deferring the proofs to the sections that follow.

4.1.1. The base locus of ψ. Every WIT-sheaf in M has stable Fourier-Mukai transform, but not every sheaf satisfies WIT. The base locus $P \subset M$ of ψ equals the set of non-WIT sheaves in M. This locus has the structure of a $\mathbb{P}^2$-bundle over $\widehat{A}\times A\times\widehat{A}$. More precisely, there is a canonical isomorphism

$$P \cong P_0 \times A \times \widehat{A}$$

where P_0 is a $\mathbb{P}^2$-bundle over $\widehat{A}$, whose fibre over a point $x \in \widehat{A}$ is the projective space of lines in

$$\operatorname{Ext}^1_A(\mathscr{P}_x|_H, \mathscr{O}_A(H)). \tag{4.1.2}$$

The sheaf in M corresponding to a triple $(\xi, a, y) \in P$, where ξ is an extension

$$\xi\colon 0 \to \mathscr{O}_A(H) \to \mathscr{E} \to \mathscr{P}_x|_H \to 0, \tag{4.1.3}$$

is obtained from $\mathscr{E}$ by translating with a and twisting with $\mathscr{P}_y$.

4.1.2. The base locus of ψ^{-1}. Every sheaf in M' satisfies WIT, but not every sheaf has stable (i.e. torsion free) Fourier-Mukai transform. The base locus $P' \subset M'$ of ψ^{-1} equals the set of sheaves in M' having Fourier-Mukai transforms with torsion. This locus has the structure of a $\mathbb{P}^2$-bundle over $\widehat{A}\times$

$A \times \widehat{A}$. More precisely, there is a canonical isomorphism

$$P' \cong P'_0 \times A \times \widehat{A}$$

where P'_0 is a $\mathbb{P}^2$-bundle over $\widehat{A}$, whose fibre over a point $x \in \widehat{A}$ is the projective space of lines in

$$\operatorname{Ext}^1_A(\mathscr{O}_A(H), \mathscr{P}_x|_H). \tag{4.1.4}$$

The sheaf in M' corresponding to a triple $(\zeta, a, y) \in P'$, where ζ is an extension

$$\zeta : 0 \to \mathscr{P}_x|_H \to \mathscr{G} \to \mathscr{O}_A(H) \to 0, \tag{4.1.5}$$

is obtained from the Fourier-Mukai transform $\widehat{\mathscr{G}}$ by translating with y and twisting with $\mathscr{P}_a$.

4.1.3. Duality. Since the canonical sheaf on A is trivial, the two Ext-spaces (4.1.2) and (4.1.4) are Serre dual. The dualities between the fibres of P and P' fit together to form an isomorphism $P' \cong \check{P}$, where

$$\check{P} \to \widehat{A} \times A \times \widehat{A}$$

is the dual projective bundle of P.

4.1.4. Limits. Let $T = \operatorname{Spec} R$ be the spectrum of a discrete valuation ring over $\mathbb{C}$, with $0 \in T$ denoting the unique closed point. Let

$$\gamma \colon T \to M$$

be a map such that $\gamma(0) \in P$ and such that (the image of) T intersects P transversally. There is then a unique

$$\lambda : T \to M'$$

coinciding with γ on $M \setminus P \cong M' \setminus P'$, and $\lambda(0) \in P'$ is the point we denoted by the limit (4.1.1) above. This point is determined by $\gamma(0)$ and the tangent vector $\gamma'(0)$ as follows:

Let $\gamma(0) \in P$ correspond to the triple

$$(\xi, a, y) \in P_0 \times A \times \widehat{A},$$

where ξ is an extension of the form (4.1.3). It is convenient to fix such an extension, which is determined only up to multiplication by a nonzero scalar. The two maps $\mathscr{O}_A(H) \to \mathscr{E}$ and $\mathscr{E} \to \mathscr{P}_x|_H$ in ξ induce a map

$$\operatorname{Ext}^1_A(\mathscr{E}, \mathscr{E}) \to \operatorname{Ext}^1_A(\mathscr{O}_A(H), \mathscr{P}_x|_H). \tag{4.1.6}$$

We identify the domain $\operatorname{Ext}^1_A(\mathscr{E}, \mathscr{E})$ with the tangent space to M at $\gamma(0)$. Then the image of $\gamma'(0)$ under the map (4.1.6) is an extension ζ of the form (4.1.5), and $\lambda(0) \in P'$ is the triple

$$(\zeta, -a, y) \in P'_0 \times A \times \widehat{A}.$$

Moreover, there is a canonical (up to the choice of ξ) embedding of the normal space $N_{P/M}(\gamma(0))$ into $\operatorname{Ext}^1_A(\mathscr{O}_A(H), \mathscr{P}_x|_H)$. Via this identification, the projection

$$T_M(\gamma(0)) \twoheadrightarrow N_{P/M}(\gamma(0))$$

is just the map (4.1.6).

Although not entirely precise, one might say that the point $\gamma(0)$ and its tangent $\gamma'(0)$ (or rather its class in the normal bundle) are swapped, and becomes the tangent $\lambda'(0)$ and the point $\lambda(0)$, in that order, when passing from M to M'.

4.2. Mukai elementary transforms

Let X be a symplectic variety, with symplectic structure σ. Assume X contains a nonsingular variety $P \subset X$ having the structure of a projective bundle $\pi\colon P \to Y$ over some variety Y. Furthermore, suppose the rank k of P, i.e. its relative dimension over Y, equals the codimension of P in X.

In this situation, the *Mukai elementary transform*

$$\phi : X \widetilde{\dashrightarrow} X' \tag{4.2.1}$$

associated to the pair (P,X) is a certain birational map ϕ to an algebraic space X'. Roughly speaking, the space X' is obtained from X by cutting away P and glueing in the dual projective bundle $\check{P}$ instead.

The Mukai elementary transform has the property that the symplectic form σ extends to a non-degenerate two-form on X'. Thus, if X' is a variety, it is symplectic.

4.2.1. Sketch of Mukai's construction. The replacement of $P \subset X$ with the dual bundle is realized by blowing up P and then blowing down the exceptional divisor E "in another direction". Thus the birational map (4.2.1) has a factorization

$$X \xleftarrow{\ f\ } \widetilde{X} \xrightarrow{\ f'\ } X' \tag{4.2.2}$$

where f and f' are blowups.

It is not hard to show that the normal bundle $N_{P/X}$ to P in X is canonically dual to the relative tangent bundle $T_{P/Y}$ of P over Y (note that, by our assumptions, they do have the same rank k). Under the identification $N_{P/X} \cong T_{P/Y}^\vee$, the dual of the canonical projection

$$T_X|_P \twoheadrightarrow N_{P/X} \cong T_{P/Y}^\vee \tag{4.2.3}$$

is just the inclusion

$$T_{P/Y} \hookrightarrow T_X|_P$$

where we have used the symplectic structure to identify the tangent bundle T_X with its dual $T_X^\vee$.

Let $f\colon \widetilde{X} \to X$ be the blowup of X with centre P. Using the isomorphism $N_{P/X} \cong T_{P/Y}^\vee$, we identify the exceptional divisor E with $\mathbb{P}(T_{P/Y})$. Then one further identifies $\mathbb{P}(T_{P/Y})$ with the incidence variety inside $P \times_Y \check{P}$. The symmetry might suggest the existence of a blowdown $f'\colon \widetilde{X} \to X'$, contracting E to $\check{P}$. Indeed, such a contraction exists in the category of algebraic spaces (or analytic spaces), by a general Castelnuovo type criterion, and thus we arrive at the diagram (4.2.2). We refer the reader to the original paper of Mukai [**26**] for the details, and for the existence of the two-form on X'.

4.2.2. A property of the Mukai elementary transform. A basic property of a blowup with nonsingular centre, is the fact that the exceptional divisor parametrizes the normal directions to the centre. Let us spell out what this means for the Mukai elementary transform $\phi : X \widetilde{\dashrightarrow} X'$.

As in Section 4.1.4, we consider a curve $\gamma\colon T \to X$, intersecting P transversally in $\gamma(0)$. We want to determine the corresponding point $\lim_{t\to 0}(\phi \circ \gamma)(t)$ in $\check{P} \subset X'$.

REMARK 4.1. Recall that, if Q denotes the projective space of lines in a vector space V, then the tangent space to Q at a line ℓ has the canonical description

$$T_Q(\ell) \cong \mathrm{Hom}_k(\ell, V/\ell).$$

As a consequence, we note that there is a canonical inclusion

$$\mathbb{P}(T_Q(\ell)) \subset \check{Q},$$

identifying the space of hyperplanes in $T_Q(\ell)$ with the space of hyperplanes in Q containing ℓ.

Let $P(y)$ be the fibre of $P \to Y$ above $y = \pi(\gamma(0))$. Letting $Q = P(y)$ and $\ell = \gamma(0)$ in the remark above, we find a canonical inclusion

$$\mathbb{P}(T_{P/X}(\gamma(0))) \subset \check{P}(y). \tag{4.2.4}$$

Under the canonical projection (4.2.3), the tangent vector $\gamma'(0) \in T_X(\gamma(0))$ is sent to a nonzero vector $v \in T^{\vee}_{P/X}(\gamma(0))$, by the transversality assumption. Thus v defines a hyperplane h in $T_{P/X}(\gamma(0))$. Via the embedding (4.2.4), we view h as a point in $\check{P}(y)$.

PROPOSITION 4.2. *Let X be a symplectic variety, containing a nonsingular $\mathbb{P}^k$-bundle $\pi\colon P \to Y$ of codimension k. Let*

$$\phi : X \dashrightarrow X'$$

be the Mukai elementary transform of X along P. Then

(1) *ϕ induces an isomorphism $X \setminus P \cong X' \setminus \check{P}$, and*
(2) *whenever $\gamma\colon T \to X$ is a curve intersecting P transversally in $\gamma(0)$, the point*

$$\lim_{t\to 0}(\phi \circ \gamma)(t) \in \check{P}(y).$$

is the hyperplane in $P(y)$ determined by $\gamma'(0)$ as above.

PROOF. Point (1) is obvious, and point (2) follows from the elementary property of the blowup $f\colon \widetilde{X} \to X$, that

$$\lim_{t\to 0}(f^{-1}\circ\gamma)(t) \in E = \mathbb{P}(N^\vee_{P/X})$$

corresponds to the line in $N_{P/X}(\gamma(0))$ spanned by the class of $\gamma'(0)$. □

Of course, we haven't done anything, just rewritten the usual property of a blowup with nonsingular centre. The importance is the following: The claim of Section 4.1.4 is that the birational map $\psi\colon M \overset{\sim}{\dashrightarrow} M'$ satisfies the *conclusion* of the proposition. The same conclusion applies to the restricted map $\phi\colon K^4A \overset{\sim}{\dashrightarrow} K_{\widehat{A}}(0,\widehat{C},-1)$, as we will see in Section 4.6.2. In this way, we bypass our inability in proving that ψ in fact *is* the Mukai elementary transformation of M along P, as the conclusion of the proposition is all we need to go on and determine the base locus of the fundamental fibration.

The author is convinced that ψ (and ϕ) is a Mukai elementary transform, and it seems even plausible that this can be deduced from the properties in the proposition. But I am not sure whether this can be done.

4.3. The non-WIT locus is a projective bundle

Let $P \subset M$ be the (closed) locus of sheaves failing WIT. Thus a sheaf $\mathscr{E} \in M$ is in P if and only if $\widehat{S}(\mathscr{E}) \neq 0$. By Lemma 3.9, the birational equivalence $\psi\colon M \overset{\sim}{\dashrightarrow} M'$ has base locus contained in P, and we will see, in Section 4.6, that the base locus in fact equals P. The purpose of this section is to give a precise description of P and, in particular, to show that it has the structure of a $\mathbb{P}^2$-bundle as claimed in Section 4.1.1. Note that this implies that the Mukai elementary transform of M in P makes sense.

LEMMA 4.3. *The locus $P \subset M$ is invariant under the natural action of $A \times \widehat{A}$.*

PROOF. A point in A acts on M by translation, whereas a homogeneous line bundle $\widehat{A} = \mathrm{Pic}^0(A)$ acts by twisting. By Proposition A.10 these operations preserve WITness. □

Recall that every sheaf in M can be written $\mathscr{I}_Z(C)\otimes\mathscr{P}_x$, where Z is a point in $Z \in A^{[4]}$ and x is a point in $\widehat{A}$. For the purpose of analysing WITness, we may, by the lemma, assume $x = 0$ and $Z \in K^4A$.

4.3.1. On a result by Maciocia. Let us begin with the observation that P is nonempty: Recall the condition in Proposition 3.7, that $\mathscr{I}_Z(C)$ fails WIT if and only if every translate of Z is contained in a curve in the linear system $|C|$. Now, if Z is a subscheme of H, then the translate $T_a^{-1}Z$ is contained in the divisor

$$D = T_a^{-1}H + T_{-a}^{-1}H$$

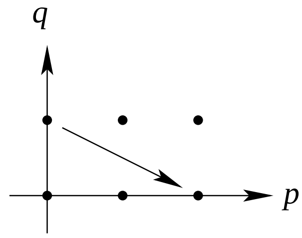

FIGURE 1. The sheet $E_2^{p,q} = R^pS(R^q\widehat{S}(\mathscr{E}))$.

which is linearly equivalent to $C = 2H$, by the theorem of the square. Thus, the sheaf $\mathscr{I}_Z(C)$ is in P whenever Z is a subscheme of H. By the same argument, or by an application of Lemma 4.3, the sheaf $\mathscr{I}_Z(C)$ is in P also when Z is a subscheme of a *translate* of H.

Under the assumption that A has Picard number 1, the converse implication also holds:

PROPOSITION 4.4 (Maciocia). *Let (A,H) denote a principally polarized abelian surface with Picard number 1, and let $Z \subset A$ be a subscheme of length 4. Then the following are equivalent:*

(1) *Z is contained in a translate $T_a^{-1}H$ of H*
(2) *$\mathscr{I}_Z(2H)$ does not satisfy WIT*

REMARK 4.5. This result has been announced by A. Maciocia [**19**, Proposition 3.2], but a proof has not been published. Maciocia has kindly sent the author unpublished notes with a sketch proof, with permission to include his result here. The idea in the following proof is due to Maciocia; the presentation is mine. I am responsible for any obfuscation of the argument.

REMARK 4.6. The author does not know whether the proposition holds without the assumption on the Picard number.

For the proof of the proposition, we will need the following:

LEMMA 4.7. *For every sheaf $\mathscr{E}$ in M there exists an exact sequence*

$$0 \to S(R^1\widehat{S}(\mathscr{E})) \to R^2S(\widehat{S}(\mathscr{E})) \to (-1)^*\mathscr{E} \to R^1S(R^1\widehat{S}(\mathscr{E}) \to 0 \qquad (4.3.1)$$

and all $R^pS(R^q\widehat{S}(\mathscr{E}))$ not occuring in this sequence vanish. Exchanging S and $\widehat{S}$, the analogous statement for a sheaf $\mathscr{F}$ in M' also holds.

PROOF. By Mukai's Theorem A.2, there exists a spectral sequence

$$E_2^{p,q} = R^pS(R^q\widehat{S}(\mathscr{E})) \Rightarrow \begin{cases} (-1)^*\mathscr{E} & \text{if } p+q=2 \\ 0 & \text{otherwise.} \end{cases}$$

Since $R^2\widehat{S}(\mathscr{E}) = 0$ for any sheaf $\mathscr{E}$ in M, the sheet $E_2^{p,q}$ of the spectral sequence has nonzero terms only in the rectangle $0 \le p \le 2$ and $0 \le q \le 1$, as illustrated in Figure 1. Thus there is only one nonzero differential, and the next sheet $E_3^{p,q}$ is the transfinite. The existence of the exact sequence (4.3.1) and the vanishing of $R^pS(\widehat{S}(\mathscr{E})$ for $p < 2$ follows when unrolling the definition of abutment of a spectral sequence.

For $\mathscr{F}$ in M' we again have $R^2S(\mathscr{F}) = 0$; hence the same argument applies. $\square$

The strategy for proving Proposition 4.4 is the following: Let Z be a subscheme of H and let $\mathscr{E} = \mathscr{I}_Z(C)$. Then it is quite easy to show that $R^2S(\widehat{S}(\mathscr{E}))$ is isomorphic to $\mathscr{O}_A(H)$. Using that H is symmetric, the middle map of sequence (4.3.1) then provides a section of $\mathscr{E}(-H) = \mathscr{I}_Z(H)$. As we will see below, this section is nonzero, and hence enables us to reconstruct the inclusion $Z \subset H$.

So the essential step in proving the proposition is to show that $R^2S(\widehat{S}(\mathscr{E}))$, if nonzero, has to be isomorphic to $\mathscr{O}_A(H)$, at least up to translation. This will follow once we can show that $\widehat{S}(\mathscr{E})$ is an invertible sheaf with first Chern class $-\widehat{H}$. The argument for this statement is based on the exact sequence

$$0 \to \widehat{S}(\mathscr{E}) \to \widehat{\mathscr{O}_A(C)} \to \widehat{\mathscr{O}_Z} \to R^1\widehat{S}(\mathscr{E}) \to 0 \tag{4.3.2}$$

induced by the short exact sequence

$$0 \to \mathscr{E} \to \mathscr{O}_A(C) \to \mathscr{O}_Z \to 0.$$

We note that $\widehat{\mathscr{O}_Z} \cong \oplus_{a \in Z} \mathscr{P}_a$, by direct computation (we observed this already in the proof of Lemma 3.6). In particular, the sheaf $\widehat{\mathscr{O}_Z}$ is poly-stable, i.e. a direct sum of stable sheaves with the same reduced Hilbert polynomial. Moreover, the sheaf $\widehat{\mathscr{O}_A(C)}$ is μ-stable: In fact, the Fourier-Mukai transform of any nondegenerate (i.e. having nonzero Euler characteristic) invertible sheaf is μ-stable [**33**, Proposition 3.2].

PROOF OF THE PROPOSITION. The implication $(1) \implies (2)$ was established in the introduction to this section. Conversely, assume (2) holds: Let

$$\mathscr{E} = \mathscr{I}_Z(C)$$

be a sheaf not satisfying WIT. In other words, assume $\widehat{S}(\mathscr{E})$ is nonzero.

In the following we will, using the assumption that the Picard number is 1, identify the Néron-Severi groups of A and $\widehat{A}$ with $\mathbb{Z}$, by sending the ample generators $H \in \mathrm{NS}(A)$ and $\widehat{H} \in \mathrm{NS}(\widehat{A})$ to $1 \in \mathbb{Z}$. In particular, we will view the first Chern class of a sheaf as an integer.

Step 1: $\widehat{S}(\mathscr{E})$ is locally free. By sequence (4.3.2), the sheaf $\widehat{S}(\mathscr{E})$ is the kernel of a map between locally free sheaves. Now, on a nonsingular surface, a sheaf is locally free if and only if it is reflexive. On the other hand, on any variety, the kernel of a homomorphism between reflexive sheaves is again reflexive. Thus we conclude that $\widehat{S}(\mathscr{E})$ is locally free.

Step 2: Bounding the first Chern class of $\widehat{S}(\mathscr{E})$. In sequence (4.3.2), the sheaves $\widehat{S}(\mathscr{E})$ and $R^1\widehat{S}(\mathscr{E})$ occur as sub- and quotient sheaves of $\widehat{\mathscr{O}_A(C)}$ and $\widehat{\mathscr{O}_Z}$, respectively. By Theorem A.14, the sheaf $\widehat{\mathscr{O}_A(C)}$ has rank 4 and first Chern class -2. Since it is μ-stable, we find

$$c_1(\widehat{S}(\mathscr{E})) \leq -1$$

and, *if we have equality*, then $\widehat{S}(\mathscr{E})$ has rank 1. Similarly, as $\widehat{\mathscr{O}_Z}$ is semi-stable and has vanishing first Chern class, the quotient sheaf $R^1\widehat{S}(\mathscr{E})$ has nonnegative first Chern class. Equivalently, by applying additivity of the first Chern class to the exact sequence (4.3.2), we have

$$c_1(\widehat{S}(\mathscr{E})) \geq -2.$$

Step 3: Ruling out $c_1 = -2$. This is the main difficulty. Assume for contradiction that the first Chern class of $\widehat{S}(\mathscr{E})$ is -2. By sequence (4.3.2), the first Chern class of $R^1\widehat{S}(\mathscr{E})$ is then zero.

Let $\mathscr{T}$ be the torsion subsheaf of $R^1\widehat{S}(\mathscr{E})$ and $\mathscr{F}$ the torsion free quotient:

$$0 \to \mathscr{T} \to R^1\widehat{S}(\mathscr{E}) \to \mathscr{F} \to 0 \tag{4.3.3}$$

Then, since $\mathscr{F}$ is a quotient of $\widehat{\mathscr{O}_Z} \cong \bigoplus_{a\in Z} \mathscr{P}_a$, the dual $\mathscr{F}^\vee$ is a subsheaf of $\bigoplus \mathscr{P}_a^\vee$. We claim that

$$c_1(\mathscr{F}^\vee) = 0 \quad \text{and} \quad \chi(\mathscr{F}^\vee) = 0. \tag{4.3.4}$$

Assuming this for a moment, we see that $\mathscr{F}^\vee$ and $\bigoplus \mathscr{P}_a^\vee$ have the same reduced Hilbert polynomials (see Section B.1). This implies, since $\bigoplus \mathscr{P}_a^\vee$ is poly-stable, that the inclusion $\mathscr{F}^\vee \subset \bigoplus \mathscr{P}_a^\vee$ splits. Dualizing again, and using that $\mathscr{F}^\vee$ and $\bigoplus \mathscr{P}_a^\vee$ are locally free, we find that the quotient $\bigoplus \mathscr{P}_a \twoheadrightarrow \mathscr{F}^{\vee\vee}$ splits. This implies that also

$$\widehat{\mathscr{O}_Z} = \bigoplus \mathscr{P}_a \twoheadrightarrow R^1\widehat{S}(\mathscr{E})$$

splits (and $R^1\widehat{S}(\mathscr{E}) \to \mathscr{F}^{\vee\vee}$ is an isomorphism), which leads to a contradiction as follows: If $R^1\widehat{S}(\mathscr{E})$ were a direct summand of $\bigoplus \mathscr{P}_a$, then it would satisfy WIT_2. Since $R^2S(R^1\widehat{S}(\mathscr{E})) = 0$ by Lemma 4.7, we would have $R^1\widehat{S}(\mathscr{E}) = 0$. But from the exact sequence (4.3.2) it follows that the rank of $R^1\widehat{S}(\mathscr{E})$ equals the rank of $\widehat{S}(\mathscr{E})$, which we assumed is nonzero. Thus we have reached a contradiction.

Let us prove the equalities (4.3.4). By sequence (4.3.3), the assumption $c_1(R^1\widehat{S}(\mathscr{E})) = 0$ implies

$$c_1(\mathscr{F}) \leq 0. \tag{4.3.5}$$

On the other hand, the sheaf $\mathscr{F}$ is a quotient of the semi-stable sheaf $\bigoplus \mathscr{P}_a$, which implies

$$c_1(\mathscr{F}) \geq 0 \quad \text{and, on equality,} \quad \chi(\mathscr{F}) \geq 0. \tag{4.3.6}$$

Similarly, since $\mathscr{F}^\vee$ is a subsheaf of $\bigoplus \mathscr{P}_a^\vee$, we have

$$c_1(\mathscr{F}^\vee) \leq 0 \quad \text{and, on equality,} \quad \chi(\mathscr{F}^\vee) \leq 0. \tag{4.3.7}$$

Now, since $\mathscr{F}$ embeds into its double dual $\mathscr{F}^{\vee\vee}$ such that the quotient $\mathscr{F}^{\vee\vee}/\mathscr{F}$ is finite, we have

$$c_1(\mathscr{F}) = c_1(\mathscr{F}^{\vee\vee}) \quad \text{and} \quad \chi(\mathscr{F}) \leq \chi(\mathscr{F}^{\vee\vee}). \tag{4.3.8}$$

Finally, since $\mathscr{F}^\vee$ is locally free,

$$c_1(\mathscr{F}^\vee) = -c_1(\mathscr{F}^{\vee\vee}) \quad \text{and} \quad \chi(\mathscr{F}^\vee) = \chi(\mathscr{F}^{\vee\vee}). \tag{4.3.9}$$

By (4.3.5) and (4.3.6) we have $c_1(\mathscr{F}) = 0$. Then using (4.3.9) and (4.3.8) we find

$$c_1(\mathscr{F}^\vee) = -c_1(\mathscr{F}^{\vee\vee}) = -c_1(\mathscr{F}) = 0.$$

This implies $\chi(\mathscr{F}^\vee) \leq 0$ by (4.3.7). Using (4.3.9), (4.3.8) and (4.3.6) we also have

$$\chi(\mathscr{F}^\vee) = \chi(\mathscr{F}^{\vee\vee}) \geq \chi(\mathscr{F}) \geq 0$$

and thus $\chi(\mathscr{F}^\vee) = 0$. We have established (4.3.4).

Step 4: Conclusion. We have shown that the first Chern class of $\widehat{S}(\mathscr{E})$ is $-\widehat{H}$, which, as we remarked under Step 2, also implies that it is an invertible sheaf. Thus there exists a point $a \in A$ such that $\widehat{S}(\mathscr{E})$ is isomorphic to $\mathscr{P}_a(-\widehat{H})$. Now note that both statements (1) and (2) are "invariant under translation", that is, each of them holds for Z if and only if it holds for a translate $T_a^{-1}Z$. Thus we are free to replace $\mathscr{E}$ by a translate, and hence, by Proposition A.10, we may assume that

$$\widehat{S}(\mathscr{E}) \cong \mathscr{O}_{\widehat{A}}(-\widehat{H}).$$

Then it follows, by definition of $\widehat{H}$, that $R^2S(\widehat{S}(\mathscr{E})) \cong \mathscr{O}_A(H)$. We conclude that the middle map of the exact sequence (4.3.1) provides a section of $\mathscr{E}(-H) \cong \mathscr{I}_Z(H)$. This section must be nonzero: Otherwise, the monomorphism on the left in sequence (4.3.1) would be an isomorphism $S(R^1\widehat{S}(\mathscr{E})) \cong R^2S(\widehat{S}(\mathscr{E}))$. This is a contradiction, since $S(R^1\widehat{S}(\mathscr{E}))$ is WIT_2, by Lemma 4.7, whereas $R^2S(\widehat{S}(\mathscr{E})) \cong \mathscr{O}_A(H)$ is WIT_0.

The existence of a nonzero section of $\mathscr{I}_Z(H)$, after translation, shows that Z is contained in a translate of H. □

4.3.2. The geometry of the non-WIT locus. Let $H^{(4)}$ denote the symmetric product, which we view as the Hilbert scheme parametrizing finite subschemes of H of length 4. By Maciocia's Proposition 4.4, the sheaves in M failing WIT are precisely the ones having the form

$$\mathscr{E} = T_a^*(\mathscr{I}_Z(C)) \otimes \mathscr{P}_x, \tag{4.3.10}$$

where $Z \in H^{(4)}$. Define

$$P = H^{(4)} \times A \times \widehat{A}.$$

We first show that P embeds into M as the non-WIT locus, and afterwards equip P with the $\mathbb{P}^2$-bundle structure announced in Section 4.1.1.

PROPOSITION 4.8. *The map $P \to M$, that sends $(Z,a,x) \in P$ to the sheaf* (4.3.10)*, is a closed immersion.*

PROOF. It is enough to verify that the translation map

$$f\colon H^{(4)} \times A \to A^{[4]}, \quad Z \mapsto T_a^{-1} Z$$

is a closed immersion. In fact, there is a commutative diagram

$$\begin{array}{ccc} H^{(4)} \times A \times \widehat{A} & \xrightarrow{f \times 1_{\widehat{A}}} & A^{[4]} \times \widehat{A} \\ \downarrow \wr & & \downarrow \wr \\ H^{(4)} \times A \times \widehat{A} & \longrightarrow & M, \end{array}$$

where the leftmost map sends (Z,a,x) to $(Z,a,x-\phi_C(a))$, and the bottom map is the one claimed to be a closed immersion.

We first show that f is a regular map, by writing down the corresponding natural transformation between functors of points. If T is a scheme, then a T-valued point of $H^{(4)} \times A$ is a pair (W,u), where $W \subset H \times T$ is a flat and finite family of degree 4 over T, and $u\colon T \to A$ is a morphism. As usual, we define "translation by u" to be the map

$$T_u\colon A \times T \to A \times T, \quad (a,t) \mapsto (a+u(t),t).$$

More precisely, we let $T_u = (m \circ (p, u \circ q), q)$, where m is the group law and p and q denote the projections. Considering W as a subscheme of $A \times T$, let Z be the inverse image of W by T_u. Then Z is a T-valued point of $A^{[4]}$, and sending (W,u) to Z defines f on T-valued points.

Secondly, we show that f is injective: If

$$T_{a_1}^{-1} Z_1 = T_{a_2}^{-1} Z_2$$

as subschemes of A, then Z_1 and Z_2 are both contained in $T_{a_1}^{-1} H \cap T_{a_2}^{-1} H$. In particular, that intersection contains at least 4 points (counting with multiplicities). But the intersection number is

$$T_{a_1}^{-1} H \cdot T_{a_2}^{-1} H = H^2 = 2$$

and H is irreducible, so we must have $T_{a_1}^{-1} H = T_{a_2}^{-1} H$. Now, since H is a principal polarization, the map

$$\phi_H\colon A \to \widehat{A}, \quad a \mapsto T_a^* H - H$$

is an isomorphism. In particular, the linear equivalence class of H, and *a fortiori* the curve H, moves under every nontrivial translation. So we must have $a_1 = a_2$, and hence also $Z_1 = Z_2$.

Thirdly, we show that f is immersive, i.e. its differential

$$df\colon T_{H^{(4)}} \times T_A \to T_{A^{[4]}}$$

is injective at every point $(Z_0, a) \in H^{(4)} \times A$. It is enough to check this for $a = 0$, as the general case can be reduced to this case by translating.

At least, the restriction of $df_{(Z_0,0)}$ to $T_{H^{(4)}}(Z_0) \times 0$ is injective: The statement is just that $H^{(4)} \to A^{[4]}$ is an immersion, which is true, since the induced map of tangent spaces can be canonically identified with the injection

$$\mathrm{Hom}_A(\mathscr{I}_{Z_0,H}, \mathscr{O}_{Z_0}) \hookrightarrow \mathrm{Hom}_A(\mathscr{I}_{Z_0,A}, \mathscr{O}_{Z_0})$$

induced by the surjection $\mathscr{I}_{Z_0,A} \twoheadrightarrow \mathscr{I}_{Z_0,H}$ from the ideal sheaf of Z_0 in A to the ideal sheaf of Z_0 in H.

Thus, to check that df is injective at $(Z_0, 0) \in H^{(4)} \times A$, it suffices to prove the following statement: If $v \in T_A(0)$ is a tangent vector and $df_{(Z_0,0)}$ sends $(0, v)$ into $T_{H^{(4)}}(Z_0)$, considered as a subspace of $T_{A^{[4]}}(Z_0)$, then v is zero.

The last statement can be proved by an infinitesimal version of the argument for the injectivity of f as follows: We view v as a map

$$v\colon \operatorname{Spec} k[\varepsilon] \to A.$$

Then we have the associated translation map

$$T_v\colon A \otimes k[\varepsilon] \to A \otimes k[\varepsilon],$$

and $df_{(Z_0,0)}$ sends $(0, v)$ to the infinitesimal family

$$Z = T_v^{-1}(Z_0 \otimes k[\varepsilon]) \subset A \otimes k[\varepsilon].$$

The hypothesis that this element of $T_{A^{[4]}}(Z_0)$ is contained in $T_{H^{(4)}}(Z_0)$ means that Z is in fact a family of subschemes of H, i.e.

$$Z \subset H \otimes k[\varepsilon].$$

But Z is contained in $T_v^{-1}(H)$, which means that the two infinitesimal families $T_v^{-1}(H)$ and $H \otimes k[\varepsilon]$ have a family of length 4 subschemes in common. This is only possible if they are equal, essentially since $H^2 < 4$. More precisely, let $s \in \mathscr{O}_H(H)$ be the section of the normal bundle to H in A corresponding to the infinitesimal deformation $T_v^{-1}(H)$. On the other hand, the trivial deformation $H \otimes k[\varepsilon]$ of H corresponds to the zero section. So if the two families $T_v^{-1}(H)$ and $H \otimes k[\varepsilon]$ have a family of length four subschemes in common, then the zero locus of s has at least length 4. But the degree of $\mathscr{O}_H(H)$ is only 2, so $s = 0$, and hence

$$T_v^{-1}(H) = H \otimes k[\varepsilon]. \tag{4.3.11}$$

We conclude that $v = 0$ as follows: The map ϕ_H sends the $k[\varepsilon]$-valued point v to the family

$$T_v^{-1}(H) - H \otimes k[\varepsilon]$$

of divisors on A. Equation (4.3.11) says that this family is trivial. As ϕ_H is an isomorphism — and in particular an immersion — we must have $v=0$, as wanted. $\square$

Next, we exhibit the promised $\mathbb{P}^2$-bundle structure on P. For this, we identify $\widehat{A}$ with the Jacobian $\mathrm{Jac}(H)$ via the canonical (restriction) map

$$\widehat{A} \xrightarrow{\sim} \mathrm{Jac}(H). \tag{4.3.12}$$

Note that this map is a closed immersion by Proposition 3.10, and hence an isomorphism.

The Abel-Jacobi map

$$H^{(4)} \to \mathrm{Jac}(H) \tag{4.3.13}$$

gives $H^{(4)}$ the structure of a $\mathbb{P}^2$-bundle over the Jacobian, and hence over $\widehat{A}$ via the isomorphism (4.3.12). Let us describe the resulting $\mathbb{P}^2$-bundle explicitly. Let

$$\mathscr{P}_H = \mathscr{P}|_{H\times\widehat{A}},$$

which is a Poincaré sheaf on $H\times\widehat{A}$, realizing the identification of $\widehat{A}$ with the Jacobian of H. To avoid a sign later on, we will, however, instead use its dual $\mathscr{P}_H^\vee$, which is an equally good Poincaré sheaf. Now, to define the Abel-Jacobi map, we need to choose an invertible sheaf of degree 4 on H. We choose $\mathscr{O}_H(C)$, and we will see in a moment that this is a good choice. Thus we view $\mathscr{P}_H^\vee \otimes p^*\mathscr{O}_H(C)$ as a universal invertible sheaf of degree 4 on H, and realize the Abel-Jacobi map as the bundle

$$P_0 = \mathbb{P}(\mathscr{C}^\vee) \to \widehat{A}, \quad \mathscr{C} = q_*(\mathscr{P}_H^\vee \otimes p^*\mathscr{O}_H(C)),$$

where p and q denote the projections from $H\times\widehat{A}$. We note that this bundle has precisely the linear systems associated to $\mathscr{P}_{-x}|_H \otimes \mathscr{O}_H(C)$ as fibres, since

$$\mathscr{C}\otimes k(x) = \mathrm{H}^0(H, \mathscr{P}_{-x}|_H \otimes \mathscr{O}_H(C))$$

by the base change theorem in cohomology.

The following reinterpretation of the fibres of P_0 concludes the verification that P_0, and hence P, has the structure announced in Section 4.1.1.

PROPOSITION 4.9. *The fibre of P_0 over $x\in\widehat{A}$ is canonically isomorphic to the projective space of lines in* $\mathrm{Ext}^1_A(\mathscr{P}_x|_H, \mathscr{O}_A(H))$.

PROOF. The fibre $P_0\otimes k(x)$ is the projective space of lines in

$$\begin{aligned}\mathrm{H}^0(H, \mathscr{P}_{-x}|_H \otimes \mathscr{O}_H(C)) &= \mathrm{Hom}_H(\mathscr{P}_x|_H, \mathscr{O}_H(C)) \\ &= \mathrm{Hom}_A(\mathscr{P}_x|_H, \mathscr{O}_H(C)).\end{aligned}$$

On the other hand, the exact sequence (recall that $C=2H$)

$$0\to\mathscr{O}_A(H)\to\mathscr{O}_A(C)\to\mathscr{O}_H(C)\to 0$$

induces a homomorphism

$$\mathrm{Hom}_A(\mathscr{P}_x|_H, \mathscr{O}_H(C)) \to \mathrm{Ext}^1_A(\mathscr{P}_x|_H, \mathscr{O}_A(H)), \tag{4.3.14}$$

which we claim is an isomorphism. It is injective, as $\mathrm{Hom}_A(\mathscr{P}_x|_H, \mathscr{O}_A(C))$ is Serre dual to $\mathrm{H}^2(A, \mathscr{P}_x|_H \otimes \mathscr{O}_A(-C)) = 0$. Thus it is enough to check that $\mathrm{Ext}^1_A(\mathscr{P}_x|_H, \mathscr{O}_A(H))$ is 3-dimensional. But it is Serre dual to

$$\mathrm{H}^1(A, \mathscr{P}_x|_H \otimes \mathscr{O}_A(-H))$$

and $\mathscr{P}_x|_H \otimes \mathscr{O}_A(-H)$ is an invertible sheaf on H of degree -2. Hence it has no global sections and Euler characteristic -3. We conclude that (4.3.14) is an isomorphism. □

Note that the canonical isomorphism in the proposition is just the following: A section $s \in \mathrm{H}^0(H, \mathscr{P}_{-x}|_H \otimes \mathscr{O}_H(C))$ defines a subscheme $Z \subset H$ with

$$\mathscr{I}_{Z,H}(C) \cong \mathscr{P}_x|_H .$$

On the other hand, we have the usual exact sequence

$$0 \to \mathscr{I}_H \to \mathscr{I}_Z \to \mathscr{I}_{Z,H} \to 0.$$

Twisting with $C = 2H$, we arrive at the extension

$$0 \to \mathscr{O}_A(H) \to \mathscr{I}_Z(C) \to \mathscr{P}_x|_H \to 0$$

which is the element in $\mathrm{Ext}^1_A(\mathscr{P}_x|_H, \mathscr{O}_A(H))$ corresponding to the section s.

REMARK 4.10. We have favoured the construction of P_0 as the bundle associated to the "relative H^0", i.e. the sheaf $\mathscr{C}$, over a construction as a relative Ext-sheaf, to avoid having to deal with base change for Ext-sheaves. In the sequel, both views of the fibres of P_0 will be useful.

We end our discussion of P_0 by noting that, if we identify A and $\widehat{A}$ via the isomorphism $\phi_H : A \xrightarrow{\sim} \widehat{A}$, then the bundle P_0 becomes simply the addition map

$$\sigma : H^{(4)} \to A \tag{4.3.15}$$

on A. This might seem odd at first, as the addition map does not depend on any choice of base point, whereas the Abel-Jacobi map depended on the choice of the degree 4 divisor $C|_H$ on H. What makes this work is the fact that under the group law on A, the zero cycle $C|_H = 2H^2$ adds to zero. In fact, writing σ for summation of zero-cycles under the group law, we have

$$\begin{aligned} \sigma(2H^2) &= \sigma(H^2) + \sigma((-1)^*H^2) \qquad \text{(since } H \text{ is symmetric)} \\ &= \sigma(H^2) - \sigma(H^2) = 0. \end{aligned}$$

We need one more ingredient to identify P_0 with (4.3.15): In addition to the restriction map $\widehat{A} \to \mathrm{Jac}(H)$, obtained by viewing $\mathrm{Jac}(H)$ as the Picard variety of H, there is also an "addition map" $\mathrm{Jac}(H) \to A$, obtained by viewing $\mathrm{Jac}(H)$

as the Albanese of H. The composition of these maps

$$\widehat{A} \to \mathrm{Jac}(H) \to A$$

is [**32**, Proposition 17.3] the *negative* of $\phi_{\widehat{H}}$. Now consider the diagram

$$\begin{array}{ccccc} P_0 & \cong & H^{(4)} & = & H^{(4)} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \sigma \\ \widehat{A} & \longrightarrow & \mathrm{Jac}(H) & \longrightarrow & A \end{array} \tag{4.3.16}$$

where the lower row is the *negative* of the restriction map, followed by the addition map. The Abel-Jacobi map in the middle uses the chosen base point $C|_H$, i.e. an element $Z \in H^{(4)}$ is sent to the class of the divisor $Z - C|_H$. Since we just observed that $C|_H$ adds to zero on A, the rightmost square commutes. On the other hand, the leftmost square commutes by construction of P_0, since we dualized the Poincaré sheaf and hence, in effect, identified $\widehat{A}$ and $\mathrm{Jac}(H)$ via the negative of the restriction map. Since the composite map $\widehat{A} \to A$ is just $\phi_{\widehat{H}}$, which is inverse to ϕ_H, we have identified the bundle $P_0 \to \widehat{A}$ with the addition map (4.3.15), as wanted.

4.3.3. Fourier-Mukai transforms of non-WIT sheaves. For later use, we calculate the sheaves $R^p\widehat{S}(\mathscr{E})$ for $\mathscr{E} \in P$, i.e. for every sheaf $\mathscr{E}$ in M failing WIT. It is enough to do the calculation for sheaves of the form

$$\mathscr{E} = \mathscr{I}_Z(C)$$

with $Z \subset H$, since, by Maciocia's Proposition 4.4, every non-WIT sheaf is obtained from a sheaf of this type by translating and twisting with a homogeneous invertible sheaf — and the Fourier-Mukai transform essentially exchanges these operations, by Proposition A.10.

PROPOSITION 4.11. *Let $Z \in A^{[4]}$ be a subscheme contained in H, let $\sigma(Z)$ denote the sum of its underlying zero-cycle under the group law, and let $x = \phi_H(\sigma(Z))$. Then*

$$\widehat{S}(\mathscr{I}_Z(C)) \cong \mathscr{O}_{\widehat{A}}(-\widehat{H})$$
$$R^1\widehat{S}(\mathscr{I}_Z(C)) \cong \mathscr{I}_{-x}(T_x^*\widehat{H})$$

where $\mathscr{I}_{-x}$ denotes the ideal of the point $-x$ in $\widehat{A}$.

Before we prove the proposition, note the following: As we saw in the previous section, whenever Z is contained in H, there is a short exact sequence

$$0 \to \mathscr{O}_A(H) \xrightarrow{\alpha} \mathscr{I}_Z(C) \xrightarrow{\beta} \mathscr{P}_x|_H \to 0 \tag{4.3.17}$$

where $x = \phi_H(\sigma(Z))$, as in the proposition.

LEMMA 4.12. *The maps α and β in the short exact sequence* (4.3.17) *induce isomorphisms*

$$\widehat{S}(\mathscr{I}_Z(C)) \cong \widehat{\mathscr{O}_A(H)}$$
$$R^1\widehat{S}(\mathscr{I}_Z(C)) \cong \widehat{\mathscr{P}_x|_H}.$$

PROOF. The sheaves $\mathscr{O}_A(H)$ and $\mathscr{P}_x|_H$ satisfy WIT with indices 0 and 1, respectively. Hence the claim follows from the long exact sequence induced by applying the Fourier-Mukai functor to sequence (4.3.17). □

From this we see that the essential part in proving Proposition 4.11 is to investigate Fourier-Mukai transforms of invertible sheaves on H of degree zero.

LEMMA 4.13. *Let (A,H) be a principally polarized abelian surface. For each $x \in \widehat{A}$, the Fourier-Mukai transform of the WIT_1-sheaf $\mathscr{P}_x|_H$ is*

$$\widehat{\mathscr{P}_x|_H} \cong \mathscr{I}_{-x}(T_x^*\widehat{H})$$

PROOF. The tensor product of $\mathscr{P}_x$ with the standard exact sequence

$$0 \to \mathscr{O}_A(-H) \to \mathscr{O}_A \to \mathscr{O}_H \to 0$$

gives the exact sequence

$$0 \to \mathscr{P}_x(-H) \to \mathscr{P}_x \to \mathscr{P}_x|_H \to 0, \tag{4.3.18}$$

to which we want to apply the Fourier-Mukai transform. By Example A.13, the homogeneous invertible sheaf $\mathscr{P}_x$ satisfies WIT_2, and

$$\widehat{\mathscr{P}_x} \cong k(-x).$$

The sheaf $\mathscr{P}_x(-H)$ also satisfies WIT_2, with

$$\widehat{\mathscr{P}_x(-H)} \cong T_x^*\widehat{\mathscr{O}_A(-H)} \cong T_x^*\mathscr{O}_{\widehat{A}}(\widehat{H})$$

by an application of Proposition A.10 and the fact [**27**, Theorem 3.13(5)] that $\widehat{\mathscr{O}_A(-H)} \cong \mathscr{O}_{\widehat{A}}(\widehat{H})$, which is almost the definition of $\widehat{H}$.

Thus, the result of applying the Fourier-Mukai functor to the short exact sequence (4.3.18) is the exact sequence

$$0 \to \widehat{\mathscr{P}_x|_H} \to T_x^*\mathscr{O}_{\widehat{A}}(\widehat{H}) \to k(-x) \to 0.$$

The lemma follows. □

PROOF OF PROPOSITION 4.11. We have

$$\widehat{S}(\mathscr{I}_Z(C)) \cong \widehat{\mathscr{O}_A(H)} \cong \mathscr{O}_{\widehat{A}}(-\widehat{H})$$

by Lemma 4.12 and the definition of $\widehat{H}$, and we have

$$R^1\widehat{S}(\mathscr{I}_Z(C)) \cong \widehat{\mathscr{P}_x|_H} \cong \mathscr{I}_{-x}(T_x^*\widehat{H})$$

by Lemma 4.12 and Lemma 4.13. □

REMARK 4.14. The author learned also the result in Proposition 4.11 from the notes by Maciocia mentioned in Remark 4.5. The calculation given above is, however, different from Maciocia's.

4.4. The dual projective bundle

Next we study WITness of sheaves in M', and stability of their Fourier-Mukai transforms.

4.4.1. Invertible sheaves on curves satisfy WIT. Here is a preliminary observation: Every $\mathscr{F} \in M'$ is torsion, so

$$R^2 S(\mathscr{F}) = 0,$$

in fact, we have $\mathrm{H}^2(A, \mathscr{F} \otimes \mathscr{P}_a) = 0$ for all $a \in A$. Next, we claim that $S(\mathscr{F}) = 0$ if and only if there exists an $a \in A$ with $\mathrm{H}^0(\widehat{A}, \mathscr{F} \otimes \mathscr{P}_a) = 0$. The "if" part follows from $S(\mathscr{F})$ being torsion free. Conversely, if $S(\mathscr{F}) = 0$, then $\mathscr{F}$ satisfies WIT_1 and its Fourier-Mukai transform has rank 1. Since $\mathscr{F}$ has Euler characteristic -1, this gives

$$\dim \mathrm{H}^0(\widehat{A}, \mathscr{F} \otimes \mathscr{P}_a) = \dim \mathrm{H}^1(\widehat{A}, \mathscr{F} \otimes \mathscr{P}_a) - 1 = 0$$

for a generic.

Now, a *generic* element $\mathscr{F} \in M'$ is an invertible sheaf of degree 3 on a nonsingular curve D in (a translate of) the linear system $|\widehat{C}|$. As we just observed, the sheaf $\mathscr{F}$ can only fail WIT_1 if $\mathscr{F} \otimes \mathscr{P}_a$ has a nonzero section for all $a \in A$. Writing $\mathrm{Jac}^3(D)$ for the Jacobian of invertible sheaves of degree 3 on D, we can rephrase this condition as follows: WITness can only fail for $\mathscr{F}$ if the image of the map

$$A \to \mathrm{Jac}^3(D), \quad a \mapsto \mathscr{F} \otimes \mathscr{P}_a,$$

which is an embedding by Proposition 3.10, is entirely contained in the Brill-Noether locus W_3. This is impossible by a result of Debarre and Fahlaoui [**7**, Corollary 3.6]: Any abelian subvariety of the Brill-Noether locus W_d has dimension at most $d/2$. It *is* true that every sheaf in M' satisfies WIT_1, but the argument just given only covers those sheaves that are supported on nonsingular curves. In other words, the problem is the presence of non-integral and singular curves in the linear system $|\widehat{C}|$.

In the following we will instead prove WITness by describing all elements in M' in terms of sheaves in M and their Fourier-Mukai transforms.

4.4.2. Every sheaf satisfies WIT. Note that, even when $\mathscr{E} \in M$ is not necessarily WIT, any quotient

$$0 \to \widehat{S}(\mathscr{E}) \to R^1\widehat{S}(\mathscr{E}) \to \mathscr{F} \to 0, \tag{4.4.1}$$

if stable, is a point in M'. We will prove that every such sheaf $\mathscr{F}$ is indeed stable, and conversely, that every $\mathscr{F} \in M'$ arises in this way.

PROPOSITION 4.15. *The points in M' are exactly the sheaves $\mathscr{F}$ that fit in an exact sequence* (4.4.1)*, for some $\mathscr{E} \in M$.*

COROLLARY 4.16. *Every sheaf $\mathscr{F} \in M'$ satisfies WIT with index* 1.

PROOF OF THE COROLLARY. Choose a short exact sequence (4.4.1). If $\mathscr{E}$ satisfies WIT_1, then so does $\mathscr{F} = \widehat{\mathscr{E}}$. If $\mathscr{E}$ fails WIT, then, by Proposition 4.11, the two sheaves $\widehat{S}(\mathscr{E})$ and $R^1\widehat{S}(\mathscr{E})$ satisfy WIT with indices 2 and 1, respectively. The result follows from the long exact sequence obtained by applying the Fourier-Mukai functor to (4.4.1). □

We will prove Proposition 4.15 in several steps. First, we use the calculations from Proposition 4.11 to determine exactly what sequences of the form (4.4.1) can look like.

LEMMA 4.17. *The following are equivalent conditions on a sheaf $\mathscr{F}$ on $\widehat{A}$.*

(1) *$\mathscr{F}$ fits in an exact sequence* (4.4.1) *for a non-WIT sheaf $\mathscr{E} \in M$.*
(2) *There exist points $x, y \in \widehat{A}$ and $a \in A$, and an effective divisor $D \in |\widehat{C}|$ containing x, such that*

$$\mathscr{F} \cong T_y^*(\mathscr{I}_{x,D}(\widehat{H})) \otimes \mathscr{P}_a$$

where $\mathscr{I}_{x,D}$ denotes the relative ideal sheaf of x in D.

In particular, for every $\mathscr{E} \in M$ and every exact sequence (4.4.1)*, the sheaf $\mathscr{F}$ is stable.*

PROOF. First suppose

$$\mathscr{E} = \mathscr{I}_Z(C), \quad Z \subset H. \tag{4.4.2}$$

Then, by Proposition 4.11, an exact sequence (4.4.1) has the form

$$0 \to \mathscr{O}_{\widehat{A}}(-\widehat{H}) \to \mathscr{I}_{-x}(T_x^*\widehat{H}) \to \mathscr{F} \to 0.$$

Thus we have

$$\mathscr{F} \cong \mathscr{I}_{-x,D}(T_x^*\widehat{H}), \quad D \in |\widehat{H} + T_x^*\widehat{H}|. \tag{4.4.3}$$

Now observe the following: On the one hand, every non-WIT sheaf in M can, by Maciocia's Proposition 4.4, be obtained from a sheaf $\mathscr{E}$ of the form (4.4.2) by translating and twisting with a homogeneous invertible sheaf. On the other hand, every sheaf satisfying condition (2) can be obtained from a sheaf $\mathscr{F}$ of the form (4.4.3) by translating and twisting. By the compatibility in Proposition A.10, between the Fourier-Mukai functor and these operations, the equivalence of the two conditions follows.

For the last statement, if $\mathscr{E}$ is not WIT, we have just seen that any $\mathscr{F}$ in an exact sequence (4.4.1) has the form (2), which clearly is stable. And, if $\mathscr{E}$ is WIT, then an exact sequence (4.4.1) is just an isomorphism $\widehat{\mathscr{E}} \cong \mathscr{F}$, and by Lemma 3.9, the sheaf $\widehat{\mathscr{E}}$ is stable. □

REMARK 4.18. Note that a sheaf $\mathscr{F}$ as in condition (2) in Lemma 4.17 is in fact invertible on D, since the ideal $\mathscr{I}_{x,D}$ of the point $x \in D$ is just $\mathscr{O}_D(-x)$.

The next step towards proving Proposition 4.15 is the following reformulation of the condition on $\mathscr{F} \in M'$ that there exists an exact sequence (4.4.1):

LEMMA 4.19. *For a sheaf $\mathscr{F} \in M'$, the following are equivalent:*

(1) $\mathscr{F}$ *fits in an exact sequence* (4.4.1) *for some* $\mathscr{E} \in M$.
(2) $\operatorname{Hom}_{\widehat{A}}(R^1\widehat{S}(\mathscr{E}), \mathscr{F}) \neq 0$ *for some* $\mathscr{E} \in M$.

PROOF. It is obvious that (1) implies (2). For the converse, let

$$\phi \colon R^1\widehat{S}(\mathscr{E}) \to \mathscr{F}$$

be a nonzero homomorphism. If $\mathscr{E}$ satisfies WIT, then both $R^1\widehat{S}(\mathscr{E}) = \widehat{\mathscr{E}}$ and $\mathscr{F}$ are stable, by Lemma 3.9. In particular, they are simple, so ϕ is an isomorphism, and the required exact sequence is just the trivial

$$0 \to 0 \to \widehat{\mathscr{E}} \cong \mathscr{F} \to 0.$$

If $\mathscr{E}$ is not WIT_1, we want to show that $\mathscr{F}$ has the form required by condition (2) of Lemma 4.17. For this, consider the short exact sequence

$$0 \to \operatorname{Ker}\phi \to R^1\widehat{S}(\mathscr{E}) \to \operatorname{Im}\phi \to 0.$$

By Proposition 4.11, the sheaf $R^1\widehat{S}(\mathscr{E})$ has first Chern class $\widehat{H}$ (considered as an element of the Néron-Severi group) and Euler characteristic zero. Hence, by additivity of these invariants, we have

$$\begin{aligned} \widehat{H} &= c_1(\operatorname{Ker}\phi) + c_1(\operatorname{Im}\phi) \\ 0 &= \chi(\operatorname{Ker}\phi) + \chi(\operatorname{Im}\phi). \end{aligned}$$

Since $\operatorname{Ker}\phi$ is a subsheaf of $R^1\widehat{S}(\mathscr{E})$, we must have, by Proposition 4.11 again,

$$\operatorname{Ker}\phi = \mathscr{I}_Z(d\widehat{H}) \otimes \mathscr{P}_a$$

for a point $a \in A$, a finite subscheme $Z \subset A$ and an integer $d \leq 1$. Here, we use that $\widehat{H}$ generates $\mathrm{NS}(\widehat{A})$. Thus we calculate

$$\begin{aligned} c_1(\operatorname{Ker}\phi) &= d\widehat{H} &\Longrightarrow\quad c_1(\operatorname{Im}\phi) &= (1-d)\widehat{H} \\ \chi(\operatorname{Ker}\phi) &= d^2 - \operatorname{len} Z &\Longrightarrow\quad \chi(\operatorname{Im}\phi) &= \operatorname{len} Z - d^2. \end{aligned}$$

On the other hand, the fact that $\operatorname{Im}\phi$ is a subsheaf of $\mathscr{F}$ has two consequences. Firstly, as $\mathscr{F}$ is pure of dimension 1, and since $c_1(\operatorname{Im}\phi) = (1-d)\widehat{H}$ and $c_1(\mathscr{F}) = 2\widehat{H}$, we can only have $d = 0$ or $d = -1$. Secondly, as $\mathscr{F}$ is stable, the criterion from Section B.1 gives

$$\frac{\chi(\operatorname{Im}\phi)}{\deg(\operatorname{Im}\phi)} \leq \frac{\chi(\mathscr{F})}{\deg(\mathscr{F})} = -\frac{1}{4} \tag{4.4.4}$$

with equality only if $\operatorname{Im}\phi = \mathscr{F}$. But the left hand side is

$$\frac{\chi(\operatorname{Im}\phi)}{\deg(\operatorname{Im}\phi)} = \frac{\operatorname{len} Z - d^2}{2(1-d)} \geq -\frac{d^2}{2(1-d)} = \begin{cases} 0 & \text{if } d = 0 \\ -1/4 & \text{if } d = -1 \end{cases}$$

where the inequality can be an equality only if $Z = \emptyset$. We conclude, by equation (4.4.4), that the expression must indeed equal $-1/4$, and hence

$$Z = \emptyset, \quad d = -1 \quad \text{and} \quad \operatorname{Im}\phi = \mathscr{F}.$$

This proves that $\operatorname{Ker}\phi \cong \mathscr{P}_b(-\widehat{H})$ for a point $b \in A$, and it follows that any $\mathscr{F}$ fitting in the short exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Ker}\phi & \to & R^1S(\mathscr{E}) & \to & \operatorname{Im}\phi & \to & 0 \\ & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \\ & & \mathscr{P}_b(-\widehat{H}) & & \mathscr{I}_x(\widehat{H}) \otimes \mathscr{P}_a & & \mathscr{F} & & \end{array}$$

has the form required by point (2) of Lemma 4.17. □

Next we show that the condition (2) of the last lemma is closed.

LEMMA 4.20. *The subset $\Lambda \subset M \times M'$ consisting of pairs $(\mathscr{E}, \mathscr{F})$ satisfying*

$$\operatorname{Hom}_{M'}(R^1S(\mathscr{E}), \mathscr{F}) \neq 0$$

is closed.

PROOF. We will apply a result of Grothendieck to obtain a coherent sheaf $\mathscr{N}$ having Λ as support.

Let $\mathscr{U}$ and $\mathscr{V}$ denote the universal families on $M \times A$ and $M' \times \widehat{A}$, respectively. Let p_{ij} denote the projections from $M \times M' \times \widehat{A}$, for example,

$$p_{12} \colon M \times M' \times \widehat{A} \to M \times M'.$$

Also write $\widehat{S}$ for the relative Fourier-Mukai functor over M, taking sheaves on $M \times A$ to sheaves on $M \times \widehat{A}$. Then there exists [**13**, III §7.7.8] a coherent sheaf $\mathscr{N}$ on $M \times M'$ and an isomorphism

$$\operatorname{Hom}_{M\times M'\times\widehat{A}}\left(p_{13}^*(R^1\widehat{S}(\mathscr{U})), p_{23}^*\mathscr{V} \otimes p_{12}^*\mathscr{M}\right) \cong \operatorname{Hom}_{M\times M'}(\mathscr{N}, \mathscr{M})$$

of functors on the category of $\mathscr{O}_{M\times M'}$-modules $\mathscr{M}$. By taking $\mathscr{M} = k(p)$ to be the residue field at the point $p = (\mathscr{E}, \mathscr{F})$ in $M \times M'$, the isomorphism becomes

$$\operatorname{Hom}_{\widehat{A}}(R^1\widehat{S}(\mathscr{E}), \mathscr{F}) \cong \operatorname{Hom}_{M\times M'}(\mathscr{N}, k(p))$$

where we have used the fact that $R^1\widehat{S}$ commutes with base change, by Remark A.9. Thus Λ is indeed the support of $\mathscr{N}$. □

PROOF OF PROPOSITION 4.15. By Lemma 4.17, every sheaf $\mathscr{F}$ sitting in an exact sequence (4.4.1) is stable, and hence is a point in M'.

For the opposite inclusion, we argue as follows. If a sheaf $\mathscr{E} \in M$ satisfies WIT, then the condition $\operatorname{Hom}_{\widehat{A}}(\widehat{\mathscr{E}}, \mathscr{F}) \neq 0$ is equivalent to $\widehat{\mathscr{E}} \cong \mathscr{F}$, as both sheaves are stable and hence simple. Thus Λ contains the graph of $\psi\colon M \dashrightarrow M'$, and, in particular, the image of Λ under the second projection is dense in M'. On the other hand, since Λ is closed, its image in M' is closed also, and hence equals the whole of M'. Thus, every $\mathscr{F} \in M'$ satisfies condition (2) of Lemma 4.19, which, by that lemma, is equivalent to the existence of a short exact sequence (4.4.1). □

4.4.3. The bundle of sheaves with non stable Fourier-Mukai transform. We have just seen that every $\mathscr{F} \in M'$ satisfies WIT_1. As its Fourier-Mukai transform $\widehat{\mathscr{F}}$ has rank 1, it is stable precisely when it is torsion free.

LEMMA 4.21. *The Fourier-Mukai transform of a sheaf $\mathscr{F} \in M'$ has torsion if and only if it sits in a short exact sequence* (4.4.1) *for a non-WIT sheaf $\mathscr{E} \in M$.*

PROOF. By Proposition 4.15, every sheaf $\mathscr{F} \in M'$ sits in a short exact sequence (4.4.1). If the sheaf $\mathscr{E}$, in that sequence, satisfies WIT, then $\mathscr{F}$ is just the Fourier-Mukai transform of $\mathscr{E}$, and thus

$$\widehat{\mathscr{F}} \cong (-1)^*\mathscr{E}$$

is torsion free.

Conversely, suppose $\mathscr{E}$ fails WIT. By translating and twisting with a homogeneous invertible sheaf, we may assume $\mathscr{E} \cong \mathscr{I}_Z(C)$ with $Z \subset H$, by Maciocia's Proposition 4.4. By Lemma 4.12, the exact sequence (4.4.1) has the form

$$0 \to \widehat{\mathscr{O}_A(H)} \to \widehat{\mathscr{P}_x|_H} \to \mathscr{F} \to 0.$$

When we apply the Fourier-Mukai functor, we arrive at the short exact sequence

$$0 \to (-1)^*\mathscr{P}_x|_H \to \widehat{\mathscr{F}} \to \mathscr{O}_A(H) \to 0$$

which shows that $\widehat{\mathscr{F}}$ has torsion in this case. □

As we saw in the proof of Lemma 4.17, any sheaf $\mathscr{F}$ sitting in an exact sequence (4.4.1), where $\mathscr{E}$ fails WIT, has the form

$$\mathscr{F} \cong \mathscr{I}_{-x,D}(T_x^*\widehat{H}),$$

where $D \in |\widehat{H} + T_x^*\widehat{H}|$ contains $-x$, up to translation and twisting with a homogeneous invertible sheaf.

We now define a projective bundle P_0' over $\widehat{A}$, with fibre over $x \in \widehat{A}$ being the linear system of divisors $D \in |\widehat{H} + T_x^*\widehat{H}|$ containing $-x$. More precisely, we let

$$P_0' = \mathbb{P}(\mathscr{D}^\vee) \to \widehat{A}, \qquad \mathscr{D} = q_*(\mathscr{I}_{\Delta'}(p^*\widehat{H} + m^*\widehat{H}))$$

where p, q and m denote the two projections and the group law on $\widehat{A} \times \widehat{A}$, and Δ' is the anti-diagonal, consisting of pairs $(x, -x)$. By the base change theorem in cohomology, the fibre of P_0' over x is the space of lines in

$$\mathscr{D}\otimes k(x)\cong \mathrm{H}^0(\widehat{A},\mathscr{I}_{-x}(\widehat{H}+T_x^*\widehat{H}))$$

as wanted. Let

$$P' = P_0'\times A\times \widehat{A}.$$

We want to show that P' embeds into M', as the locus of sheaves whose Fourier-Mukai transforms contain torsion. As we are proceeding in analogy to the development in Section 4.3.2, we will leave out the details in the proof.

PROPOSITION 4.22. *The map $P'\to M'$, sending a triple (D,a,y), with $D\in P_0'$ lying over $x\in\widehat{A}$, to the sheaf*

$$\mathscr{F} = T_y^*(\mathscr{I}_{-x,D}(T_x^*\widehat{H}))\otimes\mathscr{P}_a$$

is a closed immersion.

PROOF. It is clear that f can be defined at the level of point functors, and hence is a regular map. We proceed to showing that it is an embedding.

Firstly, f is injective: We need to check that the triple (x,a,y) and the divisor D can all be reconstructed from $\mathscr{F}$. The essential part is to determine a, as the rest can be read off Yoshioka's map $\alpha(\mathscr{F})$ and the support of $\mathscr{F}$.

To recover the point a, we use the Fourier-Mukai transform of $\mathscr{F}$. Since we have a short exact sequence

$$0\to\widehat{\mathscr{O}_A(H)}\to\widehat{\mathscr{P}_x|_H}\to\mathscr{I}_{-x,D}(T_x^*\widehat{H})\to 0$$

it follows, by using Proposition A.10, that the Fourier-Mukai transform of $\mathscr{F}$ sits in an exact sequence

$$0\to T_a^*(\mathscr{P}_{-x-y}\big|_H)\to\widehat{\mathscr{F}}\to\mathscr{P}_{\phi_H(a)-y}(H)\to 0.$$

Thus, the torsion subsheaf of $\widehat{\mathscr{F}}$ is supported on $T_a^{-1}H$, which, as we saw in the proof of Proposition 4.8, determines a.

Secondly, to show that f is immersive, we must check that an infinitesimal deformation of the data consisting of (x,a,y) and D can be reconstructed by the resulting infinitesimal deformation of $\mathscr{F}$. The argument for this is essentially the same as for the injectivity: Everything but the deformation of a can be read off the support of the deformation of $\mathscr{F}$ and the (differential of the) Yoshioka map α. Finally, the deformation of a can be recovered just as we recovered a above, using the relative Fourier-Mukai transform of the deformation of $\mathscr{F}$. □

To see that P' has the structure announced in Section 4.1.2, it remains only to identify its fibres with the space of lines in $\mathrm{Ext}^1_A(\mathscr{O}_A(H),\mathscr{P}_x|_H)$. By construction, a fibre of P' is the space of lines in

$$\mathrm{H}^0(\widehat{A},\mathscr{I}_{-x}(\widehat{H}+T_x^*\widehat{H})\cong \mathrm{Hom}_{\widehat{A}}(\widehat{\mathscr{O}_A(H)},\widehat{\mathscr{P}_x|_H}).$$

We can identify the latter vector space with $\mathrm{Ext}^1_A(\mathscr{O}_A(H), \mathscr{P}_x|_H)$ by appealing to Mukai's result [**25**, Corollary 2.4], that there is an isomorphism

$$\mathrm{Ext}^p_A(\mathscr{A}, \mathscr{B}) \cong \mathrm{Ext}^{p+i-j}_{\widehat{A}}(\widehat{\mathscr{A}}, \widehat{\mathscr{B}})$$

whenever $\mathscr{A}$ and $\mathscr{B}$ are WIT-sheaves of index i and j, respectively. This is clear if one thinks of elements of Ext-spaces as graded maps in the derived categories; one just has to figure out the right degree $p+i-j$ on the right hand side. In our situation, however, we can make the isomorphism explicit.

PROPOSITION 4.23. *There is an isomorphism*

$$\mathrm{Ext}^1_A(\mathscr{O}_A(H), \mathscr{P}_x|_H) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{A}}(\widehat{\mathscr{O}_A(H)}, \widehat{\mathscr{P}_x|_H}),$$

given by sending a short exact sequence

$$0 \to \mathscr{P}_x|_H \to \mathscr{G} \to \mathscr{O}_A(H) \to 0$$

to the induced boundary map $\delta\colon \widehat{\mathscr{O}_A(H)} \to \widehat{\mathscr{P}_x|_H}$.

PROOF. We write down the inverse map as follows: Any nonzero homomorphism $s\colon \widehat{\mathscr{O}_A(H)} \to \widehat{\mathscr{P}_x|_H}$ is injective, since the former is invertible and the latter is torsion free, by Lemma 4.13. Thus we have a short exact sequence

$$0 \to \widehat{\mathscr{O}_A(H)} \xrightarrow{s} \widehat{\mathscr{P}_x|_H} \to \mathscr{F} \to 0$$

where $\mathscr{F}$ is defined to be quotient. Then $\mathscr{F}$ satisfies WIT with index 1, and there is an induced short exact sequence

$$0 \to (-1)^* \mathscr{P}_x|_H \to \widehat{\mathscr{F}} \to (-1)^* \mathscr{O}_A(H) \to 0.$$

Applying $(-1)^*$, we obtain the required extension. It is clear that the two constructions produce inverse maps. ☐

Let us summarize: A point in the fibre of $P' \to \widehat{A} \times A \times \widehat{A}$ above (x,a,y) is a triple

$$(D,a,y) \in P' = P'_0 \times A \times \widehat{A},$$

where D is a divisor in the linear system $|\widehat{H} + T_x^*\widehat{H}|$ containing $-x$, and the corresponding sheaf in M' is

$$\mathscr{F} = T_y^*(\mathscr{I}_{-x,D}(T_x^*H)) \otimes \mathscr{P}_a.$$

The divisor D is given by a section $s \in \mathrm{H}^0(\widehat{A}, \mathscr{I}_{-x}(\widehat{H} + T_x^*H))$, defined modulo scalar multiplication. By the proposition, the section s corresponds uniquely to an extension

$$0 \to \mathscr{P}_x|_H \to \mathscr{G} \to \mathscr{O}_A(H) \to 0$$

and $\widehat{\mathscr{G}} \cong \mathscr{I}_{-x,D}(T_x^*H)$. Thus, the sheaf $\mathscr{F}$ is obtained from $\widehat{\mathscr{G}}$ by translating with y and twisting with $\mathscr{P}_a$. This concludes the verification of the description of the bundle P', and its inclusion in M', given in Section 4.1.2.

4.4.4. Duality. Recall that we have defined $P = P_0 \times A \times \widehat{A}$ and $P' = P_0' \times A \times \widehat{A}$, where

$$P_0 = \mathbb{P}(\mathscr{C}^\vee) \to \widehat{A} \qquad \mathscr{C} = q_*(\mathscr{P}_H^\vee \otimes p^*\mathscr{O}_H(C))$$
$$P_0' = \mathbb{P}(\mathscr{D}^\vee) \to \widehat{A} \qquad \mathscr{D} = q_*(\mathscr{I}_{\Delta'}(p^*\widehat{H} + m^*\widehat{H})).$$

Here, and in what follows, we use the same symbols p and q to denote the projections from any product. In the expressions above, the sheaf $\mathscr{C}$ is a push forward over the second projection from $H \times \widehat{A}$, whereas $\mathscr{D}$ is a push forward over the second projection from $\widehat{A} \times \widehat{A}$.

PROPOSITION 4.24. *The two locally free sheaves $\mathscr{C}$ and $\mathscr{D}$ on $\widehat{A}$ are dual. Hence P and P' are dual projective bundles.*

The proof has two ingredients: First we use relative Grothendieck duality to identify the dual of $\mathscr{C}$ with a suitable relative Ext-sheaf. Then we use the relative Fourier-Mukai functor to identify that Ext-sheaf with $\mathscr{D}$, using a relative version of the argument already encountered in Proposition 4.23. For this second step we need a relative version of Lemma 4.13.

LEMMA 4.25. *Consider $q\colon A \times \widehat{A} \to \widehat{A}$ as an abelian scheme over $\widehat{A}$. The sheaf $\mathscr{P}_H$ on $A \times \widehat{A}$ satisfies WIT$_1$, and its relative Fourier-Mukai transform over $\widehat{A}$ is the sheaf*

$$\mathscr{I}_{\Delta'} \otimes m^*\mathscr{O}_{\widehat{A}}(\widehat{H})$$

on $\widehat{A} \times \widehat{A}$, where Δ' is the anti-diagonal, consisting of pairs of the form $(x, -x)$, and m is the group law.

PROOF. We carry out the relative analogues of the calculations from the proof of Lemma 4.13.

There is a short exact sequence

$$0 \to \mathscr{P}(-p^*H) \to \mathscr{P} \to \mathscr{P}_H \to 0,$$

where the two first sheaves satisfy WIT$_2$ and the last satisfies WIT$_1$, by an application of Theorem A.8. Thus there is an induced short exact sequence

$$0 \to \widehat{\mathscr{P}_H} \to \widehat{\mathscr{P}(-p^*H)} \to \widehat{\mathscr{P}} \to 0. \tag{4.4.5}$$

We claim that there are isomorphisms

$$\widehat{\mathscr{P}(-p^*H)} \cong m^*\mathscr{O}_{\widehat{A}}(\widehat{H})$$
$$\widehat{\mathscr{P}} \cong \mathscr{O}_{\Delta'}$$

under which the surjection in the short exact sequence (4.4.5) is identified with the restriction map

$$m^*\mathscr{O}_{\widehat{A}}(\widehat{H}) \to \mathscr{O}_{\Delta'} \to 0. \tag{4.4.6}$$

Note that this makes sense, since the restriction of m to Δ' is the zero map, so that the restriction of $m^*\mathscr{O}_{\widehat{A}}(\widehat{H})$ to Δ' is just its structure sheaf.

By Mukai's Theorem A.2, there are isomorphisms as claimed above if and only if the Fourier-Mukai transform of the two WIT_0-sheaves $m^*\mathscr{O}_A(\widehat{H})$ and $\mathscr{O}_{\Delta'}$ are

$$\widehat{m^*\mathscr{O}_A(H)} \cong (-1)^*\mathscr{P}(-p^*H)$$
$$\widehat{\mathscr{O}_{\Delta'}} \cong (-1)^*\mathscr{P}.$$

Note that, since we view $\widehat{A}\times\widehat{A}$ as an abelian scheme over $\widehat{A}$ via second projection, the map named -1 here is the one sending (x,y) to $(-x,y)$. Hence we have

$$(-1)^*\mathscr{P} \cong \mathscr{P}^\vee.$$

The Fourier-Mukai transforms of $m^*\mathscr{O}_A(H)$ and $\mathscr{O}_{\Delta'}$ can be found by direct calculations. We only do the first one. In what follows, we use the isomorphism $\phi_H\colon A \xrightarrow{\sim} \widehat{A}$ to identify A with its dual. The Poincaré sheaf on $A\times A$ is then

$$\mathscr{P} = \mathscr{O}_{A\times A}(m^*H - p^*H - q^*H). \tag{4.4.7}$$

From $A\times A\times A$, we have the following maps: Write p_i for projection onto the i'th factor and p_{ij} for the projection onto the product of the i'th and j'th factor. Let $n\colon A\times A\times A \to A$ be the triple addition map, and let $m_{ij} = m\circ p_{ij}$. Then we have (explanation follows below):

$$\begin{aligned}
\widehat{m^*\mathscr{O}_A(H)} &\overset{(1)}{=} p_{13*}(p_{23}^*m^*\mathscr{O}_A(H)\otimes p_{12}^*\mathscr{P})\\
&\overset{(2)}{\cong} p_{13*}(m_{13}^*\mathscr{O}_A(H)\otimes m_{12}^*\mathscr{O}_A(H)\otimes p_1^*\mathscr{O}_A(-H)\otimes p_2^*\mathscr{O}_A(-H))\\
&\overset{(3)}{\cong} p_{13*}(n^*\mathscr{O}_A(H)\otimes m_{23}^*\mathscr{O}_A(-H)\otimes p_3^*\mathscr{O}_A(H))\\
&\overset{(4)}{\cong} m^*\mathscr{O}_A(-H)\otimes q^*\mathscr{O}_A(H)\otimes p_{13*}m^*\mathscr{O}_A(H)\\
&\overset{(5)}{\cong} m^*\mathscr{O}_A(-H)\otimes q^*\mathscr{O}_A(H)\\
&\overset{(6)}{\cong} \mathscr{P}^\vee(-p^*H)
\end{aligned}$$

To see this,

(1) apply the definition of the Fourier-Mukai transform of a WIT_0-sheaf,
(2) substitute (4.4.7) for the Poincaré sheaf,
(3) use the fact that the divisor

$$n^*H - m_{12}^*H - m_{13}^*H - m_{23}^*H + p_1^*H + p_2^*H + p_3^*H$$

is linearly equivalent to zero, by the theorem of the cube,
(4) apply the projection formula,

(5) use that $p_{13*}m^*\mathscr{O}_A(H)$ is free of rank $\dim \mathrm{H}^0(A,\mathscr{O}_A(H)) = 1$, and hence trivial, and
(6) substitute (4.4.7) again.

By a similar calculation, one checks that $\widehat{\mathscr{O}_{\Delta'}} \cong \mathscr{P}^\vee$ and that the map

$$\mathscr{P}^\vee(-p^*H) \to \mathscr{P}^\vee$$

induced by (4.4.6) is the canonical inclusion.

We conclude that the relative Fourier-Mukai transform of $\mathscr{P}_H$ is the kernel of the restriction map (4.4.6). This proves the lemma. □

PROOF OF PROPOSITION 4.24. The relative canonical sheaf ω_q of the projection $q\colon H\times\widehat{A}\to\widehat{A}$ is $p^*\mathscr{O}_H(H)$. Hence, by (relative) Grothendieck duality, there is an isomorphism

$$\mathscr{E}xt^1_q(p^*\mathscr{O}_H(H),\mathscr{P}_H)\cong\mathscr{C}^\vee.$$

We want to identify this Ext-sheaf with $\mathscr{D}$. Firstly, there is a tautological identification

$$\mathscr{E}xt^1_q(p^*\mathscr{O}_H(H),\mathscr{P}_H)\cong\mathscr{E}xt^1_q(p^*\mathscr{O}_A(H),\mathscr{P}_H)$$

where the right hand side denotes a relative Ext-sheaf over $q\colon A\times\widehat{A}\to\widehat{A}$. Secondly, we claim there is an isomorphism

$$\mathscr{E}xt^1_q(p^*\mathscr{O}_A(H),\mathscr{P}_H)\cong\mathscr{H}om_q(\widehat{p^*\mathscr{O}_A(H)},\widehat{\mathscr{P}_H})$$

where the right hand side denotes a relative Hom-sheaf over $q\colon\widehat{A}\times\widehat{A}\to\widehat{A}$. This can be verified locally on the base, and after restricting to an affine open $U\subset\widehat{A}$, we can just repeat the argument from Proposition 4.23, using the relative Fourier-Mukai functor over U.

We obviously have $\widehat{p^*\mathscr{O}_A(H)}\cong p^*\mathscr{O}_{\widehat{A}}(-\widehat{H})$, so, by Lemma 4.25, we have

$$\mathscr{H}om_q(\widehat{p^*\mathscr{O}_A(H)},\widehat{\mathscr{P}_H})\cong q_*(p^*\mathscr{O}_{\widehat{A}}(\widehat{H})\otimes\widehat{\mathscr{P}_H})\cong\mathscr{D}$$

and we are done. □

4.5. Limit points on curves

We prove the statements from Section 4.1.4. Throughout we let $T=\operatorname{Spec}R$ be the spectrum of a regular local 1-dimensional $\mathbb{C}$-algebra, i.e. a discrete valuation ring over $\mathbb{C}$. Furthermore, we denote by $0\in T$ the unique closed point, and $U\subset T$ is the complement of 0. Thus U is the spectrum of the quotient field of R.

4.5.1. Preparation. As in Section 4.1.4, we consider a pair of maps

$$\gamma\colon T\to M \qquad\qquad \lambda\colon T\to M'$$

intersecting P and P' transversally at $p = \gamma(0)$ and $p' = \lambda(0)$, respectively. We require that they are compatible over U in the sense that the diagram

$$\begin{array}{ccc} & \overset{\gamma}{\nearrow} & M \setminus P \\ U & & \wr\Vert \\ & \underset{\lambda}{\searrow} & M' \setminus P' \end{array}$$

commutes. Let us rephrase the setup in terms of families of sheaves.

Let $\mathscr{E}$ and $\mathscr{F}$ be the T-flat sheaves on $A \times T$ and $\widehat{A} \times T$, corresponding to γ and λ, respectively. Then the restriction of $\mathscr{E}$ to U satisfies WIT_1, and its Fourier-Mukai transform agrees with $\mathscr{F}$ over U. Thus $\mathscr{F}$ is the unique T-flat quotient

$$R^1\widehat{S}(\mathscr{E}) \twoheadrightarrow \mathscr{F}$$

that is an isomorphism over U. The points $p = \gamma(0)$ and $p' = \lambda(0)$ correspond to the fibres $\mathscr{E}_0$ and $\mathscr{F}_0$ over $0 \in T$.

We want to show that (and how) $\mathscr{F}_0$ is determined by $\mathscr{E}_0$ together with the tangent vector $\gamma'(0)$. The latter can be identified with the restriction of γ to the first order infinitesimal neighbourhood

$$\operatorname{Spec} R/(t^2) \subset T$$

where $t \in R$ is a uniformizing parameter. Note that $R/(t^2)$ is isomorphic to the ring $k[\varepsilon]$ of dual numbers (where $\varepsilon^2 = 0$), so the restriction of γ to $\operatorname{Spec} R/(t^2)$ can indeed be viewed as a $k[\varepsilon]$-valued point of M, i.e. a tangent vector.

So restrict the two families $\mathscr{E}$ and $\mathscr{F}$ to $\operatorname{Spec} R/(t^2)$, and denote the restrictions by the same symbols. Our setup is then the following:

(1) $\mathscr{E}$ and $\mathscr{F}$ are sheaves on $A \otimes k[\varepsilon]$ and $\widehat{A} \otimes k[\varepsilon]$, respectively, flat over $k[\varepsilon]$.
(2) The fibres $\mathscr{E}_0$ and $\mathscr{F}_0$ at $\varepsilon = 0$ correspond to points $p \in P$ and $p' \in P'$.
(3) The two families are related by the existence of a surjection $R^1\widehat{S}(\mathscr{E}) \twoheadrightarrow \mathscr{F}$.

We will return to the (additional) transversality assumption, that the tangent vector in $T_M(p)$ corresponding to $\mathscr{E}$ is not contained in $T_P(p)$.

Note that we did need the curve T to obtain the surjection under point (3), but once we have it, we can forget about the curve and only consider its tangent.

4.5.2. Key lemma. We start by describing $\mathscr{F}_0$ in terms of $\mathscr{E}$. Recall that the tangent space to M at p can be canonically identified with the space of self extension of $\mathscr{E}_0$,

$$T_M(p) \cong \operatorname{Ext}^1_A(\mathscr{E}_0, \mathscr{E}_0).$$

The extension corresponding to the infinitesimal deformation $\mathscr{E}$ of $\mathscr{E}_0$ is the tensor product of $\mathscr{E}$ with $0 \to k \to k[\varepsilon] \to k \to 0$,

$$0 \to \mathscr{E}_0 \to \mathscr{E} \to \mathscr{E}_0 \to 0.$$

If we apply the Fourier-Mukai functor $\widehat{S}$ to this short exact sequence, we obtain a long exact sequence, part of which is the following:

$$\cdots \to \widehat{S}(\mathscr{E}_0) \xrightarrow{\delta} R^1\widehat{S}(\mathscr{E}_0) \to \cdots$$

LEMMA 4.26. *In the above situation, if the map δ is nonzero, then its cokernel is isomorphic to $\mathscr{F}_0$.*

REMARK 4.27. We will see in Remark 4.30 that the condition $\delta \neq 0$ is equivalent to the transversality condition that the tangent vector in $T_M(p)$ corresponding to $\mathscr{E}$ is not contained in $T_P(p)$.

PROOF. We have an exact sequence

$$\widehat{S}(\mathscr{E}_0) \xrightarrow{\delta} R^1\widehat{S}(\mathscr{E}_0) \to R^1\widehat{S}(\mathscr{E}) \to R^1\widehat{S}(\mathscr{E}_0) \to 0.$$

By base change, Remark A.9, the right part of this sequence is the same thing as the tensor product of $0 \to k \to k[\varepsilon] \to k \to 0$ with $R^1\widehat{S}(\mathscr{E})$, that is

$$R^1\widehat{S}(\mathscr{E})_0 \to R^1\widehat{S}(\mathscr{E}) \to R^1\widehat{S}(\mathscr{E})_0 \to 0.$$

Introducing the quotient $R^1\widehat{S}(\mathscr{E}) \twoheadrightarrow \mathscr{F}$ and using the flatness of $\mathscr{F}$, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\widehat{S}(\mathscr{E}_0) & \xrightarrow{\delta} & R^1\widehat{S}(\mathscr{E})_0 & \longrightarrow & R^1\widehat{S}(\mathscr{E}) & \longrightarrow & R^1\widehat{S}(\mathscr{E})_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F}_0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}_0 & \longrightarrow & 0
\end{array}$$

The cokernel of δ is a sheaf in M', by Proposition 4.15. But there is an induced nonzero homomorphism

$$\operatorname{Coker}(\delta) \to \mathscr{F}_0.$$

As both sheaves are stable, they are in particular simple, so any nonzero map between them is an isomorphism, and we have the result. □

We have shown that p' is determined by p and the tangent vector $\gamma'(0)$. What remains is to relate the description of p' in Lemma 4.26 to the description in Section 4.1.4. To do this we must interpret the normal space to P at p.

4.5.3. On the normal bundle to P in M. We now bring the $\mathbb{P}^2$-bundle structure $P \to \widehat{A} \times A \times \widehat{A}$ from Section 4.3.2 into the picture. Let $P(t)$ denote the fibre above $t = (x,a,y)$, and let p be a point in $P(t)$.

We want to describe the normal space to P in M at p, together with the projection

$$T_M(p) \twoheadrightarrow N_{P/M}(p).$$

The dual of this map can be identified with the inclusion

$$T_{P(t)}(p) \hookrightarrow T_M(p)$$

by the results of Mukai quoted in Section 4.2. Thus we first aim at describing the tangent space $T_{P(t)}(p)$ to the fibre $P(t)$ and its inclusion into $T_M(p)$. Note that $A\times\widehat{A}$ acts in an obvious way on $P=P_0\times A\times\widehat{A}$, and on M by twisting and translation. The actions are compatible with the embedding $P\hookrightarrow M$, hence we may, for the purpose of analysing the inclusion $T_{P(t)}(p)\hookrightarrow T_M(p)$, assume that $t=(x,0,0)$.

Let us first make some general remarks on the tangent space to a projective space.

Consider the tangent space $T_Q(\ell)$ to the projective space Q of lines in a vector space V, at a line $\ell\subset V$. Let $k[\varepsilon]$ denote the ring of dual numbers (so that $\varepsilon^2=0$). Then $T_Q(\ell)$ is the space of rank 1 submodules of $V\otimes k[\varepsilon]$, specializing to $\ell\subset V$ at $\varepsilon=0$. Let us recall how this viewpoint fits with the usual identification

$$T_Q(\ell)\cong \operatorname{Hom}_k(\ell,V/\ell).$$

For convenience, choose a nonzero vector $v\in\ell$. Evaluation at v then defines an isomorphism from $\operatorname{Hom}_k(\ell,V/\ell)$ to V/ℓ, so we must understand how elements of V/ℓ correspond to rank 1 submodules of $V\otimes k[\varepsilon]$. Namely, a vector $\overline{w}\in V/\ell$ corresponds to the submodule of $V\otimes k[\varepsilon]$ generated by $v+\varepsilon w$, where $w\in V$ is any lifting of $\overline{w}$. Reformulating this, there is a surjection

$$V\twoheadrightarrow T_Q(\ell)$$

with kernel ℓ, sending a vector w to the submodule generated by $v+\varepsilon w$. The surjection depends on the choice of $v\in\ell$, but the induced inclusion $\mathbb{P}(T_A(\ell))\subset\mathbb{P}(V)$ does not, and is just the canonical inclusion mentioned in Remark 4.1.

Applying this to our situation, we get the following: The fibre $P(t)$ is the projective space of lines in $\operatorname{Ext}^1_A(\mathscr{P}_x|_H,\mathscr{O}_A(H))$. The point $p\in P(t)$ is a sheaf $\mathscr{E}_0$ defined by an extension

$$0\to\mathscr{O}_A(H)\xrightarrow{f}\mathscr{E}_0\xrightarrow{g}\mathscr{P}_x|_H\to 0. \tag{4.5.1}$$

and the choice of a vector $v\in\ell$ in the discussion above corresponds to fixing such an extension. Thus there is an induced surjection

$$\operatorname{Ext}^1_A(\mathscr{P}_x|_H,\mathscr{O}_A(H))\twoheadrightarrow T_{P(t)}(p).$$

On the other hand, the tangent space to M at p is canonically isomorphic to the space $\operatorname{Ext}^1_A(\mathscr{E}_0,\mathscr{E}_0)$ of self extensions of $\mathscr{E}_0$. Thus we can form a commutative diagram

$$\begin{array}{ccc}\operatorname{Ext}^1_A(\mathscr{P}_x|_H,\mathscr{O}_A(H)) & \to & \operatorname{Ext}^1_A(\mathscr{E}_0,\mathscr{E}_0)\\ \downarrow & & \wr\| \\ T_{P(t)}(p) & \subset & T_M(p)\end{array} \tag{4.5.2}$$

where the top map is defined by composition. We want to identify that map. There is an obvious candidate: As $\operatorname{Ext}^1_A(-,-)$ is a bifunctor, the two maps f and g in the extension (4.5.1) induce a map that we claim fits into the diagram.

LEMMA 4.28. *The top map in diagram* (4.5.2) *is induced by the two maps* f *and* g *in extension* (4.5.1).

PROOF. On an arbitrary scheme X, let

$$\begin{array}{lccccccccc} \xi: & 0 & \to & \mathscr{A} & \xrightarrow{f} & \mathscr{E}_0 & \xrightarrow{g} & \mathscr{B} & \to & 0 \\ \zeta: & 0 & \to & \mathscr{A} & \to & \mathscr{F}_0 & \to & \mathscr{B} & \to & 0 \end{array}$$

be two extensions of $\mathscr{O}_X$-modules. The two maps f and g induce a map

$$\operatorname{Ext}^1_X(\mathscr{B},\mathscr{A}) \to \operatorname{Ext}^1_X(\mathscr{E}_0,\mathscr{E}_0), \tag{4.5.3}$$

which we want to apply to the other extension ζ. To do this, tensor the extension

$$\xi + \varepsilon\zeta \in \operatorname{Ext}^1_{X\otimes k[\varepsilon]}(\mathscr{B}_{k[\varepsilon]},\mathscr{A}_{k[\varepsilon]})$$

with $0 \to k \to k[\varepsilon] \to k \to 0$ to get the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{A} & \longrightarrow & \mathscr{E}_0 & \longrightarrow & \mathscr{B} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{A}\otimes k[\varepsilon] & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{B}\otimes k[\varepsilon] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{A} & \longrightarrow & \mathscr{E}_0 & \longrightarrow & \mathscr{B} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where the middle row is $\xi + \varepsilon\zeta$. Then the middle *column* is the image of ζ under the map (4.5.3). I do not know an elegant proof of this, but it can be checked by a straight forward and boring calculation.

The lemma is an almost immediate consequence: Returning to our situation, where $\mathscr{A} = \mathscr{O}_A(H)$ and $\mathscr{B} = \mathscr{P}_x|_H$, let $\zeta \in \operatorname{Ext}^1_A(\mathscr{P}_x|_H, \mathscr{O}_A(H))$ be an element in the upper left corner of diagram (4.5.2). Tracing the diagram downwards, ζ is sent to the submodule of $\operatorname{Ext}^1_A(\mathscr{P}_x|_H, \mathscr{O}_A(H)) \otimes k[\varepsilon]$ generated by $\xi + \varepsilon\zeta$. Identifying

$$\operatorname{Ext}^1_A\big(\mathscr{P}_x|_H, \mathscr{O}_A(H)\big) \otimes k[\varepsilon] \cong \operatorname{Ext}^1_{A\otimes k[\varepsilon]}\big(\mathscr{P}_x|_H \otimes k[\varepsilon], \mathscr{O}_A(H)\otimes k[\varepsilon]\big),$$

we view $\xi + \varepsilon\zeta$ as an extension

$$0 \longrightarrow \mathscr{O}_A(H)\otimes k[\varepsilon] \longrightarrow \mathscr{E} \longrightarrow \mathscr{P}_x|_H \otimes k[\varepsilon] \longrightarrow 0$$

Following this element through the inclusion $T_{P(t)}(p) \subset T_M(p)$ at the bottom of diagram (4.5.2), we arrive at the infinitesimal deformation $\mathscr{E}$ of $\mathscr{E}_0$. Finally, the corresponding element in $\operatorname{Ext}^1_A(\mathscr{E}_0,\mathscr{E}_0)$ is the tensor product of $\mathscr{E}$ with the sequence $0 \to k \to k[\varepsilon] \to k \to 0$. As we have just seen, this tensor product equals the image of ζ under the map $\operatorname{Ext}^1_A(\mathscr{P}_x|_H, \mathscr{O}_A(H)) \to \operatorname{Ext}^1_A(\mathscr{E}_0, \mathscr{E}_0)$ induced by f and g. □

The lemma gives us control over the inclusion $T_{P(t)}(p) \hookrightarrow T_M(p)$. As we noted in the introduction to this section, we find the projection map $T_M(p) \twoheadrightarrow N_{P/M}(p)$ by dualizing. Now, the dual of diagram (4.5.2) can be written

$$\begin{array}{ccc} \operatorname{Ext}^1_A(\mathscr{O}_A(H), \mathscr{P}_x|_H) & \leftarrow & \operatorname{Ext}^1_A(\mathscr{E}_0, \mathscr{E}_0) \\ \cup & & \wr\| \\ N_{P/M}(p) & \leftarrow & T_M(p) \end{array} \tag{4.5.4}$$

using Serre duality. One remark must be made: We have, on the one hand, identified $T_M(p)$ with its dual, using the symplectic structure on M. On the other hand, we have identified $\operatorname{Ext}^1_A(\mathscr{E}_0, \mathscr{E}_0)$ with its dual, using Serre duality. These identifications are the same, by definition of the symplectic structure on M, as sketched in Section 1.1.3. Thus the isomorphisms in the rightmost columns of diagrams (4.5.2) and (4.5.4) are the same.

The map on the top of diagram (4.5.4) is again induced by the two maps f and g in the extension (4.5.1); this is just a reformulation of Lemma 4.28.

4.5.4. Conclusion. By collecting the loose ends, we can now verify the claims from Section 4.1.4. Let us summarize once again: Starting from the curve $\gamma\colon T \to M$ we arrived at the two infinitesimal families $\mathscr{E}$ and $\mathscr{F}$ of sheaves on A and $\widehat{A}$, respectively, with central fibres $\mathscr{E}_0$ and $\mathscr{F}_0$ corresponding to points $p \in P$ and $p' \in P'$.

We continue to assume that p is in the fibre of P above $(x, 0, 0) \in \widehat{A} \times A \times \widehat{A}$. We will, at the end of this section, take care of the necessary translation and twisting to make the reduction to this case.

We view the tangent vector $\gamma'(0)$ as the extension

$$0 \to \mathscr{E}_0 \to \mathscr{E} \to \mathscr{E}_0 \to 0. \tag{4.5.5}$$

In Lemma 4.26, we showed that, by applying the Fourier-Mukai functor to (4.5.5), the sheaf $\mathscr{F}_0$ appeared as the cokernel

$$\widehat{S}(\mathscr{E}_0) \to R^1\widehat{S}(\mathscr{E}_0) \to \mathscr{F}_0 \to 0. \tag{4.5.6}$$

What remains is to relate this construction to the diagram (4.5.4). For this we use Proposition 4.23 to replace $\operatorname{Ext}^1_A(\mathscr{O}_A(H), \mathscr{P}_x|_H)$ in that diagram with $\operatorname{Hom}_{\widehat{A}}(\widehat{\mathscr{O}_A(H)}, \widehat{\mathscr{P}_x|_H})$.

PROPOSITION 4.29. *Let*

$$0 \to \mathscr{P}_x|_H \to \mathscr{G} \to \mathscr{O}_A(H) \to 0$$

be the image of the extension (4.5.5) *under the top map in diagram* (4.5.4). *Then $\mathscr{F}_0$ is the Fourier-Mukai transform of $\mathscr{G}$.*

PROOF. Let us consider the diagram

$$\begin{array}{ccc} & \operatorname{Ext}^1_A(\mathscr{E}_0,\mathscr{E}_0) & \\ \swarrow^{r} & & \searrow^{s} \\ \operatorname{Ext}^1_A(\mathscr{O}_A(H),\mathscr{P}_x|_H) \cong \operatorname{Hom}_{\widehat{A}}(\widehat{\mathscr{O}_A(H)},\widehat{\mathscr{P}_x|_H}) & \cong & \operatorname{Hom}_{\widehat{A}}(\widehat{S}(\mathscr{E}_0),R^1\widehat{S}(\mathscr{E}_0)) \end{array} \tag{4.5.7}$$

where r is the top map in (4.5.4) and s is the map sending an extension to the induced boundary map

$$\delta\colon \widehat{S}(\mathscr{E}_0)\to R^1\widehat{S}(\mathscr{E}_0).$$

Furthermore, the isomorphisms at the bottom come from Proposition 4.23 and Lemma 4.12. We claim that the diagram commutes.

Let us trace an element $\xi\in\operatorname{Ext}^1_A(\mathscr{E}_0,\mathscr{E}_0)$ through the various maps involved: Form the diagram

$$\begin{array}{rccccccccc} \xi: & 0 & \to & \mathscr{E}_0 & \to & \mathscr{E} & \to & \mathscr{E}_0 & \to & 0 \\ & & & \| & & \uparrow & & \uparrow & & \\ f^*\xi: & 0 & \to & \mathscr{E}_0 & \to & \mathscr{E}' & \to & \mathscr{O}_A(H) & \to & 0 \\ & & & \downarrow & & \downarrow & & \| & & \\ g_*f^*\xi: & 0 & \to & \mathscr{P}_x|_H & \to & \mathscr{G} & \to & \mathscr{O}_A(H) & \to & 0 \end{array}$$

where the top right square is a pullback and the bottom left square is a pushout. Then r sends ξ to $g_*f^*\xi$. Apply $\widehat{S}$ to the last diagram to get the commutative square

$$\begin{array}{ccc} \widehat{S}(\mathscr{E}) & \xrightarrow{\delta} & R^1\widehat{S}(\mathscr{E}) \\ \wr\| & & \wr\| \\ \widehat{S}(\mathscr{O}_A(H)) & \xrightarrow{\delta'} & R^1\widehat{S}(\mathscr{P}_x|_H) \end{array} \tag{4.5.8}$$

where the vertical isomorphisms are induced by f and g, respectively, and the horizontal maps are the boundary maps in the long exact sequences induced by ξ and $g_*f^*\xi$, respectively.

Now, via the isomorphisms at the bottom of diagram (4.5.7), the element

$$g_*f^*\xi\in\operatorname{Ext}^1_A(\mathscr{O}_A(H),\mathscr{P}_x|_H)$$

corresponds to

$$\delta'\in\operatorname{Hom}_{\widehat{A}}(\widehat{\mathscr{O}_A(H)},\widehat{\mathscr{P}_x|_H}),$$

which by the commutativity of (4.5.8) again corresponds to

$$\delta\in\operatorname{Hom}_{\widehat{A}}(\widehat{S}(\mathscr{E}_0),R^1\widehat{S}(\mathscr{E}_0)).$$

By definition of s, we have $s(\xi)=\delta$, so diagram (4.5.7) commutes.

The result is now easy to deduce from Lemma 4.26: By that lemma, the cokernel of δ is $\mathscr{F}_0$. But $\mathscr{G}$ is automatically WIT_1, and the cokernel of δ' is $\widehat{\mathscr{G}}$. So we have the result. □

REMARK 4.30. With notation as in the proof, the assumption in Lemma 4.26, that $\delta = s(\xi)$ is nonzero, is equivalent with $r(\xi)$ being nonzero by the commutative diagram (4.5.7). Thus we see, using diagram (4.5.4), that the assumption $\delta \neq 0$ is equivalent to ξ being a tangent vector transversal to P.

Finally, let $p = \gamma(0)$ be a point in the fibre $P(t)$ over $t = (x,a,y)$, where a and y are no longer necessarily zero. The point p is an extension

$$\xi : 0 \to \mathscr{O}_A(H) \to \mathscr{E}_0 \to \mathscr{P}_x|_H \to 0,$$

modulo scalar multiplication, and the corresponding sheaf in M is $\mathscr{E}_0' = T_a^*\mathscr{E}_0 \otimes \mathscr{P}_y$. Let $\mathscr{E}'$ denote the infinitesimal deformation of $\mathscr{E}_0'$ corresponding to the tangent $\gamma'(0)$, and let $\mathscr{F}'$ be the infinitesimal deformation of a sheaf $\mathscr{F}_0' \in P'$, related to $\mathscr{E}'$ by a surjection

$$R^1\widehat{S}(\mathscr{E}') \twoheadrightarrow \mathscr{F}'.$$

Using the action of $A \times \widehat{A}$, we can move the curve γ, such that p is moved to the fibre above $(x,0,0)$. In terms of families of sheaves, this amounts to the following: Let $\mathscr{E} = T_{-a}^*\mathscr{E}' \otimes \mathscr{P}_{-y}$, which is an infinitesimal deformation of $\mathscr{E}_0$. There is an induced surjection

$$R^1\widehat{S}(\mathscr{E}) \twoheadrightarrow \mathscr{F},$$

where $\mathscr{F} = T_{-y}^*\mathscr{F}' \otimes \mathscr{P}_a$, by Proposition A.10 (note the sign change). We can now apply Proposition 4.29, to conclude that the fibre $\mathscr{F}_0 = T_{-y}^*\mathscr{F}_0' \otimes \mathscr{P}_a$ is the Fourier-Mukai transform of $\mathscr{G}$, with notation as in the proposition. Hence

$$\mathscr{F}_0' \cong T_y^*\widehat{\mathscr{G}} \otimes \mathscr{P}_{-a},$$

which concludes the proof of the claims from Section 4.1.4.

4.6. Consequences

4.6.1. Base loci of ψ and ψ^{-1}. By construction, the birational map ψ is regular outside P. On the other hand, if ψ were regular at $p \in P$, and γ were a curve intersection P transversally at $p = \gamma(0)$, then the point

$$p' = \lim_{t\to 0}(\psi \circ \lambda)(t)$$

in P' would be independent of the choice of γ, and in particular of its tangent at p. But it is not, by the description of ψ in Section 4.1.4, which was verified in the previous section. So P is precisely the base locus of ψ.

Along the same lines we can show that the base locus of ψ^{-1} is exactly P'. But this also follows from what we have just shown, if we remember that a birational map of symplectic varieties is biregular wherever it is defined. (This was stated in Proposition 1.11 for irreducible symplectic varieties only, but the same result holds whenever the canonical bundles of the two varieties are trivial, or in fact just nef.)

REMARK 4.31. In fact, we just proved that the base locus of a birational map satisfying the conclusions of Proposition 4.2 is exactly the bundle P.

4.6.2. Restriction to the Kummer variety. Let us identify M with $A^{[4]} \times \widehat{A}$, and consider the following diagram:

$$\begin{array}{ccc} H^{(4)} \times A & \hookrightarrow & A^{[4]} \\ \downarrow{\scriptstyle \sigma\times 1_A} & & \downarrow{\scriptstyle \sigma} \\ A \times A & \longrightarrow & A \end{array}$$

The two σ's both denote addition maps, the top map sends a pair (Z,a) to $T_a^{-1}Z \in A^{[4]}$, and the map at the bottom is

$$(a,b) \mapsto a - 4b.$$

The diagram clearly commutes, and the product of the top map with $\widehat{A}$ can be identified with $P \hookrightarrow M$. We note that the other three maps are locally trivial in the étale topology, i.e. locally isomorphic to a projection from a product.

We now restrict the diagram to the point $0 \in A$ (in the lower right corner). On the right we get, by definition, the Kummer variety K^4A, whereas on the left we find a $\mathbb{P}^2$-bundle

$$Q \to A$$

which is the restriction of $H^{(4)} \times A \to A \times A$ to

$$A \hookrightarrow A \times A, \quad a \mapsto (4a, a).$$

It follows that $P \subset M$ intersects the Kummer variety in the $\mathbb{P}^2$-bundle Q, and that this intersection is transversal (by the local triviality of the three lower maps in the diagram, pointed out above).

Similarly, one can check that $P' \subset M'$ intersects $K_{\widehat{A}}(0,\widehat{C},-1)$ transversally in a $\mathbb{P}^2$-bundle

$$Q' \to \widehat{A}.$$

It follows from the duality between P and P' that Q and Q' are dual projective bundles, when their base spaces are identified via $\phi_H : A \xrightarrow{\sim} \widehat{A}$.

Now consider the birational map $\phi : K^4A \overset{\sim}{\dashrightarrow} K_{\widehat{A}}(0,\widehat{C},-1)$. A curve $\gamma\colon T \to K^4A$, that intersects Q transversally in $\gamma(0)$, can be viewed as a curve on M intersecting P transversally, and hence the description of the point

$$\lim_{t\to 0}(\phi \circ \gamma)(t) = \lim_{t\to 0}(\psi \circ \gamma)(t)$$

in $K_{\widehat{A}}(0,\widehat{C},-1)$ from Section 4.1.4 applies. It follows that also ϕ satisfies the conclusion of Proposition 4.2 (with $Y = A$, $P = Q$ and $P' = Q'$). In particular, the base loci of ϕ and ϕ^{-1} are exactly the $\mathbb{P}^2$-bundles Q and Q', respectively.

4.6.3. The fundamental fibration.

THEOREM 4.32. *The base locus of the fundamental fibration*

$$f\colon K^4A \dashrightarrow |\widehat{C}| \cong \mathbb{P}^3$$

equals the $\mathbb{P}^2$*-bundle* $Q \to A$ *consisting of subschemes* $Z \in K^4A$ *contained in a translate of* H.

PROOF. The fundamental fibration has a factorization $f = f' \circ \phi$, where ϕ is the birational map from K^4A to $K_{\widehat{A}}(0,\widehat{C},-1)$, and f' is the regular fibration on the latter. Since ϕ is regular outside Q, so is f, and we must show that every point of Q is a base point for f.

Fix a point q in a fibre $Q(a)$ of $Q \to A$, and let $x = \phi_H(a)$. Using that $\phi\colon K^4A \overset{\sim}{\dashrightarrow} K_{\widehat{A}}(0,\widehat{C},-1)$ satisfies the conclusion of Proposition 4.2, we have the following: The locus of points in Q' arising as limits

$$\lim_{t\to 0}(\phi\circ\gamma)(t) \in Q'(x), \tag{4.6.1}$$

for varying curves γ intersecting Q transversally in $q = \gamma(0)$, is a hyperplane in $Q'(x)$. More precisely, it is the locus of points in $Q'(x) \cong Q(a)^\vee$ corresponding to hyperplanes in $Q(a)$ containing q.

The fibre $Q'(x)$ is the linear system of divisors $D \in |\widehat{C}|$ containing x. The support of the corresponding sheaf $\mathscr{F} \in M'$ is precisely that curve D. Hence the fibration

$$f'\colon K_{\widehat{A}}(0,\widehat{C},-1) \to |\widehat{C}|$$

sends the sheaf corresponding to D also to the same curve D.

By what we said above, the locus of points in $Q'(x) \subset |\widehat{C}|$, arising as limits (4.6.1), has codimension 2 in $|\widehat{C}|$, i.e. it is a pencil. It follows that the locus of points in $|\widehat{C}|$ of the form

$$\lim_{t\to 0}(f\circ\gamma)(t) \in |\widehat{C}|,$$

for varying γ intersecting Q transversally in q, is the same pencil. In particular, the fibration f is not defined at q. □

COROLLARY 4.33. *The two (projective, irreducible) symplectic varieties* K^4A *and* $K_{\widehat{A}}(0,\widehat{C},-1)$ *are birational, but not isomorphic.*

PROOF. Since A has Picard number 1, the fundamental fibration is the *unique* rational fibration on K^4A, by Proposition 1.17. By the theorem, the fundamental fibration has nonempty base locus, so K^4A admits no regular fibration. On the other hand, $K_{\widehat{A}}(0,\widehat{C},-1)$ does admit a regular fibration, given by the (Fitting) support map. Hence they are not isomorphic. □

APPENDIX A

Abelian varieties and the Fourier-Mukai transform

The Fourier-Mukai transform, introduced by Mukai in the 80s, is a way of relating sheaves on an abelian variety, or more generally an abelian scheme, to sheaves on its dual abelian variety or scheme. For convenience, we summarize the main results needed in this text. The constructions and results are collected from Mukai's two papers [**25, 27**]. Other references for Fourier-Mukai transforms are the books by Polishchuk [**32**] and Birkenhake and Lange [**2**].

Let $X \to S$ be an abelian scheme of relative dimension g over some base scheme S, and let $\widehat{X} \to S$ denote its dual. Let

$$X \xleftarrow{p} X \times_S \widehat{X} \xrightarrow{q} X$$

denote the projections, and let $\mathscr{P}$ be the Poincaré sheaf on $X \times_S \widehat{X}$, normalized such that the restrictions of $\mathscr{P}$ to $X \times 0$ and $0 \times \widehat{X}$ are trivial. We denote by $\omega_{X/S}$ the relative canonical sheaf on X over S.

DEFINITION A.1. Let $\widehat{S}$ denote the functor taking a (not necessarily quasi-coherent) $\mathscr{O}_X$-module $\mathscr{E}$ to the $\mathscr{O}_{\widehat{X}}$-module

$$\widehat{S}(\mathscr{E}) = q_*(p^*(\mathscr{E}) \otimes \mathscr{P}).$$

Also denote by S the functor taking $\mathscr{O}_{\widehat{X}}$-modules to $\mathscr{O}_X$-modules obtained by interchanging the roles of X and $\widehat{X}$.

Denote by $D(X)$ the derived category of X, i.e. the derived category of the triangulated category of complexes of (not necessarily quasi-coherent) $\mathscr{O}_X$-modules. Writing $\mathscr{K}$ for an object of $D(X)$, i.e. a complex of $\mathscr{O}_X$-modules, we have the (left) shift functor denoted by $\mathscr{K} \mapsto \mathscr{K}[n]$.

THEOREM A.2 (Mukai [**25, 27**]). The derived functor

$$R\widehat{S}\colon D(X) \to D(\widehat{X})$$

of $\widehat{S}$ is an equivalence of categories. In fact, we have a natural isomorphism

$$(RS \circ R\widehat{S})(\mathscr{K}) \cong (-1_X)^*(\mathscr{K} \otimes \omega_{X/T}^{-1})[-g].$$

REMARK A.3. Whenever X is an abelian variety over the spectrum $S = \operatorname{Spec} k$ of a field, or more generally, X is a product of such an abelian variety with an arbitrary base scheme S, the relative canonical sheaf $\omega_{X/S}$ is trivial. This simplifies the autofunctor on the right hand side in the above theorem.

REMARK A.4. The functor $R\widehat{S}$ takes complexes that are bounded, bounded below or quasi-coherent to complexes of the same kind, and induces equivalences of the corresponding full subcategories of $D(X)$ and $D(\widehat{X})$ [**25**].

To any sheaf $\mathscr{E}$ on X we can associate (up to quasi-isomorphism) the complex $R\widehat{S}(\mathscr{E})$ of sheaves on $\widehat{X}$. A favourable situation would be when $R\widehat{S}(\mathscr{E})$ is in fact just a sheaf, or more precisely, has cohomology concentrated in one degree i.

DEFINITION A.5. A sheaf $\mathscr{E}$ on X satisfies the *weak index theorem* (WIT) if there exists an integer i such that

$$R^p\widehat{S}(\mathscr{E}) = 0 \quad \text{for all } p \neq i.$$

The integer i is then called the *index* of $\mathscr{E}$. We also say that $\mathscr{E}$ is a WIT_i-sheaf.

Here, $R^p\widehat{S}$ denote the classical derived functors, i.e. $R^p\widehat{S}(\mathscr{E})$ are the cohomology sheaves of the complex $R\widehat{S}(\mathscr{E})$. We follow the convention that the zero sheaf satisfies WIT_i for all i.

DEFINITION A.6. The *Fourier-Mukai transform* of a sheaf $\mathscr{E}$ on A, satisfying the weak index theorem with index i, is the sheaf

$$\widehat{\mathscr{E}} = R^i\widehat{S}(\mathscr{E}).$$

COROLLARY A.7 (of Theorem A.2). If $\mathscr{E}$ satisfies the weak index theorem with index i, then $\widehat{\mathscr{E}}$ satisfies the weak index theorem with index $g-i$ and there is a natural isomorphism

$$\widehat{\widehat{\mathscr{E}}} \cong (-1_A)^*\mathscr{E}.$$

We may consider a sheaf $\mathscr{E}$ on X/S as a family of sheaves $\mathscr{E}_s = \mathscr{E} \otimes k(s)$ on the fibres $X_s = X \otimes k(s)$, parametrized by $s \in S$. We have the following base change result:

THEOREM A.8 (Mukai [**27**]). Let $\mathscr{E}$ be a sheaf on X, flat over S.

(1) The locus of points $s \in S$ such that $\mathscr{E}_s$ satisfies WIT is open, and the index i is locally constant.
(2) If $\mathscr{E}_s$ satisfies WIT_i for all $s \in S$, then $\mathscr{E}$ satisfies WIT_i, its Fourier-Mukai transform $\widehat{\mathscr{E}}$ is flat over S, and for every morphism $f : T \to S$ we have

$$f^*(\widehat{\mathscr{E}}) \cong \widehat{f^*(\mathscr{E})}.$$

REMARK A.9. The formation of the top non-vanishing $R^i\widehat{S}(\mathscr{E})$ (i.e. i is the highest index such that this sheaf is nonzero) always commute with base change, even when $\mathscr{E}$ does not satisfy WIT. This follows from the same property for the top non-vanishing higher direct image $R^i p_*$.

Let us now consider the case of an abelian variety A over a field k. We identify the dual abelian variety $\widehat{A}$, as a group, with the group $\mathrm{Pic}^0(A)$ of homogeneous invertible sheaves on A. If $x \in \widehat{A}$ is a point, we denote the corresponding invertible sheaf by $\mathscr{P}_x$. A useful fact is that the Fourier-Mukai functor exchanges the operations of translation and tensoring by a homogeneous invertible sheaf.

PROPOSITION A.10 (Mukai [**25**]). For each $a \in A$ and $x \in \widehat{A}$ there are natural isomorphisms

$$R\widehat{S}(T_a^*\mathscr{K}) \cong R\widehat{S}(\mathscr{K}) \otimes \mathscr{P}_{-a} \qquad R\widehat{S}(\mathscr{K} \otimes \mathscr{P}_x) \cong T_x^* R\widehat{S}(\mathscr{K})$$

of functors in $\mathscr{K}$.

DEFINITION A.11. A sheaf $\mathscr{E}$ satisfies the *index theorem* (IT) if there exists an integer i such that, for all $x \in \widehat{A}$, we have

$$H^p(A, \mathscr{E} \otimes \mathscr{P}_x) = 0 \quad \text{for all } p \neq i.$$

The integer i is then called the *index* of $\mathscr{E}$, and we say that $\mathscr{E}$ is an IT_i-sheaf.

REMARK A.12. By the base change theorem in cohomology, an IT_i-sheaf $\mathscr{E}$ is a WIT_i-sheaf, and the fibres of its Fourier-Mukai transform are

$$\widehat{\mathscr{E}}(x) \cong H^i(A, \mathscr{E} \otimes \mathscr{P}_x).$$

Since the Euler characteristic of $\mathscr{E} \otimes \mathscr{P}_x$ is independent of x, the fibre dimension of $\widehat{\mathscr{E}}$ is constant whenever $\mathscr{E}$ satisfy IT_i. Thus $\widehat{\mathscr{E}}$ is locally free.

EXAMPLE A.13. For each $a \in A$, the skyscraper sheaf $k(a)$ satisfies IT_0, and by direct computation,

$$\widehat{k(a)} \cong \mathscr{P}_a.$$

Hence, by Corollary A.7, the sheaf $\mathscr{P}_a$ satisfies WIT_2 and

$$\widehat{\mathscr{P}_a} \cong k(-a).$$

Although the index and weak index "theorems" are taken as axioms in this context, there *is* a theorem by Mumford [**28**, §16], saying that every non-degenerate invertible sheaf satisfies the index theorem (with some index). For instance, an ample invertible sheaf has index 0.

Finally, consider the case of an abelian variety A defined over the field of complex numbers. We may then associate to a sheaf or complex of sheaves on A its Chern character in the cohomology ring $H^*(A, \mathbb{Z})$. The last result of Mukai we want to quote, is the relation between the Chern character of a sheaf and that of its Fourier-Mukai transform. For this, recall the canonical duality

$$H^p(A, \mathbb{Z}) \cong H^p(\widehat{A}, \mathbb{Z})^\vee$$

between the cohomology groups of A and its dual, which through Poincaré duality gives an isomorphism

$$H^p(A,\mathbb{Z}) \cong H^{2g-p}(\widehat{A},\mathbb{Z}) \tag{A.1}$$

between the cohomology groups of complementary degree. Suppressing this isomorphism, and denoting by ch^p the component of the Chern character in degree $2p$, Mukai found the following:

THEOREM A.14 (Mukai [**27**]). Let $\mathscr{K}$ denote a complex of sheaves on a complex abelian variety. Suppressing the Poincaré isomorphism (A.1) we have

$$\mathrm{ch}^p(R\widehat{S}(\mathscr{K})) = (-1)^p\,\mathrm{ch}^{g-p}(\mathscr{K}).$$

In particular, if $\mathscr{E}$ is a sheaf satisfying WIT with index i, we have

$$\mathrm{ch}^p(\widehat{\mathscr{E}}) = (-1)^{p+i}\,\mathrm{ch}^{g-p}(\mathscr{E}).$$

APPENDIX B

Moduli of sheaves on an abelian surface

Let A be an abelian surface and fix a polarization H.

B.1. Stability

We want to spell out the meaning of stability with respect to H. The conditions for semi-stability are obtained by changing the strict inequalities to non-strict ones.

In this text, stability means Gieseker stability, which is measured by the reduced Hilbert polynomial. Recall that the reduced Hilbert polynomial is the monic polynomial obtained from the usual Hilbert polynomial by dividing with its leading coefficient. By Riemann-Roch we find that the Hilbert polynomial of a sheaf $\mathscr{E}$ on A is

$$\begin{aligned}\chi(\mathscr{E}(n)) &= \int \mathrm{ch}(\mathscr{E}(n)) \\ &= \left(\frac{H^2}{2} r(\mathscr{E})\right) n^2 + (c_1(\mathscr{E}).H)\, n + \chi(\mathscr{E}).\end{aligned}$$

where r, c_1 and χ denote the rank, first Chern class and Euler characteristic. Thus, if we let the degree $\deg(\mathscr{E})$ of $\mathscr{E}$ (with respect to H) be the intersection number $c_1(\mathscr{E}).H$, we have the following:

(1) A torsion free sheaf $\mathscr{E}$ is stable if and only if, for every proper subsheaf $\mathscr{E}' \subset \mathscr{E}$,

$$\frac{\deg(\mathscr{E}')}{r(\mathscr{E}')} \leq \frac{\deg(\mathscr{E})}{r(\mathscr{E})} \quad \text{and, on equality,} \quad \frac{\chi(\mathscr{E}')}{r(\mathscr{E}')} < \frac{\chi(\mathscr{E})}{r(\mathscr{E})}.$$

(2) A torsion sheaf $\mathscr{E}$ of pure dimension 1 is stable if and only if, for every proper subsheaf $\mathscr{E}' \subset \mathscr{E}$,

$$\frac{\chi(\mathscr{E}')}{\deg(\mathscr{E}')} < \frac{\chi(\mathscr{E})}{\deg(\mathscr{E})}.$$

Note in particular that point (1) expresses exactly how Gieseker stability refines μ-stability.

B.2. The geometry of the moduli spaces

Fixing a rank $r \geq 0$, first Chern class $c_1 \in \mathrm{NS}(A)$ and Euler characteristic χ, we denote by $M_A(r, c_1, \chi)$ the (Simpson) moduli space of stable sheaves with the given invariants.

By results of Mukai, briefly recalled in Section 1.1.3, the moduli space $M_A(r,c_1,\chi)$ is nonsingular and carries a symplectic form. Moreover, if stability and semi-stability are equivalent — as is the case for the invariants r, c_1 and χ considered in this text — then $M_A(r,c_1,\chi)$ is projective. It is not *irreducibly* symplectic, but, under certain assumptions, Yoshioka found that its Bogomolov irreducible factors are the two abelian surfaces A and $\widehat{A}$, together with an irreducible symplectic variety, here denoted $K_A(r,c_1,\chi)$. The relation between M_A and K_A is analogous to the relation between the Hilbert scheme and the Kummer variety.

More precisely, Yoshioka [**38**] defines a (regular) map

$$\alpha\colon M_A(r,c_1,\chi) \to A \times \widehat{A} \tag{B.1}$$

that can be described at the level of sets as follows, except that we take the liberty to make a sign change: Choose a representative $\mathscr{L} \in \mathrm{Pic}(A)$ in the class c_1, and also a representative $\mathscr{L}' \in \mathrm{Pic}(\widehat{A})$ in the class corresponding to c_1 via Poincaré duality (A.1). Then define $\alpha = (\delta, \widehat{\delta})$, where

$$\delta(\mathscr{F}) = \det(R\widehat{S}(\mathscr{F}))^{-1} \otimes \mathscr{L}'^{-1}$$
$$\widehat{\delta}(\mathscr{F}) = \det(\mathscr{F}) \otimes \mathscr{L}^{-1}.$$

Note that $\widehat{\delta}(\mathscr{F})$ is an element of $\mathrm{Pic}^0(A) = \widehat{A}$ and, by Theorem A.14, $\delta(\mathscr{F})$ is an element of $\mathrm{Pic}^0(\widehat{A}) = A$.

THEOREM B.1 (Yoshioka [**38**]). *Assume the triple (r,c_1,χ) is primitive in the even cohomology $H^{2*}(A,\mathbb{Z})$, and that stability and semi-stability are equivalent conditions on sheaves with these invariants. Furthermore assume that the polarization H is generic. If the dimension of $M_A(r,c_1,\chi)$ is at least* 8, *then*

(1) *$M_A(r,c_1,\chi)$ is deformation equivalent to $A^{[n]} \times \widehat{A}$ for suitable n.*
(2) *The map α in* (B.1) *is locally trivial in the étale topology.*
(3) *A fibre $K_A(r,c_1,\chi)$ of the map α is deformation equivalent to the Kummer variety K^nA. In particular, $K_A(r,c_1,\chi)$ is an irreducible symplectic variety.*

In this text, the choice of polarization H does not matter, so the genericity hypothesis is of no importance. We remark, however, that when A has Picard number one, every polarization is generic.

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