Local aspects of geometric invariant theory

Martin G. Gulbrandsen

Royal Institute of Technology, Stockholm, Sweden
E-mail address: gulbr@kth.se

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CHAPTER 1

Preview: Group varieties and actions

A group variety is the algebro-geometric analogue of a Lie group. Thus a group variety is a (not necessarily irreducible) variety G that is also a group, such that the group law

$$\mu\colon G \times G \to G, \quad (g, h) \mapsto gh$$

and the inverse

$$\iota\colon G \to G, \quad g \mapsto g^{-1}$$

are regular maps (we will consider also group *schemes*, but in this introduction we stick to varieties). The identity element of the group is a point denoted $e \in G$.

Example 1.1. Any finite group can be viewed as a group variety.

Example 1.2. The affine line $\mathbf{A}^1$ is a group variety under addition, and $\mathbf{A}^1 \setminus \{0\}$ is a group variety under multiplication. When viewed as group varieties, these are usually denoted $\mathbf{G}_a$ and $\mathbf{G}_m$ ("a" for additive and "m" for multiplicative).

An *action* of a group variety G on a variety X is a morphism

$$G \times X \to X, \quad (g, x) \mapsto gx \tag{1.1}$$

which is an action of the underlying group of G on the underlying set of points of X. Thus we require

$$ex = x, \quad g(hx) = (gh)x$$

for all $x \in X$ and $g, h \in G$.

Example 1.3. Multiplication defines an action

$$\mathbf{G}_m \times \mathbf{A}^1 \to \mathbf{A}^1$$

of the multiplicative group $\mathbf{G}_m$ on the affine line $\mathbf{A}^1$.

The main theme in these notes is that of quotients, i.e. a variety associated to an action (1.1) that deserves the name X/G. Ideally, its points should correspond to orbits in X, although we will see that it is useful to weaken this requirement. We focus here on local questions, so we assume that $X = \operatorname{Spec} A$ is affine. Viewing A as the ring of regular functions on X, it is easy to suggest a candidate quotient: A regular function on X/G should be the same thing as a regular function on X

that is constant on all orbits. These functions form a ring, which is the *invariant ring*

$$A^G = \{f \in A \mid f(gx) = f(x) \ \forall \ g \in G, x \in X\}.$$

Then it is reasonable to suggest the definition

$$X/G \stackrel{?}{=} \operatorname{Spec} A^G. \tag{1.2}$$

But here we have implicitly made the assumption that the quotient is affine (we decided what the global regular functions on the quotient should be, and then we took the spectrum of that). Moreover, it is not even clear that the right hand side is a variety, the problem being that the invariant ring may not be finitely generated. Nevertheless, the definition suggested above is the right one for an interesting class of groups, called reductive, which we will study rather intensively in the sequel. Here we just state the fact that a finite group is reductive, as long as its order is not divisible by the characteristic of the base field.

EXAMPLE 1.4. The involution $(x, y) \mapsto (-x, -y)$ defines an action of $\mathbf{Z}/(2)$ on the affine plane $\mathbf{A}^2$ (assuming the base field k has characteristic different from zero). The invariant ring

$$k[x, y]^{\mathbf{Z}/(2)}$$

clearly contains the elements

$$u = x^2, \quad v = y^2, \quad w = xy, \tag{1.3}$$

between which there is the relation

$$uv = w^2. \tag{1.4}$$

It is reasonably straight forward to verify that (1.3) and (1.4) in fact give a presentation of the invariant ring, so that the quotient is the cone

$$\mathbf{A}^2/(\mathbf{Z}/(2)) = \operatorname{Spec} k[u, v, w]/(uv - w^2).$$

EXAMPLE 1.5. Let $X = \mathrm{GL}(2)$ be the variety consisting of invertible 2×2 matrices over k and let $G \subset \mathrm{GL}(2)$ be the subvariety consisting of upper triangular matrices. Both G and X carry a group structure, but here we consider G as a group variety and X as a variety, endowed with the natural G-action of matrix multiplication from the left. Both X and G are affine, but G is not reductive: This is a typical example of what we will *not* study in this text. G contains the elements

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \qquad \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

and is in fact generated by these. Multiplication on the left by these matrices corresponds to the elementary row operations "scale first row by a", "scale second row by a" and "add a times the second row to the first row".

Now consider an arbitrary matrix

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X = \mathrm{GL}(2).$$

Since the determinant $ad - bc$ is nonzero, we have that either c or d is nonzero. If c is nonzero, then we can apply elementary row operations as follows:

(1) Scale the second row so that c becomes 1
(2) Subtract a times the second row from the first row, so that a becomes zero
(3) Scale the first row so that (the new) b becomes 1

This shows that the G-orbit containing x contains a matrix of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & s \end{pmatrix}$$

for some $s \in k$, and it is easily checked that s is unique. Similarly, if d is nonzero, the orbit contains a unique matrix of the form

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

and if both c and d are nonzero either form is possible, and then one checks that $s = t^{-1}$. This means that the projective line $\mathbf{P}^1$ parametrizes all G-orbits in X in a natural way, and suggests very strongly that, whatever we settle on as our notion of quotient, we should have $X/G \cong \mathbf{P}^1$ in this example. Note that X and G are both affine, but the quotient is projective. One can deduce from the calculations above that any G-invariant global function on X would factor through $\mathbf{P}^1$, so the invariant ring is just the constants k. Thus (1.2) would give us the very unreasonable quotient consisting of a point only. In our context, the solution is to avoid groups such as this G.

CHAPTER 2

Group schemes and actions

1. Group schemes

We fix a base scheme S, which within a couple of sections will become the spectrum $\operatorname{Spec} k$ of a field k. A group scheme over S is a group object in the category of schemes over S. This means that a group scheme is a scheme G over S, equipped with three morphisms, the group law

$$\mu\colon G \times_S G \to G,$$

the inverse

$$\iota\colon G \to G,$$

and an identity element, which is a morphism

$$\epsilon\colon S \to G.$$

(Note that, if $S = \operatorname{Spec} k$, then ϵ is a k-rational point of G. If S is something like a variety on its own, a better picture might be to view G as some sort of bundle of groups over S, with the section ϵ giving the identity element in each fibre.) These data are subject to conditions corresponding to the usual group axioms. For instance, the associativity law requires $(g_1g_2)g_3 = g_1(g_2g_3)$, or

$$\mu(\mu(g_1, g_2), g_3) = \mu(g_1, \mu(g_2, g_3))$$

for all points $g_1, g_2, g_3 \in G$. Since a map of possibly nonreduced schemes is in general not determined by its effect on points, we in fact require something stronger, namely that the diagram

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{1_G \times \mu} & G \times_S G \\ {\scriptstyle \mu \times 1_G}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ G \times_S G & \xrightarrow{\quad\mu\quad} & G \end{array} \tag{2.1}$$

commutes. Similarly, the left and right identity axioms become the commutativity of

$$\begin{array}{ccc} S \times_S G & \xrightarrow{\epsilon \times 1_G} & G \times_S G \\ & \searrow & \big\downarrow{\scriptstyle \mu} \\ & & G \end{array} \qquad \begin{array}{ccc} G \times_S S & \xrightarrow{1_G \times \epsilon} & G \times_S G \\ & \searrow & \big\downarrow{\scriptstyle \mu} \\ & & G \end{array} \tag{2.2}$$

(where the anonymous diagonal arrow is the canonical isomorphism), and the left and right inverse axioms become the commutativity of

$$\begin{array}{ccc} G & \xrightarrow{(\iota, 1_G)} & G \times_S G \\ \downarrow & & \downarrow \mu \\ S & \xrightarrow{\epsilon} & G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{(1_G, \iota)} & G \times_S G \\ \downarrow & & \mu \downarrow \\ S & \xrightarrow{\epsilon} & G \end{array} \tag{2.3}$$

(where the anonymous vertical map is the structure map for G as a scheme of S). We summarize the definition.

Definition 2.1. A *group scheme* over S is a scheme G over S, together with maps μ, ι and ϵ making the diagrams in (2.1), (2.2) and (2.3) commute.

Remark 2.2. We emphasize that if G is a group variety (a reduced separated group scheme of finite type over an algebraically closed field k), then the commutativity of the above diagrams is equivalent to the group axioms for G considered as a set (of closed points). This follows since maps between varieties are determined by their effect on closed points.

Definition 2.3. A group scheme *homomorphism* $\phi \colon G \to H$ between group schemes G and H is a morphism of schemes that is compatible with the multiplication, inverse and unit morphisms.

We will almost exclusively deal with affine group schemes $G = \operatorname{Spec} B$, defined over an affine base $S = \operatorname{Spec} R$. Thus B is an R-algebra, and the group structure is defined by three R-algebra homomorphisms

$$\mu^* \colon B \to B \otimes_R B$$
$$\iota^* \colon B \to B$$
$$\epsilon^* \colon B \to R$$

called the *comultiplication*, the *coinverse* and the *counit*. An algebra equipped with these maps, making the diagrams of algebra homomorphisms corresponding to (2.1), (2.2) and (2.3) commute, is called a Hopf algebra. Thus, to give $\operatorname{Spec}(B)$ the structure of an affine group scheme over $\operatorname{Spec}(R)$ and to give B the structure of a Hopf-algebra over R, is the same thing.[1]

We can now redo Example 1.2 in a more general setting.

Example 2.4. The affine line $\mathbf{G}_{a,R} = \operatorname{Spec} R[t]$ over any ring R can be equipped with the structure of a group scheme over R: If we identify $R[t] \otimes_R R[t] = R[t_1, t_2]$, then the comultiplication can be written

$$\mu^* \colon R[t] \to R[t_1, t_2], \quad t \mapsto t_1 + t_2.$$

[1]Later on, G will act on an affine scheme $X = \operatorname{Spec}(A)$: We have cleverly chosen symbols such that R is a Ring, A is an Algebra over R and B is a Bialgebra over R.

The coinverse and the counit are the two maps

$$\iota^*\colon R[t] \to R[t], \quad t \mapsto -t$$
$$\epsilon^*\colon R[t] \to R, \quad t \mapsto 0.$$

Similarly, we equip $\mathbf{G}_{m,R} = \operatorname{Spec} R[t, t^{-1}]$ with the group structure defined by

$$\mu^*\colon R[t, t^{-1}] \to R[t, t^{-1}] \otimes_R R[t, t^{-1}], \qquad t \mapsto t \otimes t$$
$$\iota^*\colon R[t, t^{-1}] \to R[t, t^{-1}], \qquad t \mapsto t^{-1}$$
$$\epsilon^*\colon R[t, t^{-1}] \to R, \qquad t \mapsto 1.$$

The reader is invited to check at least one of the group axioms by verifying that the required diagram is commutative.

We next show that the general linear group is a group scheme in a natural way. The only substantial input needed is the observation that the multiplication law and the inverse law are given by polynomial functions in the matrix entries and the inverse of its determinant. From this it is at least immediate that the general linear group over an algebraically closed field is a group variety. But in fact, all that is required to extend this claim to the general linear group over an arbitrary ring R, is some care with the notation.

Example 2.5. Let $R[x_{ij}]$ be the polynomial algebra in n^2 variables x_{ij}, for $1 \le i, j \le n$. Then an R-valued point of the scheme $\mathbf{A}_R^{n^2} = \operatorname{Spec} R[x_{ij}]$ can be viewed as an $n \times n$ matrix with entries from R. We let $\Delta \in R[x_{ij}]$ denote the determinant of (x_{ij}), which is a homogeneous polynomial of degree n. The open subscheme

$$\mathrm{GL}(n, R) = \operatorname{Spec} R[x_{ij}, \Delta^{-1}] \subset \mathbf{A}_R^{n^2}$$

has as R-valued points the set of invertible matrices with entries from R. This is a group. In fact, $\mathrm{GL}(n, R)$ is a group scheme over R in a natural way: The comultiplication can be defined already on $R[x_{ij}]$ by the homomorphism

$$R[x_{ij}] \to R[x_{ij}] \otimes R[x_{ij}], \quad x_{ij} \mapsto \sum_v x_{iv} \otimes x_{vj}.$$

Note how this is defined: Multiply together two copies of the matrix (x_{ij}), writing $\otimes$ for the multiplication. Then the homomorphism sends x_{ij} to entry (i, j) in this matrix. As the determinant is multiplicative, Δ is sent to $\Delta \otimes \Delta$, and hence there is an induced homomorphism

$$\mu^*\colon R[x_{ij}, \Delta^{-1}] \to R[x_{ij}, \Delta^{-1}] \otimes R[x_{ij}, \Delta^{-1}].$$

The coinverse

$$\iota^*\colon R[x_{ij}, \Delta^{-1}] \to R[x_{ij}, \Delta^{-1}]$$

sends x_{ij} to entry (i, j) in the (formal) inverse of the matrix (x_{ij}). By Cramer's rule, this is entry (j, i) in the cofactor matrix of (x_{ij}), divided by Δ. Finally, the counit is the identity matrix, i.e.

$$\epsilon^* \colon R[x_{ij}] \to R$$

sends x_{ij} to entry (i, j) in the identity matrix, which is Kronecker's δ_{ij}. The verification that this defines a Hopf algebra structure, i.e. that the diagrams (2.1), (2.2), (2.3) commute, is rather formal, and boils down to the fact that matrix multiplication fulfills the usual group laws.

Note that for $n = 1$, the group $\mathrm{GL}(1, R)$ can be identified with the multiplicative group $\mathbf{G}_{m,R}$.

EXAMPLE 2.6. The special linear group $\mathrm{SL}(n, R)$ is the closed subgroup of $\mathrm{GL}(n, R)$ defined by $\Delta = 1$, i.e.

$$\mathrm{SL}(n, R) = \operatorname{Spec} R[x_{ij}]/(\Delta - 1).$$

It is clear that the group scheme structure on $\mathrm{GL}(n, R)$ induces a group scheme structure on $\mathrm{SL}(n, R)$.

EXAMPLE 2.7. The projective linear group $\mathrm{PGL}(n, R)$ is the spectrum of the ring of degree zero elements in $R[x_{ij}, \Delta^{-1}]$,

$$\mathrm{PGL}(n, R) = \operatorname{Spec}(R[x_{ij}, \Delta^{-1}]_0).$$

The ring in question is a sub Hopf algebra of $R[x_{ij}, \Delta^{-1}]$, which means that it carries an induced Hopf algebra structure, and hence $\mathrm{PGL}(n, R)$ is a group scheme. We are brief here, as we will have more to say about this group later.

So far we have merely taken well known groups over a field, noted that the group law and the inverse law are regular maps, and then made the observation that the construction makes sense over an arbitrary base ring. In contrast, the following example makes sense only in a scheme theoretic setting.

EXAMPLE 2.8. Let k be a field of characteristic $p > 0$. Define a scheme

$$\alpha_p = \operatorname{Spec} k[t]/(t^p)$$

which can be viewed as a finite subscheme of the additive group $\mathbf{G}_a$ over k, supported at the origin, i.e. the unit for the group law. In fact, α_p is a subgroup scheme in the obvious sense: The comultiplication

$$\mu^* \colon k[t]/(t^p) \to k[t]/(t^p) \otimes_k k[t]/(t^p) \cong k[t_1, t_2]/(t_1^p, t_2^p),$$

sending $t \mapsto t_1 + t_2$ is well defined since $(t_1 + t_2)^p = t_1^p + t_2^p$ in characteristic p.

EXAMPLE 2.9. Let k be a field of arbitrary characteristic, and define for each integer n a scheme

$$\mu_n = \operatorname{Spec} k[t]/(t^n - 1)$$

This is a subgroup scheme of the multiplicative group $\mathbf{G}_m$ over k, as is easily verified. If k is algebraically closed, and its characteristic does not divide n, then μ_n is just a cyclic group of n elements, with the discrete scheme structure (i.e. a disjoint union of n copies of $\operatorname{Spec} k$). Over $k = \mathbf{C}$, the group can be depicted as n evenly spaced points on the unit circle. However, if k has characteristic $p > 0$, and we let $n = p$, then $(t^p - 1) = (t-1)^p$ and hence

$$\mu_p = \operatorname{Spec} k[t]/((t-1)^p)$$

is a nonreduced scheme supported at the unit $1 \in \mathbf{G}_m$.

It is no coincidence that we have seen nonreduced group schemes only in positive characteristic: By a theorem of Cartier, every group scheme of finite type over an algebraically closed field of characteristic zero is reduced. A related, but easier, observation is the following:

PROPOSITION 2.10. *Let G be a group variety, i.e. a separated reduced group scheme of finite type over an algebraically closed field k. Then G is nonsingular.*

PROOF. For any point $g \in G$, let t_g denote the translation map $\mu(g, -)$, i.e. the restriction of the group law to $\{g\} \times G$:

$$t_g \colon G \cong \{g\} \times G \subset G \times G \xrightarrow{\mu} G$$

Then t_g is an automorphism of G, in fact $t_{\iota(g)}$ is the inverse map. Moreover t_g sends the unit $e \in G$ to $g \in G$. Since g was arbitrary to begin with, this shows that G is either nonsingular everywhere, or singular everywhere, but the latter is impossible. $\square$

Note that the proof in fact shows much more: the local rings at any two k-rational points, on an arbitrary group scheme G over k, are isomorphic. Thus G has the same local properties everywhere. In particular, if G is nonreduced, then it has to be nonreduced everywhere. Cartier's theorem says that this is impossible for a finite type G in characteristic zero.

REMARK 2.11. Identify a scheme X with its functor $X(-)$ of points, i.e. for each scheme T, we let $X(T)$ be the set of morphisms $T \to X$. Then to give a scheme G the structure of a group scheme is equivalent to give a factorization of the functor $G(-)$ through the category of groups. Thus, informally, a group scheme G is a scheme such that $G(T)$ is a group for all T. We will not pursue this viewpoint here.

2. Actions

Let G be a group scheme over an arbitrary base scheme S, and let X be another scheme over S. Recall that if we worked with sets and not schemes, then an action of G on X would be a map

$$\sigma \colon G \times X \to X$$

also written $gx = \sigma(g, x)$, satisfying the identity law $ex = x$ and the associativity law $(gh)x = g(hx)$, for all $x \in X$ and $g, h \in G$. Again these axioms lead to commutative diagrams, so that the identity law becomes the commutativity of

$$\begin{array}{ccc} S \times_S X & \xrightarrow{\epsilon \times \mathrm{id}_X} & G \times_S X \\ & \searrow & \downarrow{\scriptstyle \sigma} \\ & & X \end{array} \tag{2.1}$$

and the associativity law becomes the commutativity of

$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{\mu \times \mathrm{id}_X} & G \times_S X \\ {\scriptstyle \mathrm{id}_G \times \sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ G \times_S X & \xrightarrow{\sigma} & X \end{array} . \tag{2.2}$$

DEFINITION 2.1. An *action* of G on X is a map σ making the diagrams (2.1) and (2.2) commute.

EXAMPLE 2.2. The group law itself defines an action $\sigma = \mu$ of any group scheme on itself. This action is called *left translation*.

Let $X = \operatorname{Spec} A$ be an affine scheme and $G = \operatorname{Spec} B$ an affine group scheme, both over an affine base $S = \operatorname{Spec} R$. Thus A and B are R-algebras, and A is also a Hopf algebra. An action σ of G on X corresponds to a *coaction* of the Hopf algebra B on A, i.e. a ring homomorphism

$$\sigma^* \colon A \to B \otimes_R A$$

fitting into the two commutative diagrams of algebra homomorphisms corresponding to (2.1) and (2.2).

EXAMPLE 2.3. Let

$$\mathbf{A}^n_R = \operatorname{Spec} R[x_1, \dots, x_n]$$

be an affine space over R. The additive group $\mathbf{G}_{a,R}$ acts on $\mathbf{A}^n_R$ by translation. More precisely, the coaction

$$\sigma^* \colon R[x_1, \dots, x_n] \to R[x_1, \dots, x_n, t]$$

sends each x_i to $x_i + t$.

EXAMPLE 2.4. The general linear group $\mathrm{GL}(n, R)$ over R acts by matrix multiplication on the affine space $\mathbf{A}^n_R$. More precisely, the coaction

$$\sigma^* \colon R[x_1, \dots, x_n] \to R[x_{ij}, \Delta^{-1}] \otimes_R R[x_1, \dots, x_n]$$

is defined by sending x_i to the i'th entry in the column vector obtained by multiplying the matrix (x_{ij}) with the column vector (x_i), writing $\otimes$ for the product. Thus

$$\sigma^*(x_i) = \sum_j x_{ij} \otimes x_j .$$

Of course, the determinant plays no role in this example (the semigroup of all, not necessarily invertible, $n \times n$ matrices does act on affine space, but this does not interest us).

Letting $n = 1$ in this example, we find the action of $\mathbf{G}_m$ on affine space by scaling. Also, by restricting to a subgroup like $\mathrm{SL}(n, R)$ of $\mathrm{GL}(n, R)$, we get an induced action.

PROPOSITION 2.5. *Let R be a ring and A an R-algebra. There is a canonical one-to-one correspondence between actions of $\mathbf{G}_{m,R}$ on $X = \operatorname{Spec} A$ and gradings $A = \bigoplus_{n \in \mathbf{Z}} A_n$ on A.*

PROOF. An action of $\mathbf{G}_{m,R}$ on X is given by a coaction

$$\sigma^* \colon A \to R[t, t^{-1}] \otimes_R A \cong A[t, t^{-1}].$$

Briefly, a coaction with

$$\sigma^*(a) = \sum_n a_n t^n \tag{2.3}$$

corresponds to the grading on A in which $a = \sum_n a_n$ is the decomposition into degree n homogeneous parts a_n.

Precisely, given a coaction σ^*, define

$$A_n = \{a \in A \mid \sigma^*(a) = at^n\}.$$

Since σ^* is an R-algebra homomorphism, it is immediate that $A_n \subset A$ is an R-submodule and that $A_n A_m \subseteq A_{n+m}$. It is also clear that $A_n \cap A_m = 0$ for distinct n and m, so to see that we have a well defined grading we only need to check that the A_n's generate A, i.e. every $a \in A$ can be written $a = \sum a_n$ with $a_n \in A_n$. So let a_n be defined by (2.3). Since the coidentity on $R[t, t^{-1}]$ sends t to 1, it follows from the identity axiom for σ^* that we have

$$a = \sum_n a_n.$$

We need to check that a_n is in fact in A_n. So let $\sigma^*(a_n) = \sum_m a_{n,m} t^m$. The associativity axiom for σ^* can now be written

$$\sum_{n,m} a_{n,m} t_1^n t_2^m = \sum_n a_n t_1^n t_2^n.$$

Comparing coefficients, we find that $a_{n,m} = 0$ for $n \neq m$ and $a_{n,n} = a_n$, so $\sigma^*(a_n) = a_n t^n$ and thus $a_n \in A_n$.

The reader will have no difficulties in verifying that each step in the argument can be reversed, giving the other direction of the correspondence. $\square$

3. Representations of affine groups

Let $G = \operatorname{Spec} B$ be an affine group scheme over a field k.

Definition 2.1. A *linear group* is a closed subgroup scheme of $\mathrm{GL}(n, k)$, i.e. a closed subscheme such that the inclusion is a group scheme homomorphism.

Since $\mathrm{GL}(n, k)$ is affine and of finite type over k, any closed subscheme is affine and of finite type. Thus every linear group is an affine group scheme of finite type. In this section we prove that the converse also holds.

Definition 2.2. A *representation* of G is a group scheme homomorphism

$$\rho\colon G \to \mathrm{GL}(n, k)$$

Definition 2.3. A *corepresentation* of the Hopf algebra B on a vector space V is a k-linear map

$$s\colon V \to B \otimes_k V$$

such that the diagrams (identity, respectively associativity)

$$\begin{array}{ccc} V & \xrightarrow{s} & B \otimes_k V \\ & \searrow & \downarrow{\scriptstyle \epsilon^* \otimes \mathrm{id}_V} \\ & & k \otimes_k V \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{s} & B \otimes_k V \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \mu^* \otimes \mathrm{id}_V} \\ B \otimes_k V & \xrightarrow{\mathrm{id}_B \otimes s} & B \otimes_k B \otimes_k V \end{array}$$

commute.

Remark 2.4. For each k-rational point in G, considered as a homomorphism $g\colon B \to k$, the corepresentation s gives a linear map

$$V \xrightarrow{s} B \otimes_k V \xrightarrow{g \otimes \mathrm{id}_V} k \otimes_k V \cong V$$

sending $v \in V$ to a vector we denote by $v^g \in V$. It follows from the axioms for a corepresentation that this defines a (right) action of the group of k-rational points in G on V. If G is a variety (meaning also that k is algebraically closed), then the action $v \mapsto v^g$ determines the corepresentation entirely: We may expand $s(v) = \sum_i b_i \otimes e_i$ in terms of a basis e_i for V, and then $v^g = \sum_i b_i(g) e_i$, where $b_i(g)$ means the evaluation of the function b_i on the point g. If we know the value of b_i at all (closed) points $g \in G$, then we know the function b_i, and hence also $s(v)$.

Example 2.5. If $\sigma\colon G \times X \to X$ is an action, then the coaction

$$\sigma^*\colon A \to B \otimes_k A$$

is a corepresentation. Here we view A as a (typically infinite dimensional) vector space. If G is a variety, then the right action associated to σ^* in the previous remark is the one sending a function $f \in A$ on X to the function $f^g(x) = f(gx)$.

If V has basis e_i, then the corepresentation is uniquely determined by elements $b_{ij} \in B$ such that

$$s(e_i) = \sum_j b_{ij} \otimes e_j$$

(the basis does not need to be finite, but of course these sums are, so there are finitely many nonzero b_{ij} for each fixed i). The two axioms then translates to the equalities

$$\epsilon^*(b_{ij}) = \delta_{ij} \tag{2.1}$$

$$\mu^*(b_{ij}) = \sum_v b_{iv} \otimes b_{vj} \tag{2.2}$$

as is easily checked by tracing the basis elements through the two commutative diagrams in Definition 2.3. Note that there are only finitely many nonzero terms in the sum appearing here. We also make the observation that, by the (right) inverse axiom for the group G, we have

$$\sum_v b_{iv} \iota^*(b_{vj}) = \delta_{ij} \tag{2.3}$$

and similarly for the left inverse axiom. These three equalities should remind the reader of the Hopf algebra sturcture on the coordinate ring of $\mathrm{GL}(n, k)$.

PROPOSITION 2.6. *If $V = k^n$ then there is a canonical one to one correspondence between corepresentations of the Hopf algebra B on V and representations*

$$\rho\colon G \to \mathrm{GL}(n, k).$$

PROOF. A representation ρ corresponds to a k-algebra homomorphism

$$\rho^*\colon k[x_{ij}, \Delta^{-1}] \to B.$$

Such a map is given by elements $b_{ij} = \rho^*(x_{ij})$, subject to the condition that the determinant of the matrix (b_{ij}) is invertible in B. For ρ to be a representation, we need ρ^* to be compatible with the counit, the comultiplication and the coinverse. By definition of the group structure on $\mathrm{GL}(n, k)$, this means that the equations (2.1), (2.2) and (2.3) should hold. But a corepresentation is also given by elements $b_{ij} \in B$ such that these three equations hold (observe that (2.3) implies that (b_{ij}) is invertible, and thus has invertible determinant). $\square$

Let us say that a sub vector space $W \subset V$ is *invariant* if $s(W) \subset A \otimes_k W$, i.e. s restricts to an induced corepresentation

$$s|_W : W \to A \otimes_k W.$$

LEMMA 2.7. *Every corepresentation $s\colon V \to A \otimes_k V$ is locally finite, i.e. every $v \in V$ is contained in an invariant finite dimensional subspace $W \subset V$.*

PROOF. The main point is just that $s(v)$ can be written as a *finite* sum

$$s(v) = \sum_{i=1}^{n} a_i \otimes v_i,$$

for $a_i \in A$ and $v_i \in V$. This expression is not uniquely determined, but we may choose one with minimal n, which implies that the a_i's are linearly independent over k. Let $W \subset V$ be the vector subspace spanned by the v_i's, which is clearly finite dimensional.

Since we have $v = \sum_i \epsilon^*(a_i)v_i$ by the identity axiom, the subspace W contains v, and it remains to see that W is invariant, i.e. that $s(W)$ is contained in $A \otimes_k W$. By the associativity axiom, we have

$$(\mathrm{id} \otimes s)(s(v)) = (\mu^* \otimes \mathrm{id})(s(v))$$

which expands to

$$\sum_i a_i \otimes s(v_i) = \sum_i \mu^*(a_i) \otimes v_i.$$

Now, for each $j = 1, \dots, n$, choose a k-linear map $\phi_j \colon A \to k$ such that $\phi_j(a_i) = \delta_{ij}$ (this defines ϕ_j uniquely on the subspace of A spanned by $a_1, \dots, a_n$, and we may for instance let ϕ_j be zero on a chosen complementary subspace). Then, applying $\phi_j \otimes \mathrm{id}_A \otimes \mathrm{id}_V \colon A \otimes_k A \otimes_k V \to A \otimes_k V$ to both sides of the last equality, we get

$$s(v_j) = \sum_i b_i \otimes v_i$$

for certain elements $b_i \in A$. This proves that $s(v_j) \in A \otimes W$, and thus W is invariant. □

THEOREM 2.8. *Every affine group scheme $G = \operatorname{Spec} A$ of finite type is linear.*

PROOF. Consider the comultiplication

$$\mu^* \colon A \to A \otimes_k A$$

as a corepresentation on A. Choose a finite generating set for A as a k-algebra. By the lemma, each generator is contained in a finite dimensional invariant subspace. The sum of all these subspaces is an invariant finite dimensional subspace $V \subset A$ that generates A as a k-algebra. Choose a basis e_i for V.

By Proposition 2.6, the restricted corepresentation $V \to A \otimes_k V$ corresponds to a representation $\rho \colon G \to \mathrm{GL}(n, k)$, given by

$$\rho^* \colon k[x_{ij}, \Delta^{-1}] \to A, \quad x_{ij} \mapsto a_{ij}.$$

By the *right* identity axiom for the Hopf algebra A, we have

$$e_i = \sum_j a_{ij} \epsilon^*(e_j)$$

and thus the image of ρ^* contains all of $V \subset A$. Since ρ^* is a k-algebra homomorphism, and V generates A, we conclude that ρ^* is surjective, which means that $\rho\colon G \to \mathrm{GL}(n,k)$ is a closed immersion. $\square$

4. More on corepresentations

We fix a group $G = \operatorname{Spec} B$ over a field k and consider corepresentations of the Hopf algebra B.

Definition 2.1. Let $s\colon V \to B \otimes_k V$ be a corepresentation. The *invariant subspace* V^G is

$$V^G = \{v \in V \mid s(v) = 1 \otimes v\}.$$

In particular, if $s = \sigma^*$ is induced by an action σ on an affine scheme $X = \operatorname{Spec} A$, then A^G is called the *invariant ring*.

Remark 2.2. If G is a variety (over an algebraically closed field k), a vector $v \in V$ is invariant if and only if v is invariant under the action of closed points in G, i.e. we have $v^g = v$ for all $g \in G$ (see Remark 2.4). In particular, we recover the definition of the invariant ring used in Chapter 1.

Definition 2.3. A *homomorphism* between two corepresentations

$$s\colon V \to B \otimes_k V, \quad t\colon W \to B \otimes_k W$$

is a vector space homomorphism $f\colon V \to W$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ s\ } & B \otimes_k V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes_k f} \\ W & \xrightarrow{\ t\ } & B \otimes_k W \end{array}$$

commutes.

Remark 2.4. The image and kernel of a homomorphism are invariant, and hence are corepresentations on their own.

Remark 2.5. We leave it to the reader to define direct sums and (finite) tensor products of corepresentations. Also, the quotient V/W of a corepresentation by an invariant subspace $W \subset V$ is a corepresentation in a natural way.

It may be slightly surprising that, given a corepresentation V, there is no obvious way to write down a "dual" corepresentation on the dual vector space $V^\vee$: At least in the case of varieties, one could consider the action of closed points of G on V, in the sense of Remark 2.4, dualize that action, and ask whether this dualized action again were induced from a corepresentation on $V^\vee$. The problem is that the dual action may not be locally finite. We circumvent this difficulty by only dualizing corepresentations on finite dimensional vector spaces.

Definition 2.6. Let $s\colon V \to B \otimes_k V$ be a corepresentation. *A dual corepresentation* is a corepresentation $s^\vee\colon V^\vee \to B \otimes_k V^\vee$ such that

$$\begin{array}{ccc} V \otimes_k V^\vee & \longrightarrow & B \otimes V \otimes_k V^\vee \\ {\scriptstyle \mathrm{ev}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{ev}} \\ k & \longrightarrow & B \otimes_k k \end{array}$$

commutes, where $\mathrm{ev}(v \otimes \alpha) = \alpha(v)$ is the evaluation map, and the top horizontal map is the corepresentation on the tensor product $V \otimes V^\vee$.

Lemma 2.7. *A finite dimensional corepresentation V has a unique dual corepresentation.*

Proof. Choose a basis v_i for V and let $v_i^\vee$ be the dual basis for $V^\vee$. Let the corepresentation s on V be given by

$$s(v_i) = \sum_j b_{ij} \otimes v_j.$$

By tracing the basis $v_i \otimes v_j^\vee$ through the diagram in the definition, we find that a dual corepresentation is necessarily given by

$$s^\vee(v_i^\vee) = \sum_j \iota^*(b_{ji}) v_j^\vee$$

and it is straight forward to verify that this does define a corepresentation. □

Definition 2.8. If V and W are two corepresentations, and V is finite dimensional, we give the vector space $\mathrm{Hom}_k(V, W)$ the corepresentation structure of $V^\vee \otimes_k W$.

Remark 2.9. The corepresentation $\mathrm{Hom}_k(V, W)$ fits into a commutative diagram analogous to the one in Definition 2.6. It can be deduced from this that the invariant subspace $\mathrm{Hom}_k(V, W)^G$ is the vector space of homomorphisms $V \to W$ of corepresentations.

Definition 2.10. Let $G = \mathrm{Spec}\, B$ be a group scheme.

(1) A corepresentation $s\colon V \to B \otimes_k V$ is *irreducible* if there is no nontrivial invariant subspace $W \subset V$.
(2) A corepresentation V is *completely reducible* if it is isomorphic to a direct sum $\bigoplus_i V_i$ of irreducible corepresentations.

Remark 2.11. By local finiteness, Lemma 2.7, an irreducible corepresentation is finite dimensional.

Lemma 2.12 (Schur's Lemma). (1) *A homomorphism $f\colon V \to W$ between irreducible corepresentations is either zero or an isomorphism.*
(2) *Suppose k is algebraically closed. Any homomorphism $f\colon V \to V$ from an irreducible corepresentation to itself is multiplication by a scalar in k.*

Proof. The image and kernel of a homomorphism f is invariant. As V and W contain no nontrivial invariant subspaces, the first claim is clear.

As V is finite dimensional, we may choose a finite basis and view f in (2) as an $n \times n$ matrix. As k is algebraically closed, its characteristic polynomial has a zero λ, which is an eigenvalue for f. Then

$$f - \lambda \colon V \to V$$

is a homomorphism of corepresentations, hence is either zero or an isomorphism. But then it is zero, since there exists an eigenvector $v \in V$ satisfying $(f - \lambda)(v) = 0$. Thus $f = \lambda$. □

The decomposition $V = \bigoplus_i V_i$ of a completely reducible corepresentation V into irreducibles V_i is in general not unique, as already a trivial representation shows. However, given such a decomposition, let $V(\mu) \subset V$ denote the direct sum of those V_i that belong to the same isomorphism class λ of corepresentations. The resulting decomposition $V = \bigoplus_\mu V(\mu)$ is called the *isotypical decomposition*. Note that, if we let $\mu = 1$ denote the trivial 1-dimensional corepresentation, then $V(1)$ is just V^G.

Proposition 2.13. *Let V be a completely irreducible corepresentation, choose a decomposition $V = \bigoplus_i V_i$ into irreducibles and let $V = \bigoplus_\mu V(\mu)$ be the associated isotypical decomposition.*

(1) *Choose a corepresentation W in the isomorphism class μ. Then $V(\mu)$ is the image of the evaluation map*

$$W \otimes \operatorname{Hom}_k(W, V)^G \to V.$$

(2) *The isotypical decomposition $V = \bigoplus_\mu V(\mu)$ is independent of the chosen decomposition into irreducibles.*

Proof. The image of the evaluation map is the vector space spanned by the elements $f(w)$, for all $w \in W$ and all homomorphisms $f \colon W \to V$ of corepresentations. Clearly $V(\mu)$ is contained in this space, since for any V_i in the class μ, we may take f to be the composition of an isomorphism $W \cong V_i$ with the inclusion $V_i \subset V$. Conversely, for any V_i *not* in μ and any homomorphism f, the composition

$$W \xrightarrow{f} V \to V_i$$

(the rightmost map being the projection) is zero, by Schur's lemma. Thus the image of the evaluation map is contained in $V(\mu)$. This proves (1).

The claim (2) follows since claim (1) gives a description of $V(\mu) \subset V$ that is independent of the chosen decomposition. □

5. Linearly reductive groups

We now turn to the for us very important class of linearly reductive affine groups.

DEFINITION 2.1. An affine group $G = \operatorname{Spec} B$ over k is *linearly reductive* if there exists an *invariant integral*

$$I \colon B \to k,$$

i.e. a k-linear function satisfying $I(1) = 1$ and left- and right-invariant in the sense that

$$\begin{array}{ccccc} B \otimes_k B & \xleftarrow{\mu^*} & B & \xrightarrow{\mu^*} & B \otimes_k A \\ \downarrow{\scriptstyle I\otimes \mathrm{id}} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle \mathrm{id}\otimes I} \\ k \otimes_k B & \longleftarrow & k & \longrightarrow & B \otimes_k k \end{array}$$

commutes.

REMARK 2.2. The definition implies that for every k-rational point $g \in G$, and every element $\phi \in B$, we have $I(\phi) = I(\phi^g)$, where ϕ^g denotes the (left) translation of ϕ by g (see Remark 2.4 and Example 2.5). Similarly, I is invariant under right translation by k-rational points of G, defined in the analogous way. If G is a variety, invariance in this sense is equivalent to the definition above, i.e. if I takes the same value on $\phi(-)$, $\phi(g(-))$ and $\phi((-)g)$ for all (closed) points $g \in G$, then the diagrams in the definition commute.

EXAMPLE 2.3. Let G be a finite group, considered as a group scheme over an arbitrary field k. Then G is linearly reductive if and only if its order n is not divisible by the characteristic of k. In fact, if this condition is satisfied,

$$I(\phi) = \frac{1}{n} \sum_{g \in G} \phi^g$$

defines an invariant integral. Conversely, if there exists an invariant integral, let ϕ be the regular function that takes the value 1 on the unit $e \in G$ and is zero everywhere else. Then the constant function 1 can be written

$$1 = \sum_{g \in G} \phi^g$$

and hence we have

$$1 = I(1) = \sum_{g \in G} I(\phi^g) = nI(\phi)$$

showing that n is invertible in k.

EXAMPLE 2.4. The multiplicative group $\mathbf{G}_m = \operatorname{Spec} k[t, t^{-1}]$ is linearly reductive. In fact

$$I(1) = 1, \quad I(t^d) = 0 \text{ for } d \neq 0$$

is an invariant integral, as is quickly verified.

EXAMPLE 2.5. The additive group $\mathbf{G}_a = \operatorname{Spec} k[t]$ is not linearly reductive: Since I would have to be invariant under translation by the (k-rational) point $1 \in \mathbf{G}_a$, we would have

$$I(t) = I(t+1) = I(t) + I(1)$$

showing $I(1) = 0$.

DEFINITION 2.6. Let $s\colon V \to B \otimes_k V$ be a corepresentation.

(1) A *Reynolds operator* for s is a k-linear invariant map

$$E\colon V \to V^G$$

which splits the inclusion $V^G \subset V$.

(2) A *natural* Reynolds operator is a choice of a Reynolds operator E for every corepresentation V, such that whenever $\phi\colon V \to W$ is a homomorphism of corepresentations, the diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ {\scriptstyle E}\downarrow & & {\scriptstyle E}\downarrow \\ V^G & \xrightarrow{\phi} & W^G \end{array}$$

commutes.

REMARK 2.7. The requirement that E is invariant means that

$$\begin{array}{ccc} V & \xrightarrow{s} & B \otimes_k V \\ {\scriptstyle E}\downarrow & & {\scriptstyle 1\otimes E}\downarrow \\ V^G & \longrightarrow & B \otimes_k V^G \end{array}$$

commutes, where the map at the bottom is the trivial corepresentation $v \mapsto 1 \otimes v$.

We observe in the next lemma that for completely reducible corepresentations, Reynolds operators are automatically natural.

LEMMA 2.8. *Suppose V is a completely reducible corepresentation.*

(1) *A Reynolds operator for V, if one exists, is unique.*

(2) *Let $f\colon V \to W$ be a homomorphism to a second (not necessarily completely reducible) corepresentation W. If V and W both admit Reynolds operators E, then the diagram*

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle E}\downarrow & & {\scriptstyle E}\downarrow \\ V^G & \xrightarrow{f} & W^G \end{array}$$

commutes.

PROOF. Consider the following claim: If U is a vector space, considered as a trivial corepresentation, and

$$F\colon V \to U$$

is an (invariant) homomorphism mapping V^G to zero, then F is zero. The claim implies the lemma: In (1) we may let F be the difference $E - E'$ between two Reynolds operators, and in (2) we may let $F = E \circ f - f \circ E$. Thus we only need to prove the claim.

Let $W \subset V$ be an irreducible corepresentation. If W is trivial, then $W \subset V^G$, and thus $f(W) = 0$. If W is nontrivial, then it cannot be embedded into the trivial corepresentation U, so F cannot be injective on W. But the kernel of F is invariant, and W is irreducible, so we must have $F(W) = 0$. By complete irreduciblity, it follows that F is zero. □

PROPOSITION 2.9. *Let $G = \operatorname{Spec} B$ be an affine group over a field k. Then G is linearly reductive if and only if the following equivalent conditions hold.*

(i) *There exists a functorial Reynolds operator on all corepresentations.*
(ii) *The functor sending a corepresentation V to the vector space V^G is exact.*
(iii) *For each finite dimensional corepresentation V and each invariant subspace $W \subset V$, there exists a complementary invariant subspace $W' \subset V$ such that $V = W \oplus W'$.*
(iv) *Every corepresentation V is completely reducible.*

PROOF. We prove that the existence of an invariant integral implies (i), then that each statement in the list implies the next one, and finally that (iv) implies the existence of an invariant integral.

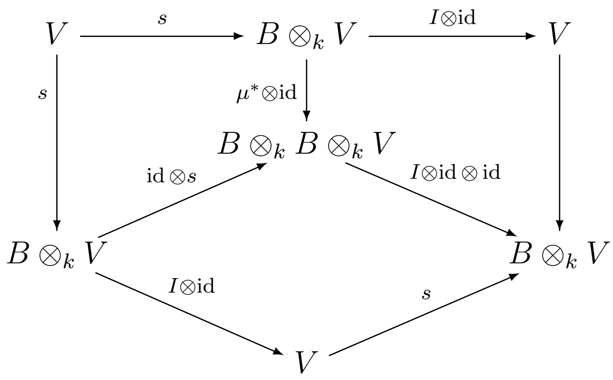
(i) Assume $I\colon B \to k$ is an invariant integral, and let V be a corepresentation. Let E be the the composition

$$E\colon V \xrightarrow{s} B \otimes_k V \xrightarrow{I \otimes \mathrm{id}} k \otimes_k V \cong V.$$

Then E is k-linear, and it follows from $I(1) = 1$ that E is the identity on V^G. The invariance of E is expressed by the outer rectangle in the diagram

$$\begin{array}{ccccc} V & \xrightarrow{\quad s \quad} & B \otimes_k V & \xrightarrow{\quad I \otimes \mathrm{id} \quad} & V \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \mathrm{id} \otimes s} & & \downarrow \\ B \otimes_k V & \xrightarrow{\mu^* \otimes \mathrm{id}} & B \otimes_k B \otimes_k V & \xrightarrow{\mathrm{id} \otimes I \otimes \mathrm{id}} & B \otimes_k V \end{array}$$

in which the left square commutes by the associativity axiom for the corepresentation, and the right square commutes by the left invariance of I. Furthermore, the image of E is in V^G if the outer pentagon in the diagram



commutes. And it does, since the commutativity of the top left trapezium is the associativity axiom, the commutativity of the top right trapezium is due to I being right invariant, and the diamond at the bottom commutes for trivial reasons. Finally, it is evident that E is natural.

(ii): The functor $V \mapsto V^G$ is in any case left exact. Furthermore, if $\phi\colon V \to W$ is surjective and $w \in W^G$, then there exists a not necessarily invariant vector $v \in V$ such that $\phi(v) = w$. If (i) is satisfied, so we have functorial Reynolds operators E, then

$$\phi(E(v)) = E(\phi(v)) = E(w) = w$$

which proves that the restriction $\phi\colon V^G \to W^G$ is surjective.

(iii) Restriction to $W \subset V$ defines a surjective map

$$\operatorname{Hom}(V, W) \to \operatorname{Hom}(W, W)$$

which is a homomorphism of corepresentations. Assuming (ii) holds, also

$$\operatorname{Hom}(V, W)^G \to \operatorname{Hom}(W, W)^G$$

is surjective. Thus we may lift $\mathrm{id}\colon W \to W$ to an invariant element $f \in \operatorname{Hom}(V, W)^G$, which means that f is a homomorphism of corepresentations which splits the inclusion $W \subset V$. It follows that the image W' of f is an invariant complement to W.

(iv) Under the hypothesis (iii) it is clear that any finite dimensional corepresentation is completely reducible. For an arbitrary corepresentation V, Zorn's lemma gives the existence of a maximal collection $\{V_i\}$ of irreducible invariant subspaces $V_i \subset V$ that is linearly independent, i.e. $\sum_i V_i = \bigoplus_i V_i$. Then (iii) together with locally finiteness shows that these V_i necessarily span all of V: For if $v \in V$ is not in their span, let $W \subset V$ be a finite dimensional invariant subspace containing v. Then there exists an invariant complement $X \subset W$ to $W \cap (\bigoplus_i V_i)$, and X contains an irreducible invariant $Y \subset X$. If we adjoin Y to the collection $\{V_i\}$ we obtain a strictly larger collection of linearly independent and invariant subspaces of V, contradicting maximality of $\{V_i\}$.

Finally, if every corepresentation is completely reducible, we show that there exists an invariant integral $I\colon B \to k$. Firstly, we may consider B itself as a corepresentation (left translation), and then $B^G = k$. Thus, any decomposition of B into irreducibles looks like $B = k \oplus (\bigoplus_i W_i)$ where k is the trivial representation and each W_i is nontrivial and irreducible. The projection onto k is evidently a *left* invariant integral. To prove that it is also right invariant, we apply the construction of a Reynolds operator considered in the first part of the proof: Namely, the composition

$$I'\colon B \xrightarrow{\mu^*} B \otimes_k B \xrightarrow{I \otimes \mathrm{id}} B$$

is k-linear, sends 1 to 1 and is left invariant. Thus both I and I' are Reynolds operators for B, considered as the left translation corepresentation. By Lemma 2.8 we have $I = I'$, which says precisely that I is right invariant also. □

REMARK 2.10. Suppose that $G = \operatorname{Spec} B$ is linearly reductive and acts on $X = \operatorname{Spec} A$. The associated corepresentation

$$\sigma^*\colon B \to B \otimes_k A$$

is a *ring homomorphism*, which has the following consequence: If $x \in A^G$ is invariant, then multiplication by x is a homomorphism of corepresentations $A \to A$. Hence, by functoriality of Reynolds operators, we must have

$$E(xy) = xE(y) \quad \text{for all } x \in A^G \text{ and } y \in A. \tag{2.1}$$

This is the *Reynolds identity.*

THEOREM 2.11. *Let $G = \operatorname{Spec} B$ be a linearly reductive group acting on an affine scheme $X = \operatorname{Spec} A$ of finite type over k. Then the invariant ring A^G is finitely generated.*

PROOF. As A is finitely generated, and any corepresentation is locally finite, we may find a finite dimensional invariant subspace $V \subset A$ that generates A as an algebra. For convenience we choose a basis for V. The corepresentation structure on V corresponds to a linear action of G on $\mathbf{A}^n = \operatorname{Spec} k[t_1, \dots, t_n]$, and we obtain a surjective map

$$\phi\colon k[t_1, \dots, t_n] \to A$$

(sending the generators t_i to the chosen basis elements in V) which is a homomorphism both of algebras and of corepresentations. By linear reductivity, “taking invariants” is exact, so

$$\phi\colon k[t_1, \dots, t_n]^G \to A^G$$

is also surjective. Hence, if $k[t_1, \dots, t_n]^G$ is finitely generated, then so is A^G.

We have reduced to the case of G acting linearly on $A = k[t_1, \dots, t_n]$. Then A^G is graded, and we let $J \subset A^G$ be the ideal generated by homogeneous elements of positive degree. Since A is Noetherian (by Hilbert's basis theorem, which was invented for this purpose), the ideal JA is finitely generated, and we may find homogeneous generators $f_1, \dots, f_m \in J$ (note that $f_i \in A^G$, but they generate the ideal JA in A). These elements generate a subalgebra

$$k[f_1, \dots, f_m] \subseteq A^G$$

and we claim that we in fact have equality. This is proved by induction on degree: Let $f \in A^G$ be homogeneous of degree $d > 0$. Then $f \in J$, so

$$f = h_1 f_1 + \cdots h_m f_m$$

where we can choose $h_i \in A$ of degree strictly less than d. Now apply the Reynolds operator $E\colon A \to A^G$, remembering that f and f_i are invariant. We find

$$\begin{aligned} f = E(f) &= E(h_1 f_1) + \cdots + E(h_m f_m) \\ &= E(h_1) f_1 + \cdots + E(h_m) f_m \end{aligned}$$

where we used the Reynolds identity in the last step. We note that, by the functoriality of Reynolds operators, E has to map the invariant subspace $A_v \subset A$, consisting of homogeneous elements of degree v, to $A_v^G \subset A^G$. Thus E preserves degree, so each $E(h_i)$ have degree strictly less than d. By induction we may assume $E(h_i) \in k[f_1, \dots, f_m]$, and then we are done. □

COROLLARY 2.12. *With assumptions as in the theorem, let* $A = \bigoplus_\mu A(\mu)$ *be the isotypical decomposition. Then each* $A(\mu)$ *is a finite module over* A^G.

PROOF. [2] The component $A(\mu)$ is the image of the evaluation map

$$W \otimes_k \operatorname{Hom}(W, A)^G \to A$$

for W a representative of λ. Hence it suffices to show that $\operatorname{Hom}(W, A)^G$ is a finite A^G-module. More generally, we show that $(V \otimes A)^G$ is a finite A^G-module for any finite dimensional corepresentation V (let $V = W^\vee$). For simplicity we choose a basis $V \cong k^n$. The corepresentation corresponds to a representation $\rho\colon G \to \mathrm{GL}(n)$, and hence a linear action of G on $\mathbf{A}^n$. Now consider the product action of G on

$$X \times \mathbf{A}^n = \operatorname{Spec} A[t_1, \dots, t_n].$$

Since the action on $\mathbf{A}^n$ is linear, the invariant ring $A[t_1, \dots, t_n]^G$ is graded, with A^G in degree 0 and $(A \otimes V)^G$ in degree 1. By (1), the invariant ring $A[t_1, \dots, t_n]^G$ is finitely generated, and it follows that its degree 1 part is finite as a module over its degree 0 part. □

[2]The presentation is borrowed from Springer: "Aktionen reduktiver Gruppen auf Varietäten" in "Algebraische Transformationsgruppen und Invariententheorie".

CHAPTER 3

Quotients

1. Categorical and good quotients

We briefly return to the general setup, with G a group scheme over an arbitrary base scheme S, and an action

$$\sigma \colon G \times_S X \to X$$

on an arbitrary scheme X over S. Let us say that a morphism $\phi \colon X \to Y$ is *invariant* if the diagram

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\sigma} & X \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \phi} \\ X & \xrightarrow{\phi} & Y \end{array}$$

commutes, where π_2 denotes second projection. For varieties, this says $\phi(gx) = \phi(x)$ for all $g \in G$ and $x \in X$.

Definition 3.1. A scheme X/G over S, together with an invariant morphism

$$\pi \colon X \to X/G$$

is a *categorical quotient* for the action σ if, for any other invariant morphism $\rho \colon X \to Y$, there exists a unique morphism $\phi \colon X/G \to Y$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X/G \\ & {\scriptstyle \rho}\searrow & \downarrow{\scriptstyle \phi} \\ & & Y \end{array}$$

commute.

Remark 3.2. The categorical quotient may not exist, but if it does, it is unique up to unique isomorphism.

Now let $S = \operatorname{Spec} R$, $G = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ all be affine. If $\phi \colon X \to Y$ is a morphism to an affine scheme $Y = \operatorname{Spec} C$, then ϕ is invariant if and only if

$$\phi^* \colon C \to A$$

satisfies $\mu^*(f(c)) = 1 \otimes f(c)$ for all $c \in C$, i.e. ϕ^* has image in A^G. It follows that $\operatorname{Spec} A^G$ has the universal property of a categorial quotient *among affine schemes.* This shows that if a categorical quotient X/G exists *and is affine* then $X/G = \operatorname{Spec} A^G$. But, as Example 1.5 shows, a categorical quotient does not have to be affine in general.

Again consider schemes over a field k. We next analyse the geometry of $\operatorname{Spec} A^G$ under the hypothesis that G is linearly reductive. The result is that $\operatorname{Spec} A^G$ is a so called *good quotient*, which implies that it is a categorical quotient. Presupposing this result, we introduce the following notation.

DEFINITION 3.3. Let G be an affine linearly reductive group acting on $X = \operatorname{Spec} A$. The *GIT quotient* is the scheme

$$X/G = \operatorname{Spec} A^G.$$

Informally, a quotient X/G should ideally parametrize orbits in X. But the fibres of $\pi\colon X \to X/G$ are necessarily closed in X, hence, if there are non closed orbits, then no quotient in this sense can exist. As a compromise between wishful thinking and reality, we may ask instead that π should separate closed orbits, and somewhat more generally, that π sends disjoint invariant closed subschemes of X to disjoint subschemes of X/G. The following result says a little bit more than this.

THEOREM 3.4. *Suppose G is an affine linearly reductive group acting on $X = \operatorname{Spec} A$ and with GIT quotient $X/G = \operatorname{Spec} A^G$. Let $\pi\colon X \to X/G$ be the map induced by the inclusion $A^G \subset A$.*

(1) *If $W \subset X$ is a closed invariant subscheme, then the image $\pi(W)$ is closed in X/G.*
(2) *We have $\pi(\bigcap_i W_i) = \bigcap_i \pi(W_i)$ for any collection of closed invariant subschemes $W_i \subset X$.*
(3) *Let $U \subset X/G$ be the principal open subset defined by the non-vanishing of an element $f \in A^G$. Then U is the GIT quotient $\pi^{-1}(U)/G$.*

REMARK 3.5. A closed subscheme $W \subset X$ is invariant if the restriction of the action $G \times W \to X$ factors through the embedding $W \subset X$. Equivalently, the ideal $I \subset A$ defining W is invariant in the sense of corepresentations. The precise meaning of $\pi(W)$ in the theorem is as follows: Assertion (1) is just topological, and says that the subset $\pi(W)$ of prime ideals in A is Zariski closed. But then it has a canonical scheme structure, since the closure $\overline{\pi(W)}$ is a scheme in a natural way in any case. Assertion (2) then holds as an equality of schemes.

REMARK 3.6. A scheme X/G satisfying the three properties in the theorem is called a *good quotient*. This notion makes sense for actions on arbitrary, not necessarily affine, schemes X, with (3) appropriately modified.

LEMMA 3.7. *If G is linearly reductive, then for any ideal $I \subset A^G$, we have*

$$IA \cap A^G = I.$$

PROOF. It is obvious that $I \subseteq IA \cap A^G$. Conversely, any element in $IA \cap A^G$ looks like

$$f = \sum_i f_i h_i$$

with $f, f_i \in A^G$ and $h_i \in A$. Apply the Reynolds operator to obtain

$$f = E(f) = \sum_i E(f_i h_i) = \sum_i f_i E(h_i)$$

and the expression on the right is clearly in I. □

PROOF OF THE THEOREM. We first establish that π itself is surjective: If $P \subset A^G$ is a prime ideal, then by Lemma 3.7, we have

$$PA \cap A^G = P$$

which implies[1] that there is a prime ideal $Q \subset A$ with $P = Q \cap A^G$. This says that π sends Q to P, so π is surjective.

Proof of (1): Suppose $W = V(I)$ for an invariant ideal $I \subset A$. The closure of the image $\overline{\pi(W)}$ is the closed subscheme defined by $I \cap A^G = I^G$. By the exactness of "taking invariants",

$$0 \to I^G \to A^G \to (A/I)^G \to 0$$

is exact, which shows that

$$\overline{\pi(W)} = \operatorname{Spec}\left((A/I)^G\right).$$

Hence we may apply the first part of the argument to conclude that

$$\pi|_W : W \to \overline{\pi(W)} = W/G$$

is surjective, and thus $\pi(W) = \overline{\pi(W)}$.

Proof of (2): In view of the first part, the claim is that

$$\sum_i (I_i^G) = (\sum_i I_i)^G \tag{3.1}$$

for any collection of invariant ideals $I_i \subset A$. The right hand side consists of invariant elements $f \in A^G$ of the form $f = \sum h_i$ with $h_i \in I_i$. Apply the Reynolds operator to find

$$f = E(f) = \sum E(h_i)$$

where $E(h_i) \in I_i^G$ since I_i is invariant and E is natural. This shows that f is in the left hand side of (3.1). The other inclusion is obvious.

Proof of (3): Since $\pi^{-1}(U) = \operatorname{Spec}(A_f)$, the claim is that

$$(A^G)_f = (A_f)^G$$

(the left hand side is the coordinate ring of U and the right hand side is the coordinate ring of the GIT quotient $\pi^{-1}(U)/G$). It is immediate

[1] see Atiyah-MacDonald Prop. 3.16

that there is an inclusion $(A^G)_f \subset (A_f)^G$. Moreover, if $h/f \in (A_f)^G$, then we use the Reynolds identity to see that

$$h/f^n = E(h/f^n) = E(h)/f^n \in (A^G)_f$$

which proves the claim. □

REMARK 3.8. The equality $(A^G)_f = (A_f)^G$ in the last part of the proof holds in fact without linear reductivity. This is because A^G is the kernel of the A^G-module homomorphism

$$A \to B \otimes_k A$$

sending a to $\sigma^*(a) - 1 \otimes a$. It follows from this that "taking invariants" commutes with tensor product $- \otimes_{A^G} M$ with any flat A^G-module M. Applying this to $M = (A^G)_f$ shows that $(A^G)_f = (A_f)^G$.

COROLLARY 3.9. *With assumptions as in the theorem, the GIT quotient X/G is a categorical quotient.*

PROOF. Let $\rho\colon X \to Y$ be an invariant map to an arbitrary scheme Y. We want to show that ρ factors through $\operatorname{Spec} A^G$. Here is the idea: We want to define ρ locally, by applying the already established universal property for GIT quotients among affine schemes. Thus we need to cover X/G with sufficiently small open affine subschemes U, such that $\pi^{-1}(U)$ is mapped by ρ to some open *affine* subset of Y.

So let $Y = \bigcup_i V_i$ be an affine open cover, and let

$$W_i = \rho^{-1}(Y \setminus V_i)$$

where $Y \setminus V_i$ is considered as a closed subscheme, for instance with the reduced scheme structure. Then $\pi(W_i)$ is closed, and we let

$$U_i = (X/G) \setminus \pi(W_i)$$

be the complement. Since V_i cover Y, we have $\bigcap W_i = \emptyset$, which implies that $\bigcap \pi(W_i) = \emptyset$. This shows that U_i cover X/G. Note that $\pi^{-1}(U_i) \subset \rho^{-1}(V_i)$.

Cover X/G with principal open subsets $U = \operatorname{Spec}((A^G)_f)$ contained in some U_i. Since $\pi^{-1}(U) = \operatorname{Spec} A_f$ is affine, with GIT quotient U, we have a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \pi\ } & U \\ & \searrow^{\rho} & \downarrow{\scriptstyle \phi} \\ & & V_i \end{array}$$

where there is a unique ϕ fitting in by the universal property for the GIT quotient U among affine schemes. The uniqueness gives that these maps $\phi\colon U \to Y$ glue to give the required map $X/G \to Y$. □

Suppose G and X are varieties. Then it follows from Theorem 3.4 that two closed points $x_1, x_2 \in X$ belong to the same fibre of the quotient map $\pi\colon X \to X/G$ if and only if their orbit closures intersect.

Moreover, each fibre contains a unique closed orbit. Thus X/G can be viewed as a parameter space for the closed orbits in X.

Note that, in contrast to part (3) of Theorem 3.4, it is not true that the GIT quotient V/G of an invariant open affine subscheme $V \subset X$ coincides with the image $\pi(V) \subset X$. Consider for instance the action of $\mathbf{G}_m$ on $\mathbf{A}^n$ by multiplication: The quotient $\mathbf{A}^n/\mathbf{G}_m = \operatorname{Spec} k$ is just a point. On the other hand, if $V = \mathbf{A}^n \setminus H$ is the complement of a hyperplane,

$$V = \operatorname{Spec} k[x_1, \dots, x_n, x_n^{-1}]$$

then the invariant ring is the polynomial ring in the $n-1$ variables x_i/x_n, and so

$$V/G = \mathbf{A}^{n-1}.$$

This is as it should be, since all orbit closures in $\mathbf{A}^n$ intersect, and the only closed orbit is the origin. In contrast, all orbits in V are closed.

Having established that sensible quotients of affine schemes by linearly reductive groups exist, we next ask for their geometric properties. This will be taken up seriously in the next section, but already now we can say something:

PROPOSITION 3.10. *Let $X/G = \operatorname{Spec} A^G$ be a GIT quotient. Let P be any of the following properties: finite type, noetherian, reduced, irreducible. If X satisfies P, then so does X/G.*

PROOF. Finite type: This is the content of Theorem 2.11, saying that if A is finitely generated, then so is A^G.

Noetherian: By Lemma 3.7, the set of ideals in A^G is a subset of the set of ideals in A, and clearly in an inclusion preserving way. Thus if the ascending chain condition holds for A, then it also holds for A^G.

Reduced: Clearly, any nilpotent element of A^G would also be a nilpotent element of A.

Irreducible: An affine variety is irreducible if and only if its coordinate ring has prime nilradical. Thus the claim follows from the observation that the nilradical in A^G is the intersection of A^G with the nilradical in A. □

In particular, if X is a variety, then so is X/G.

2. Étale slices

In this section[2], all schemes considered are of finite type over an algebraically closed field k of characteristic zero. By a theorem of Cartier, any group scheme of finite type over such a field is reduced, and hence is a nonsingular variety.

[2]This section is more sketchy than the previous ones and reflects roughly what I covered at the lecture, together with what I intended to cover. I hope to find the time to expand this part, and to add references.

We need to recall, or accept, a couple of notions regarding fibre bundles: Let G be a group scheme acting on a scheme X, and let $\pi\colon X \to S$ be an invariant morphism. Then X is a *principal G-bundle* over S if, for every point $s \in S$ there exists an étale map $U \to S$ with s in its image, such that there is a Cartesian diagram

$$\begin{array}{ccc} G \times U & \longrightarrow & X \\ \downarrow & & \downarrow \\ U & \longrightarrow & S \end{array}$$

where the map $G \times U \to X$ is equivariant, when G acts on $G \times U$ by multiplication in the first factor. In short, $X \to S$ is an "étale locally trivial G-bundle".

Now let H be a closed subgroup scheme of an affine group scheme G, and consider the action somewhat informally given by

$$H \times G \to G, \qquad (h, g) \mapsto gh^{-1}$$

(the inverse is just inserted to make this a left action; we could just as well have considered the right action $G \times H \to G$ given by multiplication). Then there exists a categorical quotient $G \to G/H$, which in fact is a principal H-bundle. This is much stronger than being a good quotient. In our context we accept this as a "general fact", which does not depend on geometric invariant theory. Note however that *if* H is linearly reductive, then G/H is necessarily the GIT quotient, but the quotient here exists in any case, and does not need to be affine, as Example 1.5 shows.

Now, with $H \subset G$ as before, suppose H acts on a scheme Y. Then define an action on the product $G \times Y$, given by letting $h \in H$ map $(g, y) \mapsto (gh^{-1}, hy)$. Also here there exists a categorical quotient, and the projection map is a principal H-bundle. This quotient $(G \times Y)/H$ is the *associated fibre bundle*, usually denoted $G \times^H Y$. The action of G on $G \times Y$, given by multiplication (from the left) in the first factor, commutes with the H-action just considered, and hence there is an induced G-action on the quotient $G \times^H Y$. Thus we have extended the H-action on Y to a G-action on the associated fire bundle. If there exists a categorical quotient Y/H for the H-action on Y, then there is a canonical isomorphism

$$(G \times^H Y)/G \cong Y/H \tag{3.1}$$

(roughly speaking, both sides are obtained from $G \times Y$ be taking the quotient with both the G- and the H-action, which commute).

Theorem 3.1 (Luna's étale slice theorem). *Let G be a linearly reductive group variety acting on a scheme X of finite type over an algebraically closed field k of characteristic zero. Let $x \in X$ be a (closed) point such that the orbit $G \cdot x \subset X$ is closed, and let $G_x \subset G$ be its stabilizer. Then there exists a locally closed subscheme $S \subset X$ containing*

x, such that the diagram

$$\begin{array}{ccc} G \times^{G_x} S & \xrightarrow{\phi} & X \\ \downarrow & & \downarrow \\ S/G_x & \xrightarrow{\overline{\phi}} & X/G \end{array}$$

is Cartesian and $\overline{\phi}$ and ϕ are étale; here ϕ is induced by the G-action on X, the vertical maps are the GIT quotients by G, and we have used the identification (3.1) *in the bottom left corner.*

REMARK 3.2. By a theorem of Matsushita, the stabilizer G_x of a point with closed orbit is linearly reductive when G is. Thus the GIT quotient S/G_x in the bottom left corner in the theorem makes sense.

Luna's theorem reduces the (étale) local geometry of the GIT quotient X/G to the local geometry of S/G_x, which typically will be much simpler. Moreover, the whole G-action of X in an étale neigbourhood of $G \cdot x$ is identified with the action on the induced fibre bundle. In the next section we give a couple of applications of Luna's theorem. In the remainder of this section we will take up a few points from the proof of Luna's theorem.

It is relatively straight forward to produce the slice: First one checks that slices behave well under restriction, in the following sense: If X is in fact an invariant closed subscheme of another scheme X' with a G-action, and $S' \subset X'$ is an étale slice (i.e. satisfies the claims in the theorem) for the G-action on X', then $S = S' \cap X$ is an étale slice for the G-action on X. Since we have seen that an arbitrary X can be embedded in $\mathbf{A}^n$, such that the G-action extends to a linear one on $\mathbf{A}^n$, it suffices to prove the therorem for linear actions on affine spaces. This goes as follows: The tangent space $T_{G\cdot x}(x)$ to the orbit at x is a G_x-invariant subspace of $T_{\mathbf{A}^n}(x)$. Since G_x is linearly reductive (Remark 3.2), there exists a complementary G_x-invariant subspace $W \subset T_{\mathbf{A}^n}(x)$. If we identify $T_{\mathbf{A}^n(x)}$ with $\mathbf{A}^n$, with origin at x, then a Zariski open subset of W is going to be the slice: It is not hard to check that the map $\phi\colon G \times^{G_x} W \to \mathbf{A}^n$ induces an isomorphism of tangent spaces at $(e, x) \in G \times^{G_x} W$ and hence is étale there. This concludes the easy part: To get Luna's theorem, we need the important "fundamental lemma" of Luna, which we discuss next.

Let $\phi\colon Y \to X$ be an equivariant morphism between affine schemes with G-actions. Suppose $y \in Y$ is a point with image $x = \phi(y)$, subject to the following conditions:

(1) ϕ is étale at y
(2) The orbits $G \cdot y$ and $G \cdot x$ are closed
(3) ϕ restricts to an isomorphism $G \cdot y \cong G \cdot x$

Let $\pi_Y\colon Y \to Y/G$ and $\pi_X\colon X \to X/G$ be the GIT quotient maps. The fundamental lemma says that in this situation, there exists a Zariski open neighbourhood $U \subset Y$ of y, of the form $\pi_Y^{-1}(V)$ for an open neighbourhood $V \subset Y/G$ of $\pi_Y(y)$, such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\phi} & X \\ \downarrow & & \downarrow \\ V = U/G & \xrightarrow{\overline{\phi}} & X/G \end{array}$$

is Cartesian and the horizontal maps are étale.

The fundamental lemma concludes the proof of Luna's theorem: The assumptions needed in the lemma are satisfied by the map $\phi \colon G \times^{G_x} W \to \mathbf{A}^n$ we have constructed. The proof is concluded by observing that, since the open subset $U \subset G \times^{G_x} W$, produced by the fundamental lemma, is the inverse image of an open subset in W/G_x, it follows that U has to be of the form $G \times^{G_x} S$ for $S \subset W$ Zariski open.

The fundamental lemma was originally stated and proved only under the additional assumption that X and Y are normal varieties. However, it can be proved quite directly, with no extra hypotheses needed. Here is a sketch:[3] Let $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ and let the closed orbits $G \cdot x$ and $G \cdot y$ correspond to ideals $I \subset R$ and $J \subset S$. Thus the image points $\pi_Y(y)$ and $\pi_X(x)$ in the GIT quotients correspond to the maximal ideals $I^G = I \cap R^G$ and $J^G = J \cap S^G$. Morally, étale local properties can be read off from completed local rings, so it is reasonable to approach the fundamental lemma by first "taking completions" in the diagram considered. Thus we consider the corresponding diagram of k-algebras

$$\begin{array}{ccc} S \otimes_{S^G} \widehat{S^G} & \longleftarrow & R \otimes_{R^G} \widehat{R^G} \\ \uparrow & & \uparrow \\ \widehat{S^G} & \longleftarrow & \widehat{R^G} \end{array} \tag{3.2}$$

where $\widehat{S^G}$ and $\widehat{R^G}$ are the completions of R and S with respect to the maximal ideals J^G and I^G, i.e. the completed local rings of $\pi_Y(y)$ and $\pi_X(x)$. Now it turns out that the horizontal arrows in this diagram are isomorphisms. From this the fundamental lemma can be deduced by standard arguments.

It is not hard to see that the invariant ring of $R \otimes_{R^G} \widehat{R^G}$ is $\widehat{R^G}$; thus if the top arrow in (3.2) is an isomorphism, then so is the bottom arrow. Establishing that the top arrow is an isomorphism is the main point in the proof of the fundamental lemma. In outline, the argument is as follows: From the étaleness of ϕ along the orbit $G \cdot y$ one can

[3] We follow Knop's appendix to the chapter by Slodowy in the book "Algebraische Transformationsgruppen und Invariantentheorie", to which the reader is referred for the details. According to Knop, the argument is a simplification of an unpublished proof also by Luna.

deduce that there is a canonical isomorphism

$$\lim_n S/J^n \cong \lim_n R/I^n \tag{3.3}$$

(these rings can be viewed as formal neighbourhoods of the orbits, so this isomorphism is quite reasonable). Now, if R were finite as a module over R^G, then extension of scalars would commute with limits, so we would have

$$\begin{aligned} R \otimes_{R^G} \widehat{R^G} &= R \otimes_{R^G} (\lim_n R^G/(I^G)^n) \\ &\cong \lim_n R/(I^G)^n R. \end{aligned}$$

A further use of the hypothetical finiteness would (more or less) show that the filtrations of R given by I^n and $(I^G)^n R$ were compatible, in the sense that the completions could be identified. Thus we could identify $\lim R/I^n$ with $R \otimes_{R^G} \widehat{R^G}$, and of course similarly for S and J. By (3.3) we would conclude that the top horizontal arrow in (3.2) were an isomorphism. Of course R is not necessarily finite over R^G, and it is not true in general that $\lim R/I^n$ is isomorphic to $R \otimes_{R^G} \widehat{R^G}$. But the isotypical components $R(\lambda)$ *are* finite over R^G by Theorem 2.11, and the argument just sketched can be applied to each isotypical component separately, and this suffices to conclude that the top arrow in (3.2) is an isomorphism.

3. Applications of Luna's theorem

THEOREM 3.1. *Let $G \times X \to X$ be an action satisfying the assumptions in Luna's Theorem 3.1. Assume that every closed point $x \in X$ has trivial stabilizer group $G_x = 1$. Then the GIT quotient $\pi\colon X \to X/G$ is a principal G-bundle.*

Note that, conversely, all stabilizer groups of a principal G-bundle are trivial.

PROOF. Let $\overline{x} \in X/G$ be a closed point, and choose a lifting $x \in X$ in the closed orbit in the fibre over $\overline{x}$ (in fact, all orbits are necessarily closed, since a non closed orbit would have orbits with nontrivial stabilizers in its closure). Let $S \subset X$ be an étale slice through x. Since the stabilizer group of x is trivial, the induced fibre bundle $G \times^{G_x} S$ is just the product $G \times S$. Thus we have a Cartesian diagram

$$\begin{array}{ccc} G \times S & \xrightarrow{\phi} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{\psi} & X/G \end{array} \tag{3.1}$$

where ψ is étale, and ϕ is G-equivariant. This says that $X \to X/G$ is a principal G-bundle. □

Corollary 3.2. *Situation as in the previous theorem. Assume in addition that X is nonsingular. Then the quotient X/G is nonsingular.*

Proof. In diagram (3.1), we may choose S to be nonsingular. But then the existence of the étale map $\psi\colon S \to X/G$ shows that X/G is nonsingular at the (arbitrarily chosen) point $\overline{x}$. □

We remark that even with nontrivial stabilizers, it may well happen that the quotient X/G is nonsingular. One finds trivial examples by replacing the G-action, with trivial stabilizers, with the induced action of a group G' with a surjective homomorphism $G' \to G$. A more interesting example is the action of the symmetric group $\mathfrak{S}_n$ on $\mathbf{A}^n$ by permutation of the coordinates. It is well known that the invariant ring is generated by the elementary symmetric functions, between which there are no relations. Thus $\mathbf{A}^n/\mathfrak{S}_n \cong \mathbf{A}^n$.

Theorem 3.3. *Let $G \times X \to X$ be an action satisfying the assumptions in Luna's Theorem 3.1. Let $x \in X$ be a (closed) point with closed orbit $G \cdot x$. Then there exists a Zariski open neighbourhood U of x such that the stabilizer group G_y of every $y \in U$ is conjugate to a subgroup of G_x.*

Proof. Let S be an étale slice through x, and, with notation as in Luna's theorem, let U be the image of ϕ. The stabilizer group in G of a point in $G \times^{G_x} S$ is equal to the stabilizer group of its image in X, since the diagram in Luna's theorem is Cartesian. One checks easily that the stabilizer group of a point (g, s) in $G \times^{G_x} S$ equals

$$g(G_x)_s g^{-1}$$

where $(G_x)_s \subseteq G_x$ denotes the stabilizer group of s under the G_x-action on S. Hence, if $y = gs$, then

$$g^{-1} G_y g = (G_x)_s \subseteq G_x$$

and we are done. □

Corollary 3.4. *Suppose in addition X is irreducible. There exists an open dense subset $U \subset X$ such that all stabilizer groups G_y for $y \in U$ are conjugate subgroups in G.*

Proof. Choose $x \in X$ such that its stabilizer group has minimal dimension and, among those stabilizers with minimal dimension, the minimal number of connected components. Now apply the previous corollary, and note any closed subgroup of G_x would either have lower dimension or the same dimension but fewer components. □

Note that the last corollary gives sense to the term "the generic stabilizer group" as a conjugacy class of subgroups of G.