VECTOR BUNDLES AND MONADS ON ABELIAN THREEFOLDS

MARTIN G. GULBRANDSEN

Abstract. The purpose of this paper is to construct examples of stable rank 2 vector bundles on abelian threefolds and to study their moduli.

More precisely, we consider principally polarized abelian threefolds (X, Θ) with Picard number 1. Using the Serre construction, we obtain stable rank 2 bundles realizing roughly one half of the Chern classes (c_1, c_2) that are a priori allowed by the Bogomolov inequality and Riemann-Roch. In the case of even c_1, we study first order deformations of these vector bundles $\mathscr{E}$, using a second description in terms of monads, similar to the ones used by Barth–Hulek on projective space. We find that all first order deformations of the bundle are induced by first order deformations of the corresponding monad, which leads to the formula

$$\dim \operatorname{Ext}^1(\mathscr{E}, \mathscr{E}) = \tfrac{1}{3}\Delta(\mathscr{E}) \cdot \Theta + 5,$$

where Δ denotes the discriminant $4c_2 - c_1^2$.

In the simplest nontrivial case (where $c_1 = 0$ and $c_2 = \Theta^2$), we construct an explicit parametrization of a Zariski open neighbourhood of $\mathscr{E}$ in its moduli space: this neighbourhood is a ruled, nonsingular variety of dimension 13, birational to a $\mathbb{P}^1$-bundle over $X \times X \times H$, where H is the Hilbert scheme (of Kodaira dimension zero) of two points on the Kummer threefold $X/(-1)$.

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1. Introduction

The geometry of moduli spaces for stable vector bundles on Calabi-Yau (in the broad sense of having trivial canonical bundle) threefolds is largely unknown, but of high interest, not least due to their relevance for string theory, and the existence of Donaldson-Thomas invariants. In lower dimension, stable vector bundles on Calabi-Yau curves (i.e. elliptic) were classified by Atiyah, and are parametrized by the same curve. Moduli spaces for stable vector bundles on Calabi-Yau surfaces (i.e. K3 or abelian) are holomorphic symplectic varieties (Mukai [14], generalizing Beauville [2], generalizing Fujiki [4]). This is a very rare geometric structure, at least on complete varieties. One may ask whether equally interesting geometries appear in higher dimension.

Here we construct examples of rank 2 vector bundles on *abelian* threefolds, and study their moduli. This work is a first step in a program where we eventually wish to study Donaldson-Thomas type invariants attached to moduli spaces for stable vector bundles on abelian threefolds. Such computations are a topic for future work; here we merely construct examples. The hope is that the abelian case may shed light also on arbitrary Calabi-Yau threefolds, but be more accessible.

Our central tool, besides the Serre construction, is monads: these are usually put to work on rational varieties, and it may be slightly surprising that they can be useful also in our context. On the other hand, we do not know whether the bundles we construct, and their moduli, show typical or exceptional behaviour.

1.1. **Notation.** We work over an algebraically closed field k of characteristic zero. Stable and semistable sheaves, and their moduli, are in the sense of Simpson [16], so that stability is measured by the normalized Hilbert polynomial. The sheaves we construct in this text will in fact have the stronger property of μ-stability in the sense of Mumford and Takemoto, which is measured by the slope.

The words line bundles and vector bundles are used as synonyms for invertible and locally free sheaves. In particular, an inclusion of vector bundles means an inclusion as sheaves, i.e. the quotient need not be locally free. We take Chern classes to live in the Chow ring modulo numerical equivalence.

Let (X, Θ) be a principally polarized abelian variety. If $x \in X$ is a point, we write $T_x \colon X \to X$ for the translation map, and define Θ_x as $T_x(\Theta) = \Theta + x$. We identify X with its dual $\operatorname{Pic}^0(X)$ by associating with x the line bundle $\mathscr{P}_x = \mathscr{O}_X(\Theta - \Theta_x)$. The normalized Poincaré line bundle on $X \times X$ is denoted $\mathscr{P}$; its restriction to $X \times \{x\}$ is $\mathscr{P}_x$.

2. The Serre construction

In this section we apply the standard Serre construction to produce rank 2 vector bundles on principally polarized abelian threefolds, including examples with small c_2. This is the content of Theorem 2.2. These examples (in the case of even c_1) will be our objects of study for the rest of this paper.

2.1. **The bundles/curves correspondence.** Let $\mathscr{E}$ be rank 2 vector bundle on a projective variety X, and let $s \in \Gamma(X, \mathscr{E})$ be a section. If the vanishing locus $V(s)$ has codimension 2, then: (1) it is a locally complete intersection, and (2) the line bundle $\omega_X \otimes \bigwedge^2 \mathscr{E}$ restricts to the canonical bundle on $V(s)$. Under a cohomological condition on $\bigwedge^2 \mathscr{E}$, the Serre construction says that any codimension two subscheme $Y \subset X$ with these two properties is of the form $V(s)$. More precisely:

Theorem 2.1. *Let X be a nonsingular projective variety with a line bundle $\mathscr{L}$ satisfying $H^p(X, \mathscr{L}^{-1}) = 0$ for $p = 1, 2$. Let $Y \subset X$ be a codimension two locally complete intersection subscheme with canonical bundle isomorphic to $(\omega_X \otimes \mathscr{L})|_Y$. Then there is a canonical isomorphism*

$$\operatorname{Hom}((\omega_X \otimes \mathscr{L})|_Y, \omega_Y) \cong \operatorname{Ext}^1(\mathscr{I}_Y \otimes \mathscr{L}, \mathscr{O}_X)$$

which is functorial in Y with respect to inclusions, and such that isomorphisms on the left correspond to locally free extensions on the right.

For the proof we refer to Hartshorne [7, Thm. 1.1 and Rem. 1.1.1], who attributes "all essential ideas" to Serre [15].

It follows that, whenever we choose an isomorphism $(\omega_X \otimes \mathscr{L})|_Y \cong \omega_Y$, the Theorem gives an extension

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{s} \mathscr{E} \longrightarrow \mathscr{I}_Y \otimes \mathscr{L} \longrightarrow 0 \tag{1}$$

with $\mathscr{E}$ locally free, and hence $Y = V(s)$ as required. We will say that $\mathscr{E}$ and Y *corresponds* if there is a short exact sequence (1).

2.2. **Construction of bundles.** For the rest of this paper, we fix a principally polarized abelian threefold (X, Θ). We assume that its Picard number (the rank of the Néron-Severi group) is 1, although this assumption is essential only to ensure stability. Thus every divisor is numerically equivalent to an integral multiple of Θ. Moreover (see e.g. Debarre [3]), an application of the endomorphism construction of Morikawa [12] and Matsusaka [11] shows that every 1-cycle is numerically equivalent to an integral multiple of $\Theta^2/2$.

So fix classes $c_1 = m\Theta$ and $c_2 = n\Theta^2/2$, where m and n are integers. If these are the Chern classes of a rank two vector bundle $\mathscr{E}$, then, by Riemann-Roch

$$\chi(\mathscr{E}) = \tfrac{1}{6}(c_1^3 - 3c_1c_2) = m^3 - \tfrac{3}{2}nm,$$

so either m or n is even. Moreover, if $\mathscr{E}$ is μ-semistable, then Bogomolov's inequality reads $m^2 \leq 2n$.

Theorem 2.2. *Let (X, Θ) be a principally polarized abelian threefold of Picard number 1, and let $c_1 = m\Theta$ and $c_2 = n\Theta^2/2$, with m and n integers. Assume*

(1) *the strict Bogomolov inequality holds, i.e. $m^2 < 2n$, and*
(2) *n is even and mn is divisible by 4.*

Then there exist μ-stable rank 2 vector bundles with Chern classes c_1 and c_2.

Remark 2.3. For each $c_1 \in \mathrm{NS}(X)$, the theorem realizes every second c_2 that is allowed by (strict) Bogomolov and Riemann-Roch. The other half seems much more subtle. In fact, we do not know any example of a rank 2 vector bundle, stable or not, that violates condition (2). The situation in which equality occurs in the Bogomolov inequality will be analysed in Proposition 2.5.

Before proving the theorem, we rephrase μ-stability for $\mathscr{E}$ as a condition on the corresponding curve Y. The argument is similar to that of Hartshorne [7, Prop. 3.1] in the case of $\mathbb{P}^3$.

Lemma 2.4. *Let (X, Θ) be as in the theorem, and $\mathscr{E}$ be a rank 2 vector bundle corresponding to a curve $Y \subset X$. Let $c_1(\mathscr{E}) = m\Theta$. Then the following are equivalent.*

(1) *$\mathscr{E}$ is μ-stable.*
(2) *$m > 0$ and Y is not contained in any translate of any divisor in the linear system $|k\Theta|$, where k is the round down of $m/2$.*

Proof. Since $\mathscr{E}$ has a section, it is clear that $m > 0$ is necessary for its μ-stability. Write $\lfloor m/2 \rfloor$ and $\lceil m/2 \rceil$ for the round down and round up of $m/2$. The bundle $\mathscr{E}$ fails μ-stability if and only if it contains a line bundle $\mathscr{P}_x(l\Theta) \subset \mathscr{E}$ with $l \geq m/2$. Since $\mathscr{P}_x(l\Theta)$ has global sections for l positive, it suffices to test with $l = \lceil m/2 \rceil$. Thus $\mathscr{E}$ is μ-stable if and only if

$$H^0(X, \mathscr{E}(-\lceil m/2 \rceil \Theta) \otimes \mathscr{P}_x)) = 0 \quad \text{for all } x \in X. \tag{2}$$

Now twist the short exact sequence (1) with $-\lceil m/2 \rceil \Theta$ and take cohomology. Since $H^p(X, \mathscr{O}_X(-\lceil m/2 \rceil \Theta))) = 0$ for $p = 0, 1$, and the determinant of $\mathscr{E}$ has the form $\mathscr{P}_a(m\Theta)$ for some $a \in X$, we find that the vanishing (2) is equivalent to the vanishing of $H^0(X, \mathscr{I}_Y(\lfloor m/2 \rfloor \Theta) \otimes \mathscr{P}_x)$ for all $x \in X$. Since Θ is ample, this is equivalent to

$$H^0(X, \mathscr{I}_Y \otimes T_x^* \mathscr{O}_X(\lfloor m/2 \rfloor \Theta)) = 0 \quad \text{for all } x \in X$$

which is condition (2). □

Proof of Theorem 2.2. Since μ-stability, and the conditions (1) and (2) in the statement of the theorem, are preserved under tensor product with line bundles, it suffices to prove the theorem for $m = 2$ and $m = 3$.

When $m = 2$, the theorem claims that there are μ-stable rank 2 bundles with $c_1 = 2\Theta$ and $c_2 = N\Theta^2$ for all integers $N \geq 2$. For this, choose N generic points $a_i \in X$ and let

$$Y = \bigcup_{i=1}^{N} Y_i, \quad Y_i = \Theta_{a_i} \cap \Theta_{-a_i}.$$

We want to apply the Serre construction to this curve.

First we claim that the Y_i's are pairwise disjoint, for a_i chosen generically. In fact, for $i \neq j$ write

$$Y_i \cap Y_j = \underbrace{(\Theta_{a_i} \cap \Theta_{a_j})}_{V} \cap \underbrace{(\Theta_{-a_i} \cap \Theta_{-a_j})}_{W},$$

where V and W have codimension 2. By an easy moving lemma for abelian varieties [10, Lemma 5.4.1], a general translate $V + x$ intersects W properly, hence empty. Thus (replacing x by a "square root" $x/2$) also $V + x$ and $W - x$ are disjoint. So Y_i and Y_j will be disjoint after a small perturbation $a_i \mapsto a_i + x$, $a_j \mapsto a_j + x$.

The normal bundle of each $Y_i \subset X$ is $\mathscr{O}_{Y_i}(\Theta_{a_i}) \oplus \mathscr{O}_{Y_i}(\Theta_{-a_i})$, hence the canonical bundle ω_{Y_i} is $\mathscr{O}_{Y_i}(\Theta_{a_i} + \Theta_{-a_i})$. The theorem of the square shows that $\Theta_{a_i} + \Theta_{-a_i}$ is linearly equivalent to 2Θ. Since the Y_i's are disjoint, we conclude that Y is a locally complete intersection with canonical bundle $\mathscr{O}_Y(2\Theta)$. The Serre construction produces a bundle $\mathscr{E}$ with determinant $\mathscr{O}_X(2\Theta)$ and second Chern class $[Y] = \sum_i [Y_i] = N\Theta^2$.

Next we show μ-stability. We claim that the only theta-translates containing Y_i are Θ_{a_i} and Θ_{-a_i}. This is a standard result: the intersection of two theta-translates are never contained in a third one. In fact, consider the Koszul complex:

$$0 \to \mathscr{O}_X(-\Theta_{a_i} - \Theta_{-a_i}) \to \mathscr{O}_X(-\Theta_{a_i}) \oplus \mathscr{O}_X(-\Theta_{-a_i}) \to \mathscr{I}_{Y_i} \to 0.$$

Twist with an arbitrary theta-translate Θ_x and apply cohomology to obtain an isomorphism

$$H^0(X, \mathscr{O}_X(\Theta_x - \Theta_{a_i})) \oplus H^0(X, \mathscr{O}_X(\Theta_x - \Theta_{-a_i})) \cong H^0(X, \mathscr{I}_{Y_i}(\Theta_x)).$$

Thus Θ_x contains Y_i if and only if $x = \pm a_i$ as claimed. It follows that, for $N \geq 2$, no theta-translate contains Y, and so $\mathscr{E}$ is μ-stable by Lemma 2.4.

In the case $m = 3$, we take

$$Y = \bigcup_{i=1}^{N} Y_i, \quad Y_i = D_i \cap \Theta_{-2a_i}$$

for N generic points $a_i \in X$ and generic divisors $D_i \in |2\Theta_{a_i}|$. A similar argument to the one above shows that the Serre construction produces a μ-stable rank 2 vector bundle with determinant $\mathscr{O}_X(3\Theta)$ and second Chern class $2N\Theta^2$, for each $N \geq 2$. □

Recall that a vector bundle $\mathscr{E}$ is *semihomogeneous* if it is translation invariant up to twist: for every $x \in X$, there exists a line bundle $\mathscr{L} \in$

$\mathrm{Pic}^0(X)$ such that $T_x^*(\mathscr{E})$ is isomorphic to $\mathscr{E} \otimes \mathscr{L}$. Semihomogeneous bundles are well understood thanks to work of Mukai [13].

Proposition 2.5. *Let (X, Θ) be as in the Theorem, and let $c_1 = m\Theta$ and $c_2 = n\Theta^2/2$ satisfy $m^2 = 2n$, i.e. equality occurs in the Bogomolov inequality. Then $\mathscr{E}$ is a non simple, semihomogeneous vector bundle.*

Proof. Semihomogenous bundles of rank r are numerically characterized (Yang [17]) by the property that the all the Chern roots equal c_1/r. This means that the Chern character takes the form $ch = r\exp(c_1/r)$, or, equivalently, the total Chern class is $c = (1 + c_1/r)^r$. If $r = 2$, this is equivalent to $c_1^2 = 4c_2$. Thus $\mathscr{E}$ is semihomogeneous.

By Mukai [13], *simple* semihomogeneous vector bundles are classified, up to twist by homogeneous line bundles, by the element $\delta = c_1/r$ in $\mathrm{NS}(X)\otimes\mathbb{Q}$. But m is even, since $m^2 = 2n$, so there exist line bundles with class $c_1/2$. This rules out the possibility that $\mathscr{E}$ is simple. □

Remark 2.6. Mukai [13] shows that every semihomogeneous bundle is semistable, and it is simple if and only if it is stable. Thus the bundles $\mathscr{E}$ in Proposition 2.5 are semistable, but not stable. Mukai also shows that the Harder-Narasimhan filtration of a semihomogeneous vector bundle with $\delta = c_1/r$ has factors that are simple semihomogeneous bundles with the same invariant δ. In our (rank 2) situation, this shows that $\mathscr{E}$ is an extension of line bundles (possibly split) with first Chern class $c_1(\mathscr{E})/2$.

2.3. **The curves $\Theta_a \cap \Theta_{-a}$.** For later use, we make two observations regarding the curve obtained by intersecting two general theta-translates, which was used as input for the Serre construction above.

Lemma 2.7. *There is a Zariski open subset $U \subset X$ such that $\Theta \cap \Theta_x$ is a nonsingular irreducible curve for all $x \in U$.*

Proof. Let $\Theta^{\mathrm{ns}} \subset \Theta$ denote the nonsingular locus, and let

$$d\colon \Theta^{\mathrm{ns}} \times \Theta^{\mathrm{ns}} \to X$$

be the difference map, so that $d^{-1}(x) \cong \Theta^{\mathrm{ns}} \cap \Theta_x^{\mathrm{ns}}$. Generic smoothness (see Hartshorne [6, III 10.8]) applied to d shows that $\Theta^{\mathrm{ns}} \cap \Theta_x^{\mathrm{ns}}$ is nonsingular for generic $x \in X$.

It is well known that the surface Θ is normal, in particular it is nonsingular in codimension one. (In fact it has at most one singular point, in which case X is the Jacobian of a hyperelliptic curve.) It follows that $\Theta \cap \Theta_x$ does not intersect the singular locus of Θ, nor of Θ_x, for x outside a certain divisor in X. Thus $\Theta \cap \Theta_x = \Theta^{\mathrm{ns}} \cap \Theta_x^{\mathrm{ns}}$ is nonsingular for generic x.

Finally $\Theta \cap \Theta_x$ is an ample divisor on the normal surface Θ, hence it is connected (see Hartshorne [6, III 7.9]). □

Lemma 2.8. *Let a and b be two points in X and define $Y_a = \Theta_a \cap \Theta_{-a}$ and $Y_b = \Theta_b \cap \Theta_{-b}$. Then, for a and b generic, no divisor in $|2\Theta|$ contains both Y_a and Y_b.*

Proof. Begin by imposing the conditions on a and b that Y_a and Y_b are disjoint irreducible curves, and also that the two curves $\Theta_a \cap \Theta_{\pm b}$ are irreducible. Assume there is a divisor $D \in |2\Theta|$ containing both Y_a and Y_b. We will prove the lemma by producing a curve C such that $C \cap \Theta_b = C \cap \Theta_{-b}$, and then deduce that b is not generic.

First we observe that D meets $\Theta_a \cap \Theta_b$ properly. As the latter is irreducible, it suffices to verify that it is not contained in D. In fact, one checks that the linear subsystem of $|2\Theta|$, consisting of divisors containing $\Theta_a \cap \Theta_b$, is the pencil spanned by $\Theta_a + \Theta_{-a}$ and $\Theta_b + \Theta_{-b}$ (for this, determine $H^0(\mathscr{I}_{\Theta_a \cap \Theta_b}(2\Theta))$ using the Koszul resolution). The only element of this pencil containing Y_a is $\Theta_a + \Theta_{-a}$, and the only element containing Y_b is $\Theta_b + \Theta_{-b}$, so no element contains both.

In particular, D and Θ_a intersect properly, so $D \cap \Theta_a$ is a curve containing Y_a. Since $D \cap \Theta_a$ has cohomology class $2\Theta^2$, and Y_a has class Θ^2, there is another effective 1-cycle C of class Θ^2 such that

$$D \cap \Theta_a = Y_a + C$$

as 1-cycles. We saw above that $D \cap \Theta_a$ meets Θ_b properly, so we consider the 0-cycle

$$D \cap \Theta_a \cap \Theta_b = Y_a \cap \Theta_b + C \cap \Theta_b.$$

The left hand side contains $Y_b \cap \Theta_a$. Since Y_a and Y_b are disjoint, this means that $C \cap \Theta_b$ contains $Y_b \cap \Theta_a$, i.e. their difference is an effective cycle. But these are 0-cycles of the same degree, so they are equal. None of the arguments given distinguish between b and $-b$, so we find that also $C \cap \Theta_{-b}$ equals $Y_b \cap \Theta_a$. Thus we have established

$$C \cap \Theta_b = C \cap \Theta_{-b}.$$

To conclude, we apply the endomorphism construction of Morikawa [12] and Matsusaka [11], which we briefly recall. The endomorphism $\alpha = \alpha(C, \Theta)$ associated to C and Θ is defined by

$$\alpha(x) = \sum (C \cdot \Theta_x) - \sum (C \cdot \Theta)$$

where each term means the sum, using the group law, of the points in the intersection cycle appearing. This is well defined as a point in X, although the intersection cycle is only defined up to rational equivalence. The constant term is included to force $\alpha(0) = 0$, i.e. to make α a group homomorphism. We have just established that C intersects Θ_b and Θ_{-b} properly, and the two intersections are equal already as cycles. In particular $\alpha(b) = \alpha(-b)$, so all we need to know to prove the lemma is that α is not constant, so that $\alpha(2b) \neq 0$ defines a nonempty Zariski open subset. But in fact, a theorem of Matsusaka [11] tells us that α is multiplication by 2 (the intersection number $C \cdot \Theta = 3!$ divided by $\dim X = 3$), so the condition required is just that $4b \neq 0$, i.e. b is not a 4-torsion point. $\square$

2.4. **Parameter count.** As an immediate consequence of Lemma 2.8, we find that, if $\mathscr{E}$ corresponds to a curve Y with at least two components of the form $\Theta_{a_i} \cap \Theta_{-a_i}$, for sufficiently general points a_i, then the short exact sequence

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{s} \mathscr{E} \longrightarrow \mathscr{I}_Y(2\Theta) \longrightarrow 0,$$

shows that $H^0(X, \mathscr{E})$ is spanned by s.

We can deform the bundle $\mathscr{E}$ by varying the data in the Serre construction, and a heuristic parameter count goes as follows: the choice of the points $a_i \in X$ contributes $3N$ dimensions, and the choice of an isomorphism $\omega_Y \cong \mathscr{O}_Y(2\Theta)$ contributes N dimensions, since Y has N connected components. Thus we get a $4N$-dimensional family of vector bundles $\mathscr{E}$ with a section s. As we just saw, the section is unique up to scale, so if we forget the section, we are left with $4N - 1$ parameters. In addition there are two obvious ways of deforming a bundle on an abelian variety: translation by points, and twist by line bundles in $\mathrm{Pic}^0(X)$. Including these in the count, and assuming all the parameters to be independent, we conclude that the Serre construction gives a family of vector bundles of dimension $4N + 5$.

In contrast, we show in Section 5 that the space of first order infinitesimal deformations of $\mathscr{E}$ has dimension $8N - 3$. The two dimensions $4N + 5$ and $8N - 3$ coincide only in the first nontrivial case $N = 2$, and in this case, as we detail in Section 6, the Serre construction gives an open subset of the corresponding moduli space. We see no way of computing the number of infinitesimal deformations of $\mathscr{E}$ directly from the Serre construction (even in the $N = 2$ case), and will employ a different viewpoint involving monads.

3. Monads

In this section we rephrase the Serre construction of vector bundles used in Theorem 2.2, in the case of even c_1, in terms of certain monads. This new viewpoint is then used to analyse first order deformations: we will show that every first order deformation of the vector bundle is induced by a first order deformation of the monad.

Definition 3.1 (Barth–Hulek [1]). A *monad* is a composable pair of maps of vector bundles

$$\mathscr{A} \xrightarrow{\phi} \mathscr{B} \xrightarrow{\psi} \mathscr{C}$$

such that $\psi \circ \phi$ is zero, ψ is surjective and ϕ is an embedding of vector bundles (i.e. injective as a homomorphism of sheaves, and with locally free cokernel).

Thus $\mathscr{E} = \operatorname{Ker}(\psi)/\operatorname{Im}(\phi)$ is a vector bundle.

We will also use chain complex notation $(M^\bullet, d)$ for monads, so that $M^{-1} = \mathscr{A}$, $M^0 = \mathscr{B}$, $M^1 = \mathscr{C}$ and M^i is zero otherwise, and the differential d consists of two nonzero components $d^{-1} = \phi$ and $d^0 = \psi$. Thus $M^\bullet$ is exact except in degree zero, where its cohomology is $\mathscr{E} = H^0(M^\bullet)$. By a family of monads over some base S, we mean a monad on $S \times X$.

3.1. **Decomposable monads.** Consider rank 2 vector bundles $\mathscr{E}$ with trivial determinant $\bigwedge^2\mathscr{E} \cong \mathscr{O}_X$ on the principally polarized abelian threefold (X, Θ). From the construction in Theorem 2.2, we have a series of such vector bundles, such that $\mathscr{E}(\Theta)$ corresponds to a curve $Y = \bigcup_i Y_i$, where $Y_i = \Theta_{a_i} \cap \Theta_{-a_i}$.

We now show that, corresponding to the decomposition of Y into its connected components Y_i, there is a way of building up $\mathscr{E}$ from the Koszul complexes[1]

(3)

$$\xi_i\colon \quad 0 \longrightarrow \mathscr{O}_X(-\Theta) \xrightarrow{\begin{pmatrix} \vartheta_i^+ \\ \vartheta_i^- \end{pmatrix}} \mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i} \xrightarrow{\begin{pmatrix} \vartheta_i^- & -\vartheta_i^+ \end{pmatrix}} \mathscr{I}_{Y_i}(\Theta) \longrightarrow 0$$

where $\vartheta_i^\pm$ are nonzero global sections of $\mathscr{O}_X(\Theta_{\pm a_i})$. This can be conveniently phrased in terms of a monad.

Proposition 3.2. *Let $a_1, \dots, a_N \in X$ be generically chosen points and $Y_i = \Theta_{a_i} \cap \Theta_{-a_i}$. Then $\mathscr{E}(\Theta)$ corresponds to $Y = \bigcup_{i=1}^N Y_i$ if and only if $\mathscr{E}$ is isomorphic to the cohomology of a monad*

$$(N-1)\mathscr{O}_X(-\Theta) \xrightarrow{\phi} \bigoplus_{i=1}^N (\mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i}) \xrightarrow{\psi} (N-1)\mathscr{O}_X(\Theta)$$

where, if we decompose ϕ and ψ into pairs

$$\phi^\pm\colon (N-1)\mathscr{O}_X(-\Theta) \to \bigoplus_{i=1}^N \mathscr{P}_{\pm a_i}$$
$$\psi^\pm\colon \bigoplus_{i=1}^N \mathscr{P}_{\pm a_i} \to (N-1)\mathscr{O}_X(\Theta)$$

then we have

$$\phi^\pm = \begin{pmatrix} \vartheta_1^\pm & & & \\ & \vartheta_2^\pm & & \\ & & \ddots & \\ & & & \vartheta_{N-1}^\pm \\ \vartheta_N^\pm & \vartheta_N^\pm & \dots & \vartheta_N^\pm \end{pmatrix}, \quad \psi^\pm = \pm(\phi^\mp)^\vee$$

for nonzero sections $\vartheta_i^\pm \in \Gamma(X, \mathscr{O}_X(\Theta_{\pm a_i}))$.

Proof. If $\mathscr{E}(\Theta)$ and Y correspond, there is an extension

$$\xi\colon \quad 0 \longrightarrow \mathscr{O}_X(-\Theta) \longrightarrow \mathscr{E} \longrightarrow \mathscr{I}_Y(\Theta) \longrightarrow 0.$$

[1] Here and elsewhere, whenever $f\colon \mathscr{F}_1 \to \mathscr{F}_2$ is a homomorphism of sheaves, we use the same symbol to denote any twist $f\colon \mathscr{F}_1(D) \to \mathscr{F}_2(D)$.

Giving such an extension is, by Theorem 2.1, equivalent to giving an isomorphism $\mathscr{O}_Y(2\Theta) \cong \omega_Y$. The obvious decomposition

$$\operatorname{Hom}(\mathscr{O}_Y(2\Theta), \omega_Y) \cong \textstyle\bigoplus_{i=1}^N \operatorname{Hom}(\mathscr{O}_{Y_i}(2\Theta), \omega_{Y_i})$$

gives, when applying Theorem 2.1 also to each Y_i, a corresponding decomposition

$$\operatorname{Ext}^1(\mathscr{I}_Y(\Theta), \mathscr{O}_X(-\Theta)) \cong \textstyle\bigoplus_{i=1}^N \operatorname{Ext}^1(\mathscr{I}_{Y_i}(\Theta), \mathscr{O}_X(-\Theta)), \tag{4}$$

which sends ξ to an N-tuple of extensions ξ_i. Each $\operatorname{Hom}(\mathscr{O}_{Y_i}(2\Theta), \omega_{Y_i})$ is one dimensional, since Y_i is connected, so $\operatorname{Ext}^1(\mathscr{I}_{Y_i}(\Theta), \mathscr{O}_X(-\Theta))$ is one dimensional, too. This shows that each ξ_i is of the form (3).

From the functoriality in Theorem 2.1, it follows that the inclusion of each direct summand in (4) is the natural map, induced by the inclusion $\mathscr{I}_Y \subset \mathscr{I}_{Y_i}$. Thus ξ is obtained from the ξ_i's by pulling them back over this inclusion of ideals, and adding the results in $\operatorname{Ext}^1(\mathscr{I}_Y(\Theta), \mathscr{O}_X(-\Theta))$. By definition of (Baer) addition in ext-groups, this means that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N\mathscr{O}_X(-\Theta) \arrow[r] \arrow[d, "\beta"'] & \bigoplus_{i=1}^N(\mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i}) \arrow[r] \arrow[d] & \bigoplus_{i=1}^N \mathscr{I}_{Y_i}(\Theta) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{O}_X(-\Theta) \arrow[r] \arrow[d, equal] & \mathscr{F} \arrow[r] & \bigoplus_{i=1}^N \mathscr{I}_{Y_i}(\Theta) \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{O}_X(-\Theta) \arrow[r] & \mathscr{E} \arrow[r] \arrow[u, hook] & \mathscr{I}_Y(\Theta) \arrow[r] \arrow[u, hook, "\alpha"] & 0
\end{tikzcd}$$

where the top row is $\bigoplus_i \xi_i$, the bottom row is ξ, the top left square is pushout over the N-fold addition β, and the bottom right square is pullback along the inclusion α. This diagram presents $\mathscr{E}$ as the middle cohomology of a complex

$$\operatorname{Ker}(\beta) \xrightarrow{\phi} \textstyle\bigoplus_{i=1}^N(\mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i}) \xrightarrow{\psi} \operatorname{Coker}(\alpha).$$

Now identify $\operatorname{Ker}(\beta)$ with $(N-1)\mathscr{O}_X(-\Theta)$ by means of the monomorphism

$$(N-1)\mathscr{O}_X \to N\mathscr{O}_X, \quad (f_1, \dots, f_{N-1}) \mapsto (f_1, \dots, f_{N-1}, -\textstyle\sum_i f_i)$$

and similarly identify $\operatorname{Coker}(\alpha)$ with $(N-1)\mathscr{O}_X(\Theta)$ by means of the epimorphism

$$N\mathscr{O}_X \to (N-1)\mathscr{O}_X, \quad (f_1, \dots, f_N) \mapsto (f_1 - f_N, \dots, f_{N-1} - f_N)$$

(the latter is surjective even when restricted to $\bigoplus_i \mathscr{I}_{Y_i}$ because the Y_i's are pairwise disjoint). Via these identifications, the homomorphisms ϕ and ψ are represented by the matrices as claimed — except that $\vartheta_N^{\pm}$ appears with opposite sign, but we are free to rename $\vartheta_N^{\pm}$ to $-\vartheta_N^{\pm}$. □

Definition 3.3. A monad is *decomposable* if it is isomorphic, as a complex, to a monad of the form appearing in Proposition 3.2.

With this terminology, a rank 2 vector bundle $\mathscr{E}$ can be resolved by a decomposable monad if and only if $\mathscr{E}(\Theta)$ corresponds to a disjoint union $Y = \bigcup_i Y_i$, where $Y_i = \Theta_{a_i} \cap \Theta_{-a_i}$, via the Serre construction.

Remark 3.4. The symmetry seen in the decomposable monads is no accident, but reflects the self duality of $\mathscr{E}$ corresponding to the natural pairing $\wedge$ on $\mathscr{E}$ with values in $\bigwedge^2(\mathscr{E}) \cong \mathscr{O}_X$. See Barth–Hulek [1].

4. Digression on the hyperext spectral sequence

Our basic aim is to understand first order deformations of the bundles $\mathscr{E}$ appearing as the cohomology of a decomposable monad. The strategy is to analyse $\operatorname{Ext}^1(\mathscr{E}, \mathscr{E})$ using the first hyperext spectral sequence associated to the monad. This is in principle straight forward, but requires some honest calculation. As preparation, we collect in this section a few standard constructions in homological algebra, for ease of reference. We fix an abelian category $\mathcal{A}$ with enough injectives and infinite direct sums, and denote by $K(\mathcal{A})$ the homotopy category of complexes and by $D(\mathcal{A})$ the derived category.

4.1. **The spectral sequence.** Let $(M^\bullet, d_M)$ and $(N^\bullet, d_N)$ denote complexes in $\mathcal{A}$, and assume that $N^\bullet$ is bounded from below. The *first hyperext spectral sequence* is a spectral sequence

$$E_1^{pq} = \textstyle\bigoplus_i \operatorname{Ext}^q(M^i, N^{i+p}) \Rightarrow \operatorname{Ext}^{p+q}(M^\bullet, N^\bullet). \tag{5}$$

Briefly, take a double injective resolution $N^\bullet \to I^{\bullet\bullet}$ with $I^{\bullet\bullet}$ concentrated in the upper half plane (for instance a Cartan-Eilenberg resolution), and form the double complex $\operatorname{Hom}^{\bullet\bullet}(M^\bullet, I^{\bullet\bullet})$. The required spectral sequence is the first spectral sequence associated to this double complex.

4.2. **The edge maps.** Along the axis $q = 0$, the first sheet of the spectral sequence (5) has the usual hom-complex $\operatorname{Hom}^\bullet(M^\bullet, N^\bullet)$. Its cohomology is

$$E_2^{p,0} = \operatorname{Hom}_{K(\mathcal{A})}(M^\bullet, N^\bullet[p])$$

where the right hand side denotes homotopy classes of morphisms of complexes. Since all differentials emanating from $E_r^{p,0}$ for $r \geq 2$ vanish, there are canonical *edge maps*

$$E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \subset \operatorname{Ext}^p(M^\bullet, N^\bullet).$$

View the right hand side as the group $\operatorname{Hom}_{D(\mathcal{A})}(M^\bullet, N^\bullet[p])$ of morphisms in the derived category. Then it is straight forward to verify that the edge map is in fact the canonical map

$$\operatorname{Hom}_{K(\mathcal{A})}(M^\bullet, N^\bullet[p]) \to \operatorname{Hom}_{D(\mathcal{A})}(M^\bullet, N^\bullet[p]).$$

4.3. **Differentials at E_2.** For $q = 1$, it is convenient to view elements of $\operatorname{Ext}^1(M^i, N^{i+p})$ as extensions, in the sense of short exact sequences, and this viewpoint leads to the following interpretation of the differentials d_2^{p1} at the E_2-level:

Lemma 4.1. *Let $\xi \in E_1^{p1}$ be given as a collection of extensions*

$$\xi_i \colon 0 \to N^{i+p} \to X^i \to M^i \to 0.$$

(1) *We have $d_1^{p1}(\xi) = 0$ if and only if there are maps f^i such that the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & N^{i+p-1} \arrow[r, "d_N"] \arrow[d] & N^{i+p} \arrow[r, "d_N"] \arrow[d] & N^{i+p+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & X^{i-1} \arrow[r, "f^{i-1}"] \arrow[d] & X^i \arrow[r, "f^i"] \arrow[d] & X^{i+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & M^{i-1} \arrow[r, "d_M"] & M^i \arrow[r, "d_M"] & M^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

(2) *When $d_1^{p1}(\xi) = 0$ and (f^i) is as above, ξ represents an element of E_2^{p1}, and the differential*

$$d_2^{p1} \colon E_2^{p1} \to E_2^{p+2,0} = \operatorname{Hom}_{K(\mathcal{A})}(M^\bullet, N^\bullet[p+2])$$

sends ξ to the morphism having components $M^{i-1} \to N^{i+p+1}$ induced by $f^i \circ f^{i-1}$. In particular $d_2^{p1}(\xi) = 0$ if and only if there exists a collection (f^i) making the middle row in the diagram in (1) a complex.

Proof. This is straight forward, although tedious, to verify directly from the construction of the spectral sequence. □

4.4. **Serre duality.** Let X be a scheme of pure dimension d over a field, with a dualizing sheaf ω_X such that Grothendieck-Serre duality holds. Let $M^\bullet$ be a bounded below complex of coherent $\mathscr{O}_X$-modules. We obtain two hyperext spectral sequences (5): one abutting to $\operatorname{Ext}^n(\mathscr{O}_X, M^\bullet) = H^n(X, M^\bullet)$, which we denote by E, and one abutting to $\operatorname{Ext}^n(M^\bullet, \omega_X)$, which we denote by $\hat{E}$. Then E is nothing but the first hypercohomology spectral sequence, and the E_1-levels of E and $\hat{E}$ are Grothendieck-Serre dual. We need to know that the duality extends to all sheets.

Lemma 4.2. *The two spectral sequences E and $\hat{E}$ are dual in the following sense:*

(1) *There are canonical dualities between the vector spaces E_r^{pq} and $\hat{E}_r^{-p,d-q}$ for all p, q, r, extending the Grothendieck-Serre duality between $H^q(X, M^p)$ and $\operatorname{Ext}^{d-q}(M^p, \omega_X)$ for $r = 1$.*

(2) *The differentials*

$$d_r^{pq}\colon E_r^{pq} \to E_r^{p+r,q-r+1}$$
$$\hat{d}_r^{-p-r,d-q+r-1}\colon \hat{E}_r^{-p-r,d-q+r-1} \to \hat{E}_r^{-p,d-q}$$

are dual maps.

Proof. This seems to be standard. We include a sketch, following Herrera–Lieberman [8] (who work in a slightly different context). Firstly, for any three complexes $L^\bullet$, $M^\bullet$, $N^\bullet$, the Yoneda pairing

$$\operatorname{Ext}^i(L^\bullet, M^\bullet) \times \operatorname{Ext}^j(M^\bullet, N^\bullet) \to \operatorname{Ext}^{i+j}(L^\bullet, N^\bullet)$$

can be defined on hyperext groups by resolving $M^\bullet$ and $N^\bullet$ by injective double complexes, and taking the double hom complex. On this "resolved" level, the Yoneda pairing is given by composition, and there is an induced pairing of hyperext spectral sequences in the appropriate sense, which specializes to the usual Yoneda pairing between ext-groups of the individual objects L^l, M^m, N^n at the E_1-level. Specialize to the situation $L^\bullet = \mathscr{O}_X$ and $N^\bullet = \omega_X$ to obtain a morphism of spectral sequences from E to the dual of $\hat{E}$. At the E_1-level this is the Grothendieck-Serre duality map, hence an isomorphism, which is enough to conclude that it is an isomorphism of spectral sequences [5, Section 11.1.2]. □

Remark 4.3. If $M^\bullet$ and $N^\bullet$ denote two complexes of vector bundles, then we may apply the Lemma to the complex $(M^\bullet)^\vee \otimes N^\bullet$ to obtain a duality between the two hyperext spectral sequences abutting to $\operatorname{Ext}^n(M^\bullet, N^\bullet)$ and $\operatorname{Ext}^n(N^\bullet, M^\bullet \otimes \omega_X)$, respectively.

5. Deformations of decomposable monads

We now apply the homological algebra from the previous section to analyse first order deformations of vector bundles $\mathscr{E}$ which can be resolved by a decomposable monad. Firstly, we find that deformations obtained by varying the isomorphism $\omega_Y \cong \mathscr{O}_Y(2\Theta)$ in the Serre construction coincide with the deformations obtained by varying the differential in the monad, while keeping the objects fixed. Secondly, and this is the nontrivial part, we find that all first order deformations of $\mathscr{E}$ can be obtained by also deforming the objects in the monad, and there are more of these deformations than those obtained by varying Y in the Serre construction. Since the objects in the monad are sums of line bundles, their first order deformations are easy to understand, so we are able to compute the dimension of $\operatorname{Ext}^1(\mathscr{E}, \mathscr{E})$, in Theorem 5.7.

5.1. **Calculations in the spectral sequence.** Let $\mathscr{E}$ be the rank 2 vector bundle arising as the cohomology of a decomposable monad

$$M^\bullet\colon\ \mathscr{A} \xrightarrow{\phi} \mathscr{B} \xrightarrow{\psi} \mathscr{C}$$

given explicitly in Proposition 3.2.

$$\begin{tikzcd}
E_1^{-2,3} \arrow[r] & E_1^{-1,3} \arrow[r] & E_1^{0,3} & & \\
 & & E_1^{0,2} & & \\
 & & E_1^{0,1} & & \\
 & & E_1^{0,0} \arrow[r] & E_1^{1,0} \arrow[r] & E_1^{2,0}
\end{tikzcd}$$

FIGURE 1. The first sheet in the spectral sequence for $\operatorname{Ext}^i(\mathscr{E},\mathscr{E})$

The hyperext spectral sequence from Section 4.1 gives

$$E_1^{pq} = \bigoplus_i \operatorname{Ext}^q(M^i, M^{i+p}) \Rightarrow \operatorname{Ext}^{p+q}(\mathscr{E},\mathscr{E}). \tag{6}$$

Using that ample line bundles on X have sheaf cohomology concentrated in degree 0, whereas anti-ample line bundles have sheaf cohomology in top degree, we see that the nonzero terms in the first sheet have the shape depicted in Figure 1. It follows that all differentials at level E_r vanish for $r = 3$ and $r > 4$. Also, the duality of Section 4.4 shows that each term E_r^{pq} is dual to $E_r^{-p,3-q}$, and similarly for the differentials. In this section we analyse the E_2-sheet, and get as a consequence that the spectral sequence in fact degenerates at the E_3-level.

5.1.1. *The objects E_2^{pq}.* By duality, it suffices to consider the lower half of Figure 1. The only nonzero differentials in this area, at the E_1-level, are in the lower row $q = 0$. We observed in Section 4.2 that the cohomology groups of this row are the groups of morphisms $M^\bullet \to M^\bullet[p]$ modulo homotopy.

Lemma 5.1. *The dimensions of $E_2^{p,0}$ for $p = 0, 1, 2$ are 1, $N-1$ and $6(N-1)^2 - N + 2$, respectively.*

In the proof we will use the following notation: for any map $f\colon \mathscr{A} \to \mathscr{B}$, its *transpose* $f^t\colon \mathscr{B} \to \mathscr{A}^\vee = \mathscr{C}$ is

$$f^t = f^\vee \circ \iota,$$

where $\iota\colon \mathscr{B} \xrightarrow{\sim} \mathscr{B}^\vee$ is the direct sum over all i of the skew symmetric

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i} \to \mathscr{P}_{-a_i} \oplus \mathscr{P}_{a_i}.$$

Thus $\psi = \phi^t$.

Proof. The vector spaces in question are the cohomologies of the complex

$$0 \longrightarrow E_1^{0,0} \xrightarrow{d_1^{0,0}} E_1^{1,0} \xrightarrow{d_1^{1,0}} E_1^{2,0} \longrightarrow 0,$$

where

$$\begin{aligned}\dim E_1^{0,0} &= \dim\big(\operatorname{Hom}(\mathscr{A},\mathscr{A})\oplus\operatorname{Hom}(\mathscr{B},\mathscr{B})\oplus\operatorname{Hom}(\mathscr{C},\mathscr{C})\big)\\ &= 2(N-1)^2+2N\\ \dim E_1^{1,0} &= \dim\big(\operatorname{Hom}(\mathscr{A},\mathscr{B})\oplus\operatorname{Hom}(\mathscr{B},\mathscr{C})\big)\\ &= 4N(N-1)\\ \dim E_1^{2,0} &= \dim\operatorname{Hom}(\mathscr{A},\mathscr{C})\\ &= 8(N-1)^2\end{aligned}\tag{7}$$

(use that $\Gamma(X,\mathscr{O}_X(\Theta_{\pm a_i}))$ has dimension 1, and $\Gamma(X,\mathscr{O}_X(2\Theta))$ has dimension 8). Thus it suffices to compute the dimensions of the kernels of the two differentials $d_1^{0,0}$ and $d_1^{1,0}$, i.e. the vector spaces of morphisms of degree 0 and 1 from the monad to itself.

One checks immediately that any morphism $M^\bullet\to M^\bullet$ (of degree 0) is multiplication with a scalar, so

$$\dim E_2^{0,0}=1.\tag{8}$$

Next we compute the dimension of the space of morphisms $M^\bullet\to M^\bullet[1]$. Since $\mathscr{C}=\mathscr{A}^\vee$, such a morphism is given by an element of

$$\operatorname{Hom}(\mathscr{A},\mathscr{B})\oplus\operatorname{Hom}(\mathscr{B},\mathscr{A}^\vee),$$

which we may write as $(\mu,-\nu^t)$, where both μ and ν are homomorphisms $\mathscr{A}\to\mathscr{B}$. The sign on $-\nu^t$ is inserted to compensate for the sign on the differential in the shifted complex $M^\bullet[1]$; thus $(\mu,-\nu^t)$ defines a morphism $M^\bullet\to M^\bullet[1]$ if and only if $\nu^t\circ\phi=\phi^t\circ\mu$.

As in Proposition 3.2, we decompose these homomorphisms into pairs $\mu^\pm$ and $\nu^\pm$, and then

$$\begin{aligned}\nu^t\circ\phi &= (\nu^-)^\vee\circ\phi^+-(\nu^+)^\vee\circ\phi^-\\ \phi^t\circ\mu &= (\phi^-)^\vee\circ\mu^+-(\phi^+)^\vee\circ\mu^-.\end{aligned}\tag{9}$$

Choosing generators $\vartheta_i^\pm\in\Gamma(X,\mathscr{O}_X(\Theta_{\pm a_i}))$, we may represent μ by a matrix with entries $\mu_{ij}^\pm\vartheta_i^\pm$, where $\mu_{ij}^\pm$ are scalars. Similarly for ν. Then the two compositions (9) are given by $(N-1)\times(N-1)$ scalar matrices with entries

$$\begin{aligned}(\nu^t\circ\phi)_{ij} &= (\mu_{ij}^+-\mu_{ij}^-)\vartheta_i^+\vartheta_i^-+(\mu_{Nj}^+-\mu_{Nj}^-)\vartheta_N^+\vartheta_N^-\\ (\phi^t\circ\mu)_{ij} &= (\nu_{ji}^+-\nu_{ji}^-)\vartheta_j^+\vartheta_j^-+(\nu_{Ni}^+-\nu_{Ni}^-)\vartheta_N^+\vartheta_N^-.\end{aligned}\tag{10}$$

Recall that the Kummer map $X\to|2\Theta|$ sends $a_i\in X$ to the divisor $\Theta_{a_i}+\Theta_{-a_i}$. This implies that, for sufficiently general points a_i, and $i\neq j$, the three elements $\vartheta_i^+\vartheta_i^-$, $\vartheta_j^+\vartheta_j^-$ and $\vartheta_N^+\vartheta_N^-$ are linearly independent in $\Gamma(X,\mathscr{O}_X(2\Theta))$. It follows easily that the two expressions in (10) coincide for all i and j if and only if there are equalities of scalar

$(N-1)\times(N-1)$ matrices

$$(\mu_{ij}^+) - (\mu_{ij}^-) = (\nu_{ij}^+) - (\nu_{ij}^-) = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_{N-1} \\ c_N & c_N & \cdots & c_N \end{pmatrix},$$

where $c_1, \dots, c_N$ are arbitrary scalars. Thus the vector space of morphisms $M^\bullet \to M^\bullet[1]$ has a basis corresponding to the (μ_{ij}^+), (ν_{ij}^+) and (c_i), hence has dimension $2N(N-1)+N$. The expressions for $\dim E_2^{p,0}$ follow from this, together with (7) and (8). $\square$

5.1.2. *The differentials d_2^{pq}.* The only nonzero differentials at the E_2-level are $d_2^{0,1}$ and its dual $d_2^{-2,3}$. So it suffices to analyse $d_2^{0,1}$. This is, by Lemma 4.1, an obstruction map for equipping first order infinitesimal deformations of the objects M^i with differentials, and will henceforth be denoted ω.

The domain

$$E_2^{0,1} = \bigoplus_i \operatorname{Ext}^1(M^i, M^i) \tag{11}$$

of $\omega = d_2^{0,1}$ is canonically isomorphic to a direct sum of a large number of copies of $H^1(X, \mathscr{O}_X)$. More precisely, for each i and j from 1 to $N-1$, apply the bifunctor $\operatorname{Ext}^1(-,-)$ to the i'th projection $\mathscr{A} \to \mathscr{O}_X(-\Theta)$ in the first argument and the j'th inclusion $\mathscr{O}_X(-\Theta) \to \mathscr{A}$ in the second argument. This defines an inclusion

$$f_{ij}\colon H^1(X, \mathscr{O}_X) \cong \operatorname{Ext}^1(\mathscr{O}_X(-\Theta), \mathscr{O}_X(-\Theta)) \hookrightarrow \operatorname{Ext}^1(\mathscr{A}, \mathscr{A})$$

and clearly the direct sum of all the f_{ij}'s is an isomorphism. Similarly, for all i and j from 1 to $N-1$, we define inclusions

$$h_{ij}\colon H^1(X, \mathscr{O}_X) \cong \operatorname{Ext}^1(\mathscr{O}_X(\Theta), \mathscr{O}_X(\Theta)) \hookrightarrow \operatorname{Ext}^1(\mathscr{C}, \mathscr{C})$$

whose direct sum is an isomorphism. Finally, for all i from 1 to N, and each sign $\pm$, define inclusions

$$g_i^\pm\colon H^1(X, \mathscr{O}_X) \cong \operatorname{Ext}^1(\mathscr{P}_{\pm a_i}, \mathscr{P}_{\pm a_i}) \hookrightarrow \operatorname{Ext}^1(\mathscr{B}, \mathscr{B})$$

induced by projection to and inclusion of the summand $\mathscr{P}_{\pm a_i}$ of $\mathscr{B}$. Note that also the direct sum of the $g_i^\pm$'s is an isomorphism, since $\operatorname{Ext}^1(\mathscr{P}_x, \mathscr{P}_y) = H^1(X, \mathscr{P}_{y-x})$ vanishes unless $x = y$.

The obstruction map ω takes values in homotopy classes of morphisms $M^\bullet \to M^\bullet[2]$. Such a morphism is given by a single homomorphism from $\mathscr{A}$ to $\mathscr{C}$, which can be presented as an $(N-1)\times(N-1)$ matrix with entries in $\Gamma(X, \mathscr{O}_X(2\Theta))$. We now give such a matrix representative for the homotopy class $\omega(\xi)$, for any element ξ in each summand $H^1(X, \mathscr{O}_X)$ of $E_2^{0,1}$.

Lemma 5.2. *For every i, the boundary map of the long exact cohomology sequence associated to the Koszul complex*

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\begin{pmatrix}\vartheta_i^+\\ \vartheta_i^-\end{pmatrix}} \mathscr{O}_X(\Theta_{a_i}) \oplus \mathscr{O}_X(\Theta_{-a_i}) \xrightarrow{\begin{pmatrix}\vartheta_i^- & -\vartheta_i^+\end{pmatrix}} \mathscr{I}_{Y_i}(2\Theta) \longrightarrow 0$$

induces an isomorphism

$$H^0(X, \mathscr{I}_{Y_i}(2\Theta))/\langle \vartheta_i^+\vartheta_i^-\rangle \cong H^1(X, \mathscr{O}_X)$$

where $\langle \vartheta_i^+\vartheta_i^-\rangle$ denotes the one dimensional vector space spanned by the section $\vartheta_i^+\vartheta_i^-$.

Proof. Since $H^1(X, \mathscr{O}_X(\Theta_{\pm a_i})) = 0$, there is an induced right exact sequence

$$\begin{matrix} H^0(X, \mathscr{O}_X(\Theta_{a_i})) \\ \oplus \\ H^0(X, \mathscr{O}_X(\Theta_{-a_i})) \end{matrix} \longrightarrow H^0(X, \mathscr{I}_{Y_i}(2\Theta)) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow 0.$$

Each summand $H^0(X, \mathscr{O}_X(\Theta_{\pm a_i}))$ is spanned by $\vartheta_i^\pm$, which is sent to $\mp\vartheta_i^+\vartheta_i^-$ in $H^0(X, \mathscr{I}_{Y_i}(2\Theta))$. □

Proposition 5.3. *Let $\xi \in H^1(X, \mathscr{O}_X)$. The obstruction map ω does the following on each summand in its domain:*

(1) *Lift ξ to sections u and v of $\mathscr{I}_{Y_i}(2\Theta)$ and $\mathscr{I}_{Y_N}(2\Theta)$, respectively, using the lemma. Then $\omega(f_{ij}(\xi))$ is represented by the $(N-1) \times (N-1)$ matrix having j'th column (the transpose of)*

$$\begin{pmatrix} v & \cdots & v & \underset{\substack{\uparrow \\ \text{(entry } i)}}{u+v} & v & \cdots & v \end{pmatrix}$$

and zeros everywhere else.

(2) *Lift ξ to a section u of $\mathscr{I}_{Y_i}(2\Theta)$. If $i \neq N$, then $\omega(g_i^\pm(\xi))$ is represented by the $(N-1) \times (N-1)$ matrix having u at entry (i, i), and zeros everywhere else. The remaining case $\omega(g_N^\pm(\xi))$ is represented by the $(N-1) \times (N-1)$ matrix having all entries equal to u.*

(3) *Lift ξ to sections u and v of $\mathscr{I}_{Y_j}(2\Theta)$ and $\mathscr{I}_{Y_N}(2\Theta)$, respectively. Then $\omega(h_{ij}(\xi))$ is represented by the $(N-1) \times (N-1)$ matrix having i'th row*

$$\begin{pmatrix} v & \cdots & v & \underset{\substack{\uparrow \\ \text{(entry } j)}}{u+v} & v & \cdots & v \end{pmatrix}$$

and zeros everywhere else.

Proof. Notation: In the commutative diagrams that follow, we will use dotted arrows roughly to indicate maps we are about to fill in by some construction.

Part 1: We view ξ as an extension in $\operatorname{Ext}^1(\mathscr{O}_X(-\Theta), \mathscr{O}_X(-\Theta))$. Consider the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_X(-\Theta) \arrow[r] \arrow[d, hook] \arrow[dr, "\phi_i"] & \mathscr{F} \arrow[r] \arrow[d, dashed, "\hat{\phi}_i"] & \mathscr{O}_X(-\Theta) \arrow[r] \arrow[d, dashed] & 0 \\
 & \mathscr{A} \arrow[r, "\phi"'] & \mathscr{B} \arrow[r, "\psi"'] & \mathscr{C} &
\end{tikzcd}
\tag{12}
$$

where the top row is the extension ξ, the bottom row is the monad, and the leftmost vertical map is inclusion of the i'th summand, so in terms of matrices, ϕ_i is the i'th column of ϕ. Suppose we can find a map $\hat{\phi}_i$ making the left part of the diagram commute. Then it is straight forward to verify, using Lemma 4.1, that $\omega(f_{ij}(\xi))$ is represented by the induced vertical map on the right, precomposed with projection $\mathscr{A} \to \mathscr{O}_X(-\Theta)$ on the j'th summand.

Thus we seek $\hat{\phi}_i$. The assumption that u is a lifting of ξ, means that there is a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_X(-\Theta) \arrow[r, "\begin{pmatrix} \vartheta_i^+ \\ \vartheta_i^- \end{pmatrix}"] & \mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i} \arrow[r, "\begin{pmatrix} \vartheta_i^- & -\vartheta_i^+ \end{pmatrix}"] & \mathscr{I}_{Y_i}(\Theta) \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{O}_X(-\Theta) \arrow[r] \arrow[u, equal] & \mathscr{F} \arrow[r] \arrow[u, "\tilde{u}"] & \mathscr{O}_X(-\Theta) \arrow[r] \arrow[u, "u"'] & 0
\end{tikzcd}
$$

in which the rightmost square is a pullback. Similarly, the section v fits in the pullback diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_X(-\Theta) \arrow[r, "\begin{pmatrix} \vartheta_N^+ \\ \vartheta_N^- \end{pmatrix}"] & \mathscr{P}_{a_N} \oplus \mathscr{P}_{-a_N} \arrow[r, "\begin{pmatrix} \vartheta_N^- & -\vartheta_N^+ \end{pmatrix}"] & \mathscr{I}_{Y_N}(\Theta) \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{O}_X(-\Theta) \arrow[r] \arrow[u, equal] & \mathscr{F} \arrow[r] \arrow[u, "\tilde{v}"] & \mathscr{O}_X(-\Theta) \arrow[r] \arrow[u, "v"'] & 0
\end{tikzcd}
$$

Now define $\hat{\phi}_i$ to be

$$
(\tilde{u}, \tilde{v}) \colon \mathscr{F} \longrightarrow (\mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i}) \oplus (\mathscr{P}_{a_N} \oplus \mathscr{P}_{-a_N})
$$

followed by the appropriate inclusion to $\mathscr{B}$. One verifies immediately that $\hat{\phi}_i$ extends ϕ_i in (12), and that the induced map in the rightmost part of that diagram is given by the vector as claimed in part 1.

Part 2: We view ξ as an extension in $\operatorname{Ext}^1(\mathscr{P}_{\pm a_i}, \mathscr{P}_{\pm a_i})$. Consider the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathscr{P}_{\pm a_i} \arrow[r] \arrow[d, "\psi_i^\pm"'] & \mathscr{G} \arrow[r] \arrow[dl, dashed] & \mathscr{P}_{\pm a_i} \arrow[r] & 0 \\
 & \mathscr{C} & & \mathscr{A} \arrow[u, "\phi_i^\pm"'] \arrow[ul, dashed] &
\end{tikzcd}
$$

where the top row is ξ, and the two vertical maps $\phi_i^{\pm}$ and $\psi_i^{\pm}$ denote the i'th row of $\phi^{\pm}$ and the i'th column of $\psi^{\pm}$. Suppose we can find dotted arrows making the two triangles commute. Then one verifies, using Lemma 4.1, that the composition of the two dotted arrows is a representative for $\omega(g_i^{\pm}(\xi)))$.

The problem reduces to seeking s and t fitting in a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{P}_{\pm a_i} \arrow[r] \arrow[d, "\vartheta_i^{\mp}"'] & \mathscr{G} \arrow[r] \arrow[dl, dashed, "t"] & \mathscr{P}_{\pm a_i} \arrow[r] & 0 \\
& \mathscr{O}_X(-\Theta) & & \mathscr{O}_X(\Theta) \arrow[u, "\mp\vartheta_i^{\pm}"'] \arrow[ul, dashed, "s"'] &
\end{tikzcd} \tag{13}$$

as follows: if $i < N$, then $\mp(0, \ldots, 0, s, 0, \ldots, 0)$ would lift $\phi_i^{\pm}$ and $\pm(0, \ldots, 0, t, 0, \ldots, 0)$ would extend $\psi_i^{\pm}$. Their composition is the matrix having $t \circ s$ in entry (i, i), and zeros elsewhere. If $i = N$, then similarly $\mp(s, \ldots, s)$ and $\pm(t, \ldots, t)$ would be the required lift and extension. Their composition is the matrix having all entries equal to $t \circ s$. Thus part 2 of the proposition will be established once we have constructed such maps s and t having composition $t \circ s = u$.

Now use that ξ is the pullback of the Koszul complex for Y_i along u. This enables us to construct the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_X(-\Theta) \arrow[rr, "\begin{pmatrix} \vartheta_i^+ \\ \vartheta_i^- \end{pmatrix}"] \arrow[dd, equal] \arrow[dr, "\vartheta_i^{\mp}"'] & & \mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i} \arrow[rr, "\begin{pmatrix} \vartheta_i^- & -\vartheta_i^+ \end{pmatrix}"] \arrow[dl] & & \mathscr{I}_{Y_i}(\Theta) \arrow[r] & 0 \\
& & \mathscr{P}_{\mp a_i} \arrow[dd, equal] & & \mathscr{P}_{\mp a_i} \arrow[ul] \arrow[ur, "\mp\vartheta_i^{\pm}"'] & & \\
0 \arrow[r] & \mathscr{O}_X(-\Theta) \arrow[rr] \arrow[dr, "\vartheta_i^{\mp}"'] & & \mathscr{G}' \arrow[rr] \arrow[uu] \arrow[dl, dashed] & & \mathscr{O}_X(-\Theta) \arrow[r] \arrow[uu, "u"'] & 0 \\
& & \mathscr{P}_{\mp a_i} & & \mathscr{P}_{\mp a_i}(-2\Theta) \arrow[uu, "u"] \arrow[ul, dashed] \arrow[ur, "\mp\vartheta_i^{\pm}"'] & &
\end{tikzcd}$$

as follows: the top row is the Koszul complex, and the two unlabelled diagonal arrows in the top part are the canonical inclusion of and projection to the summand $\mathscr{P}_{\mp a_i}$. Pull back along u to get the short exact sequence in the lower part of the diagram. Thus this sequence coincides with ξ, twisted by $\mathscr{P}_{\mp a_i}(-\Theta)$. There are now uniquely determined dotted arrows making the diagram commute, and their composition is u. Up to twist by $\mathscr{P}_{\pm a_i}(\Theta)$, the lower part of the diagram is thus the required diagram (13). This ends the proof of part 2.

Part 3 is essentially dual to part 1, and is left out. $\square$

By Lemma 2.8, we have

$$H^0(\mathscr{I}_{Y_i}(2\Theta)) \oplus H^0(\mathscr{I}_{Y_j}(2\Theta)) = H^0(\mathscr{O}_X(2\Theta)) \tag{14}$$

for all $i \neq j$ (the Lemma gives an inclusion of the left hand side into the right hand side, and by Riemann-Roch, the two sides have the same dimension). This decomposition of sections of $\mathscr{O}_X(2\Theta)$, together with the explicit description of ω in the proposition, enables us to conclude:

Corollary 5.4. *The obstruction map ω is surjective.*

Proof. We show that any $(N-1)\times(N-1)$ matrix of sections of $\mathscr{O}_X(2\Theta)$ represents an element in the image of ω. Let i and j be indices between 1 and $N-1$.

Let u be a section of $\mathscr{I}_{Y_i}(2\Theta)$. Let $\xi \in H^1(X, \mathscr{O}_X)$ be its image under the boundary map in Lemma 5.2. By the Proposition,

$$\omega(f_{ij}(\xi) - f_{ii}(\xi) + g_i^{\pm}(\xi))$$

is represented by the matrix having u as entry (i, j) and zeros elsewhere. Similarly, start with a section u of $\mathscr{I}_{Y_j}(2\Theta)$ instead, then

$$\omega(h_{ij}(\xi) - h_{ii}(\xi) + g_i^{\pm}(\xi))$$

is represented by the matrix having u as entry (i, j) and zeros elsewhere. By (14) it follows that the image of ω contains an element represented by any matrix (s_{ij}) of sections of $\mathscr{O}_X(2\Theta)$, subject to the condition that s_{ii} is a section of $\mathscr{I}_{Y_i}(2\Theta)$.

Now start with an arbitrary section v of $\mathscr{I}_{Y_N}(2\Theta)$ and let ξ be its image in $H^1(X, \mathscr{O}_X)$. Then the representative of $\omega(f_{ii}(\xi))$ in the Proposition is equivalent, modulo matrices (s_{ij}) of the form already obtained, to the matrix having v as element (i, i) and zeros elsewhere. Apply (14) with $j = N$ to conclude that any matrix of sections of $\mathscr{O}_X(2\Theta)$, with no restriction on the diagonal elements, represents an element in the image of ω. $\square$

Corollary 5.5. *The spectral sequence* (6) *degenerates at E_3.*

Proof. The previous corollary implies $E_3^{2,0} = 0$. By duality also $E_3^{-2,3} = 0$. It follows from the shape of the first sheet, Figure 1, that all differentials vanish at the E_3-level and beyond. $\square$

5.2. **First order deformations.** From the calculations in the previous section, we can understand infinitesimal deformations of the vector bundle $\mathscr{E}$ in terms of its monad. Let $k[\epsilon]$ be the ring of dual numbers. By a first order deformation of $M^\bullet$, we mean a monad over $X \otimes_k k[\epsilon]$, with $M^\bullet$ as fibre over $\epsilon = 0$, modulo isomorphism.

Theorem 5.6. *Let $M^\bullet$ be a decomposable monad with cohomology $\mathscr{E}$. The vector spaces of first order infinitesimal deformations of $M^\bullet$ and of $\mathscr{E}$ are isomorphic via the natural map, sending a first order deformation of $M^\bullet$ to its cohomology.*

Proof. Since the hyperext spectral sequence associated to the monad degenerates at E_3, and the only E_3^{pq} terms with $p+q=1$ are $E_3^{0,1}$ and $E_3^{1,0}$, there is a short exact sequence

$$0 \longrightarrow E_3^{1,0} \longrightarrow \operatorname{Ext}^1(\mathscr{E},\mathscr{E}) \longrightarrow E_3^{0,1} \longrightarrow 0. \tag{15}$$

Let $D(M^\bullet)$ be the vector space of first order deformations of $M^\bullet$. Thus the claim is that the natural map $D(M^\bullet) \to \operatorname{Ext}^1(\mathscr{E},\mathscr{E})$ is an isomorphism.

Now $E_3^{0,1}$ is the kernel of the obstruction map $\omega = d_2^{0,1}$. By Lemma 4.1, this is the space of those first order deformations of the objects in $M^\bullet$, that allow the differential d_M to extend (non uniquely) to the deformed objects. Via this identification, $D(M^\bullet) \to E_3^{0,1}$ is the natural forgetful map, so it is surjective. Moreover, its kernel is the space of first order deformations of the differential in $M^\bullet$, keeping the objects fixed. It remains to see that this space maps isomorphically to $E_3^{1,0}$.

By the shape of the spectral sequence (Figure 1) we have $E_3^{1,0} = E_2^{1,0}$, and, by Lemma 5.1, this is

$$E_2^{1,0} = \operatorname{Hom}_{K(X)}(M^\bullet, M^\bullet[1]).$$

The inclusion of $E_2^{1,0}$ into $\operatorname{Ext}^1(\mathscr{E},\mathscr{E})$ is the edge map discussed in Section 4.2. This map factors canonically through $D(M^\bullet)$: associate to a map $f\colon M^\bullet \to M^\bullet[1]$ of complexes the first order deformation $\phi(f)$ with objects $M^\bullet \otimes_k k[\epsilon]$ and differential $d_M \otimes 1 + f \otimes \epsilon$. The resulting diagram

$$\begin{tikzcd} \operatorname{Hom}_{K(X)}(M^\bullet, M^\bullet[1]) \arrow[r, hook] \arrow[dr, "\phi"'] & \operatorname{Ext}^1(\mathscr{E},\mathscr{E}) \\ & D(M^\bullet) \arrow[u] \end{tikzcd}$$

commutes up to sign. Thus ϕ is injective, and, moreover, its image in $D(M^\bullet)$ is exactly the space of first order deformations of the differential in $M^\bullet$, with constant objects: any differential on $M^\bullet \otimes_k k[\epsilon]$, specializing to d_M for $\epsilon = 0$, has the form $d_M \otimes 1 + f \otimes \epsilon$ for some f satisfying

$$(d_M \otimes 1 + f \otimes \epsilon)^2 = 0.$$

Since $d_M^2 = 0$ and $\epsilon^2 = 0$, this says that $f \circ d_M + d_M \circ f = 0$, i.e. f defines a morphism $M^\bullet \to M^\bullet[1]$. This gives the required identification between $E_3^{1,0}$ and deformations of the differential. $\square$

Next, we give the dimension formula for $\operatorname{Ext}^1(\mathscr{E},\mathscr{E})$, which we phrase in a twist invariant way.

Theorem 5.7. *Let $\mathscr{E}$ be a rank 2 vector bundle obtained as the cohomology of a decomposable monad. Then*

$$\dim \operatorname{Ext}^1(\mathscr{E},\mathscr{E}) = \tfrac{1}{3}\Delta(\mathscr{E}) \cdot \Theta + 5$$

where Δ denotes the discriminant $4c_2 - c_1^2$.

Proof. Consider again the short exact sequence (15).

The space $E_3^{0,1}$ is the kernel of the map $\omega = d_2^{0,1}$ studied in Section 5.1.2. Its domain (11) has dimension

$$(2(N-1)^2 + 2N)\dim H^1(X, \mathscr{O}_X) = 6(N-1)^2 + 6N$$

and its codomain $E_2^{2,0}$ has dimension $6(N-1)^2 - N + 2$, by Lemma 5.1. Since ω is surjective, the dimension of its kernel $E_3^{0,1}$ is thus $7N - 2$. Moreover, the dimension of $E_3^{1,0} = E_2^{1,0}$ is $N - 1$ by the same Lemma, so $\operatorname{Ext}^1(\mathscr{E}, \mathscr{E})$ has dimension $8N - 3$.

On the other hand, we know from the Serre construction that $\mathscr{E}(\Theta)$ has Chern classes $c_1 = 2\Theta$ and $c_2 = N\Theta^2$, and thus discriminant $(4N-4)\Theta^2$. The formula follows. □

Remark 5.8. The space of first order deformations obtained by varying the isomorphism $\omega_Y \cong \mathscr{O}_Y(2\Theta)$, coincides with the space of first order deformations of the differential in $M^\bullet$. In fact, it is trivial that the former is contained in the latter, and these spaces have the same dimension $N - 1$.

The dimension formula shows that, for $N > 2$, there are more first order deformations of $\mathscr{E}$ than there are parameters in the Serre construction (even including translation and twist, as in Section 2.4). We leave open the questions whether there are honest (non infinitesimal) deformations of $\mathscr{E}$ not induced by deformations of the monad, and whether there are such deformations of the monad not induced by the Serre construction.

6. Birational description of $M(0, \Theta^2)$

As before, let (X, Θ) be a principally polarized abelian threefold with Picard number 1. We write $M(c_1, c_2)$ for the coarse moduli space of stable rank 2 vector bundles on X with the indicated Chern classes.

The main point in the preceding section is that all first order infinitesimal deformations of the vector bundles constructed in Section 4.4, in the case of even c_1, can be realized as first order infinitesimal deformations of a monad. In this section we show that in the first nontrivial example, corresponding to $N = 2$, this statement holds not only infinitesimally, but Zariski locally: by deforming the monad, we realize a Zariski open neighbourhood of the vector bundle in its moduli space.

Theorem 6.1. *Let $\mathscr{E}$ be the rank 2 cohomology vector bundle of a decomposable monad, as in Proposition 3.2 for $N = 2$. Then, Zariski locally around $\mathscr{E}$, the moduli space $M(0, \Theta^2)$ is a ruled, nonsingular variety of dimension 13.*

More precisely, there is a Zariski open neighbourhood around $\mathscr{E}$ which is isomorphic to a Zariski open subset of a $\mathbb{P}^1$-bundle over $X \times X \times H$, where H is the Hilbert scheme of two points in the Kummer threefold $X/(-1)$.

Remark 6.2. The Hilbert scheme H in the Theorem has Kodaira dimension zero [9, Theorem 11.1.2]. In particular $M(0, \Theta^2)$ is not rational.

In the proof below we will realize the parametrization of $M(0, \Theta^2)$ using the language of monads. In terms of the Serre construction, we could proceed as follows: Let $(b, b', Z) \in X \times X \times H$. We view Z as an unordered pair $\{\pm a_1, \pm a_2\}$ of points in X modulo sign, to which we associate the curve $Y = Y_1 \cup Y_2$, with $Y_i = \Theta_{a_i} \cap \Theta_{-a_i}$. Choose an isomorphism $\alpha\colon \mathscr{O}_Y(2\Theta) \cong \omega_Y$ modulo scale — this corresponds to a (general) point in the $\mathbb{P}^1$-bundle in the Theorem. Then the Serre construction associates to (Y, α) a rank two vector bundle $\mathscr{E}(\Theta)$, and we send the whole data set (b, b', Z, α) to $T_b^*(\mathscr{E}) \otimes \mathscr{P}_{b'} \in M(0, \Theta^2)$. For the details, we prefer monads, although this is a matter of taste:

Proof. Firstly, we parametrize decomposable monads

$$\mathscr{O}_X(-\Theta) \xrightarrow{\phi} \bigoplus_{i=1}^2 (\mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i}) \xrightarrow{\psi} \mathscr{O}_X(\Theta) \tag{16}$$

by an open subset of a $\mathbb{P}^1$-bundle P over H,

The isomorphism class of (16), as a complex, determines the element $\{\pm a_1, \pm a_2\}$ in the Hilbert scheme H, and for fixed a_i's, the differential

$$\phi = (\vartheta_1^+, \vartheta_1^-, \vartheta_2^+, \vartheta_2^-), \quad \psi = (\vartheta_1^-, -\vartheta_1^+, \vartheta_2^-, -\vartheta_2^+)$$

can be parametrized as follows: let

$$V = \Gamma(X, \mathscr{O}_X(\Theta)) \oplus \Gamma(X, \mathscr{O}_X(\Theta))$$

and associate to each pair $(\vartheta_1, \vartheta_2) \in V$, with nonzero components, the monad with differential given by $\vartheta_i^+ = T_{\pm a_i}^*(\vartheta_1)$. Then it is easy to verify that every monad (16) is isomorphic to one associated to a pair $(\vartheta_1, \vartheta_2)$, and a second pair $(\vartheta_1', \vartheta_2')$ gives an isomorphic monad if and only if there is a $\lambda \in \mathbb{G}_m$ such that

$$(\vartheta_1', \vartheta_2') = (\lambda\vartheta_1, \lambda\vartheta_2) \quad \text{or} \quad (\vartheta_1', \vartheta_2') = (\lambda\vartheta_1, -\lambda\vartheta_2).$$

Thus, for fixed a_i's, the monads (16) are parametrized one-to-one by the quotient of

$$\mathbb{P}^1 \setminus \{0, \infty\} \subset \mathbb{P}^1 = \mathbb{P}(V^\vee)$$

by the involution $(u : v) \mapsto (u : -v)$. The quotient of $\mathbb{P}^1$ by this involution is again a $\mathbb{P}^1$. Order the a_i's temporarily as $(\pm a_1, \pm a_2) \in K \times K$. Then it follows that monads (16) are parametrized in a two-to-one fashion by an open subset in $\mathbb{P}^1 \times K \times K$. Exchanging $\pm a_1 \leftrightarrow \pm a_2$ and $\vartheta_1 \leftrightarrow \vartheta_2$ corresponds to the involution exchanging the two factors of $K \times K$ and acting by $(u : v) \mapsto (v : u)$ in $\mathbb{P}^1$. The quotient of the trivial $\mathbb{P}^1$-bundle $\mathbb{P}^1 \times K \times K$ by this involution gives the required $\mathbb{P}^1$-bundle $P \to H$ over an open subset of the Hilbert scheme.

By construction, an open subset $U \subset P$ parametrizes decomposable monads — it is straight forward to see that there is a "universal" monad on $U \times X$, whose fibres over closed points in U are in bijective correspondence with monads (16). The cohomology of the universal monad is a family of vector bundles, and we may shrink U if necessary to ensure that it is a family of stable vector bundles. It defines a rational map

$$P \dashrightarrow M(0, \Theta^2). \tag{17}$$

Now let $(b, b') \in X \times X$ act on monads by translation $T_b^*(-)$ and twist $\mathscr{P}_{b'} \otimes -$. Apply this to (16): the result is a monad of the form

$$\mathscr{P}_{b'-b}(-\Theta) \longrightarrow \textstyle\bigoplus_{i=1}^2 (\mathscr{P}_{a_i+b} \oplus \mathscr{P}_{-a_i+b}) \longrightarrow \mathscr{P}_{b'+b}(\Theta). \tag{18}$$

Since we can read off (b, b') from the isomorphism class of this monad, we conlude that $X \times X$ acts freely on monads of this form. Moreover, for any two monads $M_1^\bullet$ and $M_2^\bullet$ of the form (18), with cohomology vector bundles $\mathscr{E}_1$ and $\mathscr{E}_2$, the first hyperext spectral sequence gives an isomorphism $\operatorname{Hom}(M_1^\bullet, M_2^\bullet) \xrightarrow{\sim} \operatorname{Hom}(\mathscr{E}_1, \mathscr{E}_2)$ (there are no homotopies, since $E_1^{-1,0}$ vanishes). It follows that $M_1^\bullet$ and $M_2^\bullet$ are isomorphic as complexes if and only if $\mathscr{E}_1$ and $\mathscr{E}_2$ are isomorphic vector bundles. The rational map (17) combined with the $X \times X$-action,

$$P \times X \times X \dashrightarrow M(0, \Theta^2)$$

is thus generically injective (on closed points). By Theorems 5.6 and 5.7 it is also generically étale, and hence birational. □

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Stord/Haugesund University College, Norway
E-mail address: `martin.gulbrandsen@hsh.no`