Kinetic Energy of Rigid Bodies
Purpose: Kinetic energy

Focus on

- Kinetic energy of rigid bodies
- Concept of virtual work and virtual displacement
- Classification of constraints
- D'Alembert's Principle
- Examples
Kinetic energy of a rigid body

Let’s consider a rigid body of mass $m$ in plane motion.

$$T = \frac{1}{2} m \bar{v}^2 + \frac{1}{2} \sum \Delta m_i \mathbf{v}_i^2; \quad |\mathbf{v}'| = |\boldsymbol{\omega} \times \mathbf{r}'|$$

$$= \frac{1}{2} m \bar{v}^2 + \frac{1}{2} \left( \sum_{i} r_i^2 \Delta m_i \right) \omega^2$$

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- Kinetic energy of a rigid body can be separated into:
  - the kinetic energy of motion of the mass center $C$ and
  - the kinetic energy of rotation of the body about $C$.

$$T_{Rot} = \frac{1}{2} \sum \Delta m_i \left( r_i \omega \right)^2 = \frac{1}{2} \sum \left| \boldsymbol{\omega} \times r_i \right|^2 \Delta m_i$$

$$= \frac{1}{2} \left[ I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 - 2I_{xy} \omega_x \omega_y - 2I_{yz} \omega_y \omega_z - 2I_{zx} \omega_z \omega_x \right]$$

$$T = \frac{1}{2} m \bar{v}^2 + \frac{1}{2} \left[ I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 - 2I_{xy} \omega_x \omega_y - 2I_{yz} \omega_y \omega_z - 2I_{zx} \omega_z \omega_x \right]$$
Kinetic energy of a rigid body ...

If axes of coordinates coincide with the principal axes $x'\ y'\ z'$ of the body

$$T = \frac{1}{2} m \overrightarrow{v}^2 + \frac{1}{2} \left[ I_{x'}\overrightarrow{\omega}_x^2 + I_{y'}\overrightarrow{\omega}_y^2 + I_{z'}\overrightarrow{\omega}_z^2 \right]$$

- For a rigid body rotating about a fixed axis through $O$.

$$T = \frac{1}{2} \sum \Delta m_i v_i^2 = \frac{1}{2} \sum \Delta m_i (r_i \omega)^2 = \frac{1}{2} \left( \sum r_i^2 \Delta m_i \right) \omega^2$$

$$= \frac{1}{2} I_o \omega^2$$

- In principal coordinates $x'\ y'\ z'$

$$T = \frac{1}{2} \left[ I_{x'}\overrightarrow{\omega}_x^2 + I_{y'}\overrightarrow{\omega}_y^2 + I_{z'}\overrightarrow{\omega}_z^2 \right]$$
Conservation of Work and Energy

When a rigid body, or a system of rigid bodies, moves under the action of conservative forces, the sum of the kinetic energy \((T)\) and that of the potential energy \((V)\) of the system remains constant.

\[
T_1 + V_1 = T_2 + V_2
\]

In the plane motion of a rigid body, the **kinetic energy** of the body should include both the **translational term** and the **rotational term**

\[
T = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2
\]
Conservation of Work and Energy …

Conservative forces, - forces that do work, for example, gravitation, external forces that lead to displacement, ...

 Forces acting on rigid bodies which do no work are not conservative,

Example:

 Forces applied to fixed points:
 - reactions at a frictionless pin when the supported body rotates about the pin.

 Forces acting in a direction perpendicular to the displacement of their point of application:
 - reaction at a frictionless surface to a body moving along the surface
 - weight of a body when its center of gravity moves horizontal

Friction force at the point of contact of a body rolling without sliding on a fixed surface.
Conservation of Work and Energy ...

- **Example 1:** Let’s express the work of conservative forces as a change in potential energy, the principle of work and energy becomes

\[ T_1 + V_1 = T_2 + V_2 \]

- **Consider the slender rod of mass \( m \).**

\[
T_1 = 0, \quad V_1 = 0 \\
T_2 = \frac{1}{2} m \dot{v}_2^2 + \frac{1}{2} l\omega_2^2 \\
= \frac{1}{2} m \left( \frac{1}{2} l\omega \right)^2 + \frac{1}{2} \left( \frac{1}{12} ml^2 \right) \omega^2 = \frac{1}{2} \frac{ml^2}{3} \omega^2 \\
V_2 = -\frac{1}{2} Wl \sin \theta = -\frac{1}{2} mgl \sin \theta \\
T_1 + V_1 = T_2 + V_2 \\
0 = \frac{1}{2} \frac{ml^2}{3} \omega^2 - \frac{1}{2} mgl \sin \theta \\
\omega = \left( \frac{3g}{l \sin \theta} \right)^{1/2}
\]
Example 2

A wheel is freely rotating about a horizontal axis O and an ideal string is wrapped on it has a small mass $m$ attached at its end as shown. The wheel has a moment of inertia $I$ and radius $R$. Find the expression for the speed of the mass after it has fallen through a distance $h$.

**Sol.:** Using energy theorem

$$T_1 + V_1 = T_2 + V_2$$

$$0 + mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$$

$$v = \sqrt{\frac{2mghR^2}{I + mR^2}}$$
Example 3

A slender rod of mass \( m = 15 \text{ kg} \) and length \( L = 2.5 \text{ m} \) is pivoted at point O, which is 0.5 m from end B. End A of the rod is pressed against a spring with stiffness of \( k = 300 \text{ N/mm} \) until the spring is compressed to \( x = 40 \text{ mm} \), where the rod can be assumed to be in horizontal position. If the rod is released from this position, determine the angular velocity when the rod passes a vertical position above O. What are the reaction forces at O at this moment?

Answer: \( \omega_2 \approx 4 \text{ rad/s} \)

\( R_x = 0 \) and \( R_y = 328 \text{ N} \) \( \uparrow \)
D’Alembert’s Principle

Equilibrium equation in statics: \( \sum F = 0 \)

\( \Rightarrow \) can be used to solve for three unknowns in 3D space

Equilibrium equation of motion in dynamics: \( \sum F = m.a \)

where \( \sum F \) is the sum of the external forces acting on the particle, \( m \) is the particle mass (constant), and \( a \) is the acc. of the particle relative to an inertial ref. frame.

Rewriting the equation of motion: \( \sum F - m.a = 0 \)

Conversion of the system into a dynamics equilibrium \( \Rightarrow \) D’Alembert’s principle

External forces – inertia forces = 0

For a body in rotational motion: \( \sum T = I.\alpha \) and \( \sum T - I.\alpha = 0 \)

where \( \sum T \) is the sum of external torques acting on the body,

\( I \) is the mass moment of inertia of the body with respect to the rotating axis, and \( \alpha \) is the angular acceleration of the body.

The principle is applied to solve problems for a body simultaneously undergoing translation and rotation. It greatly simplifies complicated dynamic problems in mechanics.
Virtual work and virtual displacement

For a system of $N$ particles

- Let the position of each particle be given by $x_1, x_2, \ldots x_{3N}$
- Let’s consider that there are $3N$ forces $F_1, F_2, \ldots F_{3N}$ acting in the direction of each coordinate
- Let the system at a given instant is subjected to small displacements $\delta x_1, \delta x_2, \ldots \delta x_{3N}$ in the direction of each coordinate
- Work done by the forces: $\delta W = \sum F_i \delta x_i$

OR, in vector notation

$$\delta W = \sum F_i \delta r_i$$

Note: • There is no passage of time for the virtual displacement $\delta x$
• The forces remain constant.
Example

Use the virtual work method and determine the relationship between the torque $M$ applied to the crank $R$ and the force $F$ applied to the slider in the shown mechanism.

For system in static equilibrium

$$\delta W = \sum F \delta x = M \delta \theta + F \delta x = 0$$

where $x = R \cos \theta + L \cos \phi$

$$h = R \sin \theta = L \sin \phi \Rightarrow \sin \phi = \frac{R}{L} \sin \theta$$

Trigonometric relation: $\cos \phi = \left(1 - \sin^2 \phi \right)^{\frac{1}{2}}$

$$\Rightarrow \cos \phi = \left(1 - \left(\frac{R}{L}\right)^2 \sin^2 \theta \right)^{\frac{1}{2}}$$

Substituting and simplifying

$$\Rightarrow M = FR \sin \theta \left\{1 + \frac{Ros \theta}{L \sqrt{1 - \left(\frac{R}{L}\right)^2 \sin^2 \theta}}\right\}$$
Classification of constraints

- Holonomic constraint and
- Non-holonomic constraint

If a constraint can be written as an equality function of the form

\[
f(r_1, \ldots, r_n, t) = 0 \text{ or } f(q_1, q_2, \ldots, q_N, t) = 0 \] (where \( q_i \) are generalized coordinates), then it is called holonomic constraint.

A rigid body has such types of constraints and existence of holonomic constraints allows elimination of some variables.

Constraints that can not be expressed in the above form are called non-holonomic constraints.

For example, motion of a gas in a container.
Classification of constraints …

Equality constraints involve only generalized coordinates and time (holonomic), while non-holonomic constraints depend on generalized coordinates and higher derivatives (velocities), as well as time.

Inequalities do not constrain the position in the same way as equality constraints do, thus they are non-holonomic.

Other classifications

A geometric constraint restricts the configurations that can be achieved during motion.

A kinematic constraint only restricts the velocities that can be acquired at a given position. The system can, however, occupy any position desired.
Constraints and degrees of freedom (DOF)

A DOF is an important element in describing the dynamics of a system consisting of multiple lumped parameters.

Number of degrees of freedom of a system

- the minimum number of variables to completely specify the position of every particle in the system
- Number of kinematically independent configuration constraints or variables required to describe completely the motion of a system

NB: A rigid body in (unconstrained) space has 6 DOF \((r, \Omega)\)

The number of degrees of freedom of a particle/lumped mass gets reduced if it is subjected to constraints. For instance, a rigid body in a 3D space has 3 DOF constrained, hence its motion is defined by 3 DOF.
Constraints and degrees of freedom (DOF)

Examples

- Rolling contact
- Pinned joint
  - (Generally only applied to a rigid body)

- Fixed joint
  - (Generally not applied to a rigid body motion)
Summary and Questions

In this lecture we focused on KE of a rigid body, a.o.

- Kinetic energy of rigid bodies: Translational and rotational KE
- KE of rigid body when coordinates of motion coincide with the principal axes of the body
- Concept of virtual work and virtual displacement
- DOF and classification of constraints
- D'Alembert's Principle

Next: Intro. generalized coordinates, derivation of Lagrange's equation from D'Alembert's principle.