«Complex Analysis & Dynamical Systems VI»
dedicated to the **60th Birthday of Professor David Shoikhet**

Boundary behaviour of one-parameter semigroups and evolution families

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One-parameter semigroup in $\mathbb{D}$

Definition

A *one-parameter semigroup in* $\mathbb{D} := \{z : |z| < 1\}$ is a continuous homomorphism from $\left(\mathbb{R}_{\geqslant 0}, +\right)$ to $\left(\mathrm{Hol}(\mathbb{D}, \mathbb{D}), \circ\right)$. In other words, a *one-parameter semigroup* is a family $(\phi_t)_{t \geqslant 0} \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ such that

(i) $\phi_0 = \mathrm{id}_{\mathbb{D}}$;

(ii) $\phi_{t+s} = \phi_t \circ \phi_s = \phi_s \circ \phi_t$ for any $t, s \geqslant 0$;

(iii) $\phi_t(z) \to z$ as $t \to +0$ for any $z \in \mathbb{D}$.

One-parameter semigroups appear, *e.g.* in:

- iteration theory in $\mathbb{D}$ as *fractional iterates*;
- operator theory in connection with *composition operators*;
- *embedding problem* for time-homogeneous stochastic processes;
- as flows of *semicomplete autonomous holomorphic vector fields*.

Fixed points of 1-param. semigroups

Definition

A *boundary fixed point (BFP)* of $\phi \in \mathrm{Hol}(\mathbb{D},\mathbb{D})$ is a point $\sigma \in \mathbb{T} := \partial\mathbb{D}$ at which

$$\angle \lim_{z\to\sigma} \phi(z) = \sigma.$$

The *multiplier* at the BFP σ is $\lambda(\sigma) = \phi'(\sigma) := \angle \lim_{z\to\sigma} \big(\phi(z) - \sigma\big)/\big(z - \sigma\big)$

$$= \liminf_{z\to\sigma} \frac{1 - |\phi(z)|}{1 - |z|}.$$

If $\lambda(\sigma) \neq \infty$, then the BFP σ is said to be *regular (BRFP)*.

Definition

For a fixed point $z_0 \in \mathbb{D}$, the multiplier is $\lambda(z_0) = \phi'(z_0)$.

Fixed points — continued

Definition

The *Denjoy – Wolff point* (*DW-point*) τ of $\phi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\mathrm{id}_{\mathbb{D}}\}$ is the unique fixed point $\tau \in \overline{\mathbb{D}}$ (in the interior or boundary sense) at which the multiplier $|\lambda(\tau)| \leqslant 1$.

In what follows we will assume that

all one-parameter semigroups (ϕ_t) we consider are *not conjugated to rotation*, *or, equivalently,* that $\phi_t \neq \mathrm{id}_{\mathbb{D}}$ for all $t > 0$.

Remark

Elements ϕ_t, $t > 0$, of a 1-parameter semigroup (ϕ_t) share the same:

- Denjoy – Wolff point;
- interior and boundary fixed points;
- BRFPs.

Theorem 1 (Contreras, Díaz-Madrigal, Pommerenke, 2004; **P. Gum.**, ArXiv:1211.3965)

Let (ϕ_t) be a one-parameter semigroup in $\mathbb{D}$. Then:

(i) for all $t \geqslant 0$ and **every** $\boldsymbol{\sigma \in \mathbb{T}}$ there exists the angular limit

$$\phi_t(\sigma) := \angle \lim_{z\to\sigma} \phi_t(z).$$

(ii) moreover, for each $\sigma \in \mathbb{T}$ and each Stolz angle S at σ the continuity of $\phi_t|_{S\cup\{\sigma\}}$ at σ is locally uniform w.r.t. $t \geqslant 0$;

(iii) the family of functions (*"trajectories"*)

$$\Big\{[0,+\infty) \ni t \mapsto \phi_t(z) : z \in \overline{\mathbb{D}}\Big\}$$

is uniformly equicontinuous;

(iv) $\phi_{t+s}(z) = \phi_t\big(\phi_s(z)\big)$ holds also for all $z \in \partial\mathbb{D}$.

Remark

Theorem 1 does NOT imply existence of unrestricted limits

$$\lim_{\mathbb{D}\ni z\to\sigma} \phi_t(z), \quad \sigma \in \mathbb{T}.$$

Theorem 2 (Contreras, Díaz-Madrigal, Pommerenke, 2004; **P. Gum.**, ArXiv:1211.3965)

Let (ϕ_t) be a one-parameter semigroup in $\mathbb{D}$ and $\sigma \in \mathbb{T}$ its boundary fixed point. Then:

(UnrLim) for any $t \geqslant 0$ there exists the unrestricted limit

$$\lim_{\mathbb{D}\ni z\to\sigma} \phi_t(z) \quad [\text{clearly} = \sigma],$$

(EqCont) the continuity of $\phi_t|_{\mathbb{D}\cup\{\sigma\}}$ at σ is locally uniform w.r.t. $t \geqslant 0$.

Angular and unrestricted limits

Some remarks on Theorem 2.

- Contreras, Díaz-Madrigal, and Pommerenke proved (UnrLim) for the case of the DW-point $\tau \in \mathbb{D}$.
- Their main idea was to show that the Kœnigs function of (ϕ_t) is continuous at BFPs (as a map to $\overline{\mathbb{C}}$).
- For the case of $\tau \in \mathbb{T} := \partial\mathbb{D}$:
  - ☺ the method of C. – D.-M. – P. works for BFPs $\sigma \in \mathbb{T} \setminus \{\tau\}$,
  - ☹ but it fails for $\sigma = \tau$, because in fact the Kœnigs function does NOT need to be continuous at the boundary DW-point.

These results can be found in

P. Gumenyuk, *Angular and unrestricted limits of one-parameter semigroups in the unit disk.* Preprint, 32pp. ArXiv:1211.3965

Evolution families

Definition (Bracci, Contreras, Díaz-Madrigal, 2012)

A family $(\varphi_{s,t})_{0\leqslant s\leqslant t} \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$ is an *evolution family* of order $d \in [1,+\infty]$ if

EF1. $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$ for all $s \geqslant 0$; EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ if $0 \leqslant s \leqslant u \leqslant t$;

EF3. for any $z \in \mathbb{D}$ there exists a function $k_z \in L^d_{\mathrm{loc}}([0,+\infty))$ s.t.

$$\left|\varphi_{s,u}(z) - \varphi_{s,t}(z)\right| \leqslant \int_u^t k_z(\xi)d\xi, \quad 0 \leqslant s \leqslant u \leqslant t. \tag{1}$$

Remarks

☞ This notion is a *non-autonomous generalization* of one-parameter semigroups. Indeed, if (ϕ_t) is a one-parameter semigroup, then $\varphi_{s,t} := \phi_{t-s}$, $t \geqslant s \geqslant 0$, is an evolution family of order $d = +\infty$.

☞ It comes from the much-celebrated *Loewner Theory*.

Evolution families — continued

Remarks

- In contrast to one-parameter semigroups, *every univalent* $\phi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ can be embedded into an evolution family.
- Conversely, all elements of every evolution family $(\varphi_{s,t})$ are *univalent functions*.
- Evolution families can be described by means of a certain non-autonomous semicomplete ODE.

This ODE, known as the general *Loewner ODE*, is of the form

$$\frac{d}{dt}\varphi_{s,t}(z) = G\big(\varphi_{s,t}(z), t\big), \quad t \geqslant s; \quad \varphi_{s,t}(z)\big|_{t=s} = z. \tag{2}$$

The function G in the r.h.s. is referred to as a *Herglotz vector field*.

Herglotz vector fields

Definition (Bracci, Contreras, Díaz-Madrigal, 2012)

A function $G : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is said to be a *Herglotz vector field* of order $d \in [1, +\infty]$, if:

(i) for a.e. $t \geqslant 0$ fixed, the function $G(\cdot, t)$ is an infinitesimal generator of some one-parameter semigroup in $\mathbb{D}$, *i.e.* [Berkson – Porta, 1978]

$$G(z, t) = (\tau_t - z)(1 - \overline{\tau_t} z) p_t(z), \tag{3}$$

where $\tau_t \in \overline{\mathbb{D}}$ and $p_t \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ with $\mathrm{Re}\, p_t \geqslant 0$;

(ii) for each $z \in \mathbb{D}$ fixed, the function $G(z, \cdot)$ is measurable on $[0, +\infty)$;

(iii) for each compact set $K \subset \mathbb{D}$ there exists a non-negative function $k_K \in L^d_{\mathrm{loc}}\big([0, +\infty)\big)$ such that

$$\sup_{z \in K} |G(z, t)| \leqslant k_K(t) \quad \text{for a.e. } t \geqslant 0.$$

Evolution families and Hergtlotz VFs

Theorem (Bracci, Contreras, Díaz-Madrigal, 2012)

Let $(\varphi_{s,t}) \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$, $d \in [1, +\infty]$.
Then $(\varphi_{s,t})$ is an evolution family of order d $\iff$ there exists a Herglotz vector field G of the same order d s.t.
for any $s \geqslant 0$, $z \in \mathbb{D}$, the function $w = w_{z,s}(t) := \varphi_{s,t}(z)$ is the positive trajectory of the general Loewner ODE

$$dw/dt = G\big(w(t), t\big), \quad t \geqslant s; \quad w(s) = z. \tag{4}$$

Theorem (Bracci, Contreras, Díaz-Madrigal, 2012)

In the above theorem, the correspondence between the evolution families and Herglotz vector fields is one-to-one and onto.

F. Bracci, **M.D. Contreras**, **S. Díaz-Madrigal** and **P. Gumenyuk**, *Boundary regular fixed points in Loewner Theory*. Preprint, 28pp.
ArXiv:1303.5216

BRFPs of evolution families

Theorem 3 (Bracci, Contreras, Díaz-Madrigal, **P. Gum.**; ArXiv '13)

Let $(\varphi_{s,t})$ be an evolution family, G its Herglotz vector field and $\sigma \in \mathbb{T}$. Then the following two assertions are "$\Longleftrightarrow$":

(i) σ is a BRFP of $\varphi_{s,t}$ for each $s \geqslant 0$ and $t \geqslant s$;

(ii) the following two conditions hold:

(ii.1) for a.e. $t \geqslant 0$, $G(\cdot, t)$ has a BRNP at σ, *i.e.* there exists

$$G'(\sigma, t) := \angle \lim_{z \to \sigma} \frac{G(z,t)}{z - \sigma} =: \ell(t) \neq \infty; \tag{5}$$

(ii.2) the function ℓ is of class L^1_{loc} on $[0, +\infty)$.

Moreover, if the assertions above hold, then $\ell(t) \in \mathbb{R}$ and

$$\varphi'_{s,t}(\sigma) = \exp \int_s^t \ell(t')\, dt' \quad \text{whenever } 0 \leqslant s \leqslant t. \tag{6}$$

Remarks on Theorem 3

- For the autonomous case, *i.e.* for one-parameter semigroups, it was proved by Contreras, Díaz-Madrigal & Pommerenke, 2006.
- Analogous characterization of evolution families with the common DW-point was given by Bracci, Contreras & Díaz-Madrigal, 2012.
- Asymmetry in Theorem 3:

(i) σ is a BRFP of all $\varphi_{s,t}$'s $\Longrightarrow$ $\varphi'_{s,t}(\sigma)$ is AC_{loc} in s and t

(ii.1) σ is a BRNP of $G(\cdot,t)$ for a.e. $t \geqslant 0$ $\not\Longleftrightarrow$ (ii.2) $\ell(t) := G'(\sigma,t)$ is L^1_{loc}

- Comparison with the case of the DW-point: [curious]

If σ is the DW-point of every $\varphi_{s,t} \neq \mathrm{id}_{\mathbb{D}}$, then ℓ is of class L^d_{loc}, while for the common BRFP σ, we only have $\ell \in L^1_{\mathrm{loc}}$ $\left[\ell^+ \in L^d_{\mathrm{loc}}\right.$ but $\left.\ell^- \in L^1_{\mathrm{loc}}\right]$.

Regular contact points

Definition

A point $\sigma \in \mathbb{T}$ is said to be a *regular contact point* of an evolution family $(\varphi_{s,t})$ if it is a regular contact point of $\varphi_{0,t}$ for all $t \geqslant 0$,

i.e., for all $t \geqslant 0$,

$$\exists\, \varphi_{0,t}(\sigma) := \angle \lim_{z\to\sigma} \varphi_{0,t}(z) \in \mathbb{T} \quad \text{and}$$

$$\varphi'_{0,t}(\sigma) := \angle \lim_{z\to\sigma} \frac{\varphi_{0,t}(z) - \varphi_{0,t}(\sigma)}{z - \sigma} \in \mathbb{C}.$$

We studied regular contact points of evolution families and obtain a partial analogue of Theorem 3.

Regular contact points

Theorem 4 (Bracci, Contreras, Díaz-Madrigal, P. Gum.; ArXiv '13)

[Rough formulation]

Let $(\varphi_{s,t})$ be an evolution family, G its Herglotz vector field.
Suppose $\sigma \in \mathbb{T}$ is a regular contact point of $(\varphi_{s,t})$.

Then for any $t \geqslant 0$,

$$\varphi_{0,t}(\sigma) = \sigma + \int_0^t G\big(\varphi_{0,s}(\sigma), s\big)\, ds \quad \text{and}$$

$$\varphi'_{0,t}(\sigma) = \exp \int_0^t G'\big(\varphi_{0,s}(\sigma), s\big)\, ds.$$

[in the angular sense]

THANK YOU !!!