Parametric representation of univalent self-maps with given boundary fixed points

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One of the most classical object of study
in *Geometric Function Theory* is the

Class $\mathcal{S}$

By $\mathcal{S}$ we denote the class of all *holomorphic functions*

$$f : \mathbb{D} \xrightarrow{\text{into}} \mathbb{C}, \quad \mathbb{D} := \{z : |z| < 1\},$$

which are

- *univalent* in $\mathbb{D}$, and
- normalized by the condition $f(0) = 0$, $f'(0) = 1$.

The study of $\mathcal{S}$ is difficult in many aspects, in particular, because:

- there is no natural linear structure in the class $\mathcal{S}$;
- the class $\mathcal{S}$ is even not a convex set in $\mathrm{Hol}(\mathbb{D}, \mathbb{C})$.

Theorem (Parametric Representation) [Loewner, 1923; Kufarev, 1943; Pommerenke, 1965-75; Gutlyanski, 1970]

$$\mathcal{S} = \Big\{ \mathbb{D} \ni z \mapsto f[p](z) = \lim_{t\to+\infty} e^t \varphi_{0,t}(z) :$$
$$[0,+\infty) \ni t \mapsto \varphi_{0,t}(z) =: w(t) \text{ solves } (1) \Big\}$$

Loewner–Kufarev ODE

$$\frac{dw}{dt} = -w(t)\, p(w(t), t), \quad w(0) = z \in \mathbb{D}, \tag{1}$$

where $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is a *classical Herglotz function*, *i.e.*

- $p(z, \cdot)$ is measurable for all $z \in \mathbb{D}$;
- $p(\cdot, t)$ is holomorphic for all $t \geqslant 0$;
- $\operatorname{Re} p > 0$ and $p(0, t) = 1$ for all $t \geqslant 0$.

Representation of $\mathcal{U}_0(\mathbb{D})$

Thus the Parametric Representation is the *surjective map* $p \mapsto f[p]$ from the *convex cone* $\mathcal{P}_0$ of all classical Herglotz functions onto $\mathcal{S}$.

Notation: $\mathcal{U}\ (\mathbb{D}) := \big\{\varphi \in \mathrm{Hol}(\mathbb{D},\mathbb{D})\colon \varphi \text{ is univalent}\big\}$,
$\mathcal{U}_0(\mathbb{D}) := \big\{\varphi \in \mathcal{U}(\mathbb{D})\colon \varphi(0)=0,\ \varphi'(0)>0\big\}$ — semigroups!

Theorem ("Parametric Representation folklore")

Let $\varphi : \mathbb{D} \to \mathbb{C}$. Then $\varphi \in \mathcal{U}_0(\mathbb{D}) \iff \boxed{\varphi = \varphi_{0,T}}$,
where $T := -\log \varphi'(0)$ and $t \mapsto \varphi_{0,t}(z) =: w(t)$ is the solution to

$$dw(t)/dt = -w(t)\, p(w(t),t), \quad w(0) = z \in \mathbb{D},$$

for some classical Herglotz function p.

Simple idea to represent $\mathcal{U}(\mathbb{D})$

For $\varphi \in \mathcal{U}(\mathbb{D})$, write $\varphi = L \circ \varphi_0$, where $\varphi_0 \in \mathcal{U}_0(\mathbb{D})$ and $L \in \mathrm{M\ddot{o}b}(\mathbb{D})$.

Not every simple idea turns out to be productive.

Decomposition $\varphi = L \circ \varphi_0$ does not allow to study infinitesimal structure of subsemigroups in $\mathcal{U}(\mathbb{D},\mathbb{D})$ and $\mathrm{Hol}(\mathbb{D},\mathbb{D})$.

Examples of semigroups w.r.t. $\cdot \circ \cdot$

- Probability generating functions of Galton – Watson processes
  $\mathcal{H}_{\mathrm{gen}} := \left\{\varphi(z) = \sum_{n=0}^{+\infty} p_n z^n \colon p_n \geqslant 0,\ \sum_{n=0}^{+\infty} p_n = 1\right\}$;
- $\mathcal{H}_\infty(\mathbb{H}) := \left\{\varphi \in \mathrm{Hol}(\mathbb{H},\mathbb{H}) \colon \mathrm{Im}\,\varphi(z) \geqslant \mathrm{Im}\, z\right\}$, $\mathbb{H} := \{z \colon \mathrm{Im}\, z > 0\}$,
  $\mathcal{H}_\infty(\mathbb{H}) = \{\mathrm{id}_{\mathbb{H}}\} \cup \left\{\varphi \in \mathrm{Hol}(\mathbb{H},\mathbb{H}) \colon \varphi^{\circ n} \xrightarrow{n\to\infty} \infty\right\}$;
- The reciprocal Cauchy transforms of probabil. measures with finite variance and mean zero $\mathcal{H}_{\mathrm{CT}}(\mathbb{H}) := \left\{\varphi(z) = \left(\int_{\mathbb{R}} \frac{d\mu(x)}{z-x}\right)^{-1} :\right.$
  $\left.\mu \geqslant 0,\ \mu(\mathbb{R}) = 1,\ \int_{\mathbb{R}} x^2\, d\mu(x) < +\infty,\ \int_{\mathbb{R}} x\, d\mu(x) = 0\right\} \subset \mathcal{H}_\infty(\mathbb{H})$;

Examples of semigroups — CONTINUED

- $\mathcal{H}_{\mathrm{SLE}}(\mathbb{H}) := \Big\{\varphi \in \mathrm{Hol}(\mathbb{H},\mathbb{H})$ meromorphic at ∞:
$$\varphi(z) = z + \sum_{n=1}^{+\infty} c_n/z^n,\ c_n \in \mathbb{R}\Big\} \subset \mathcal{H}_{\mathrm{CT}}(\mathbb{H});$$
- $\mathcal{H}_{0,1}(\mathbb{D}) := \Big\{\varphi \in \mathrm{Hol}(\mathbb{D},\mathbb{D})$ with $\varphi(0) = 0$ and BRFP at $1\Big\}$;
- "Unival. analogues": $\mathcal{U}_{\mathrm{gen}}$, $\mathcal{U}_\infty(\mathbb{H})$, $\mathcal{U}_{\mathrm{CT}}(\mathbb{H})$, $\mathcal{U}_{\mathrm{SLE}}(\mathbb{H})$, $\mathcal{U}_{0,1}(\mathbb{D})$.

Definition (BRFP)

$\sigma \in \mathbb{T}$ is a *boundary regular fixed point (BRFP)* of $\varphi \in \mathrm{Hol}(\mathbb{D},\mathbb{D})$ if

$$\angle\lim_{z\to\sigma} \varphi(z) = \sigma, \quad \varphi'(\sigma) := \angle\lim_{z\to\sigma} \frac{\varphi(z)-\sigma}{z-\sigma} \neq \infty.$$

Infinitesimal structure

V. V. Goryainov, 1987, 1992, 1996, 2002(2), 2011: *infinitesimal structure* and *parametric representation* of semigroups in $\mathrm{Hol}(\mathbb{D},\mathbb{D})$

Definition (one-parameter semigroups)

A *one-parameter semigroup* $(\phi_t) \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$ is a continuous semigroup homomorphism

$$\big([0,+\infty),\cdot+\cdot\big) \ni t \mapsto \phi_t \in \big(\mathrm{Hol}(\mathbb{D},\mathbb{D}),\cdot\circ\cdot\big).$$

One-parameter semigroups as semiflows

For any one-parameter semigroup (ϕ_t) there exists $G \in \mathrm{Hol}(\mathbb{D},\mathbb{C})$ s.t.

$$\frac{d\phi_t(z)}{dt} = G(\phi_t(z)), \quad \phi_0(z) = z, \quad \text{for all } z \in \mathbb{D} \text{ and all } t \geqslant 0. \tag{4}$$

Definition: G is called the *(infinitesimal) generator* of (ϕ_t).

Definition (Infinitesimal structure of a semigroup)

By the *infinitesimal structure* $\mathcal{G}[U]$ of a (sub)semigroup $U \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$ we mean the *set of all infinitesimal generators of one-parameter semigroups* $(\phi_t) \subset U$.

Examples

- ☞ $\mathcal{G} := \mathcal{G}[\mathrm{Hol}(\mathbb{D},\mathbb{D})] = \mathcal{G}[\mathcal{U}(\mathbb{D})] = \Big\{G(z) = (\tau - z)(1 - \overline{\tau} z)p(z) : \tau \in \overline{\mathbb{D}}, \operatorname{Re} p \geqslant 0\Big\}$ (Berkson – Porta formula)
- ☞ $\mathcal{G}[\mathcal{U}_0(\mathbb{D})] = \Big\{G(z) = -zp(z) : \operatorname{Re} p \geqslant 0, \operatorname{Im} p(0) = 0\Big\}$

$$\mathcal{U}_0(\mathbb{D}) := \Big\{\varphi \in \mathrm{Hol}(\mathbb{D},\mathbb{D}) : \varphi \text{ univalent}, \varphi(0) = 0, \varphi'(0) > 0\Big\}$$

Analogy with Lie groups

Denote: (ϕ_t^G) the one-param. semigroup generated by $G \in \mathcal{G}$.

Analogue of the Lie exponential map

$$\mathcal{G}[U] \ \ni \ G \ \mapsto \mathrm{Exp}_{\mathrm{Lie}}(G) := \phi_1^G \ \in \ U \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D}) \text{ (subsemigroup)}$$

☹ Unfortunately, typically $\mathrm{Exp}_{\mathrm{Lie}}\big(\mathcal{G}[U]\big) \neq U, \neq \mathcal{O}_U(\mathrm{id}_{\mathbb{D}})$.

Loewner's idea: Instead of (ϕ_t)'s satisfying the autonomous ODE

$$d\phi_t(z)/dt = G(\phi_t(z)), \ \ t \geqslant 0, \quad \phi_0(t) = 0, \tag{6}$$

consider two-parameter families $(\varphi_{s,t})_{t \geqslant s \geqslant 0}$, generated by its *non-autonomous analogue*:

$$d\varphi_{s,t}(z)/dt = G\big(\varphi_{s,t}(z), t\big), \ \ t \geqslant s \geqslant 0, \quad \varphi_{s,s}(z) = z \in \mathbb{D}, \tag{7}$$

where $G(\cdot, t) \in \mathcal{G}[U]$ for a.e. $t \geqslant 0$
(plus a "*'reasonable assumption*" regarding dependence on t).

Development of Loewner's idea

- The ODE $$\frac{d\varphi_{s,t}(z)}{dt} = G\big(\varphi_{s,t}(z), t\big), \ \ t \geqslant s \geqslant 0, \ \ \varphi_{s,s}(z) = z, \qquad (7)$$
  is called the *general Loewner equation*;
- the functions $G : [0, +\infty) \ni t \mapsto G(\cdot, t) \in \mathcal{G}$
  are called *Herglotz vector fields*.
- the families $(\varphi_{s,t})$ are usually referred to as *evolution families*.

Definition

We say that a semigroup $U \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ *admits the (Loewner-type) Parametric Representation* if the union $\mathcal{R}[U]$ of all evolution families $(\varphi_{s,t})$ generated by Herglotz vector fields G
with $G(\cdot, t) \in \mathcal{G}[U]$ for a.e. $t \geqslant 0$ coincides with U.

This means that one can reconstruct the semigroup U from its infinitesimal structure $\mathcal{G}[U]$ using the general Loewner ODE (7).
But, *a priori*, this depends on the *strict definition* of Herglotz v. f.'s.

- ✔ V.V. Goryainov, for several *concrete choices* of the semigroup $U \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$, gave precise definitions of Herglotz vector fields, *specific for each* U, and established Parametric Representations in each case.
- ✔ Recently, another general approach has been suggested by F. Bracci, M.D. Contreras and S. Díaz-Madrogal, 2008
  [*J. Reine Angew. Math.* **672**(2012), 1–37]

Definition (Bracci *et al*)

A function $G : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is a *Herglotz vector field* if:

(i) $G(\cdot, t) \in \mathcal{G}$ for a.e. $t \geqslant 0$;

(ii) $G(z, \cdot)$ is measurable on $[0, +\infty)$ for every $z \in \mathbb{D}$;

(iii) for any compact set $K \subset \mathbb{D}$,
$M_K(t) := \sup_K |G(\cdot, t)|$ is locally integrable on $[0, +\infty)$.

General approach

Definition (evolution families — *intrinsic definition*) Bracci *et al*

A family $(\varphi_{s,t})_{t\geqslant s\geqslant 0} \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$ is called an *evolution family* if:

(i) $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$ for all $s \geqslant 0$;

(ii) $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $t \geqslant u \geqslant s \geqslant 0$;

(iii) for any $z \in \mathbb{D}$, the maps $[s,+\infty) \ni t \mapsto \varphi_{s,t}(z)$ are locally absolutely continuous *uniformly w.r.t.* $s \geqslant 0$.

Theorem (Bracci *et al*)

The general Loewner ODE

$$d\varphi_{s,t}(z)/dt = G\big(\varphi_{s,t}(z),\, t\big), \quad t \geqslant s \geqslant 0, \quad \varphi_{s,s}(z) = z, \tag{7}$$

establishes an (essentially) *1-to-1 correspondence* between *Herglotz vector fields* G and *evolution families* $(\varphi_{s,t})$.

This includes uniqueness and global existence for solutions to (7).

Note: (7) is to be understood as a Carathéodory ODE.

Problem statement

Problem: *construct a Loewner-type parametric representation for semigroups formed by univalent self-maps with given fixed points.*

Let $\mathcal{F}$ be a finite set of points on $\mathbb{T} := \partial\mathbb{D}$.

First family of semigroups

$$\mathcal{U}(\mathbb{D},\mathcal{F}) := \Big\{\varphi \in \mathcal{U}(\mathbb{D}) : \text{each } \sigma \in \mathcal{F} \text{ is a BRFP of } \varphi\Big\}$$

"BRFP"="*boundary regular fixed point*": $\angle\lim_{z\to\sigma} \varphi(z) = \sigma$, $\varphi'(\sigma) \neq \infty$.

Let $\tau \in \overline{\mathbb{D}} \setminus \mathcal{F}$.

Second family of semigroups

$$\mathcal{U}_\tau(\mathbb{D},\mathcal{F}) := \{\mathrm{id}_\mathbb{D}\} \cup \Big\{\varphi \in \mathcal{U}(\mathbb{D},\mathcal{F}) \setminus \{\mathrm{id}_\mathbb{D}\} : \tau \text{ is the DW-point of } \varphi\Big\}$$

"DW-point"="*Denjoy – Wolff point*":

- for $\tau \in \mathbb{D}$ simply means $\varphi(\tau) = \tau$;
- for $\tau \in \mathbb{T}$ means that $\varphi^{\circ n} \to \tau$ l.u. in $\mathbb{D}$ as $n \to +\infty$.

Motivation

- Loenwer's idea potentially can work in the general setting of an abstract semigroup with "compatible diffeology".
  However, no criteria for such a semigroup to admit a parametric representation is known.
  So it is interesting to study more examples.

- In Geometric Function Theory there has been considerable interest to study self-maps with given BRFP's

  H. Unkelbach, 1938, 1940; C. Cowen, Ch. Pommerenke, 1982;

  Ch. Pommerenke, A. Vasil'ev, 2000; J.M. Anderson, A. Vasil'ev, 2008;

  M. Elin, D. Shoikhet, N. Tarkhanov, 2011;

  V.V. Goryainov [talk at Steklov Math. Inst., Moscow, 26/12/2011];

  A. Frolova, M. Levenshtein, D. Shoikhet, A.Vasil'ev, ArXiv:1309.3074, 2013.

- The infinitesimal structure of $\mathcal{U}(\mathbb{D},\mathcal{F})$ and $\mathcal{U}_\tau(\mathbb{D},\mathcal{F})$ is well-studied.

- The following result: [*see next slide*]

Theorem (Bracci, Contreras, Díaz-Madrigal, **P. Gum.**, 2013)

Let $(\varphi_{s,t})$ be an evolution family with Herglotz vector field G.
Let $\sigma \in \mathbb{T}$. Then **TFAE**:

(i) σ is a BRFP of $\varphi_{s,t}$ for all $t \geqslant s \geqslant 0$;

(ii) G satisfies:

(ii.1) for a.e. $t \geqslant 0$ fixed, $G(\cdot, t) \in \mathcal{G}[\mathcal{U}(\mathbb{D}, \{\sigma\})]$ [Contreras, Díaz-Madrigal, Pommerenke, 2006] $\iff \exists \angle \lim_{z \to \sigma} \dfrac{G(t,z)}{z - \sigma} =: \lambda(t) \neq \infty$;

(ii.2) $\lambda(t)$ is locally integrable on $[0, +\infty)$.

If the above conditions hold, then $\varphi'_{s,t}(\sigma) = \exp(\int_s^t \lambda(\xi)\, d\xi)$.

Let U be our semigroup, $\mathcal{U}(\mathbb{D}, \mathcal{F})$ or $\mathcal{U}_\tau(\mathbb{D}, \mathcal{F})$.
This theorem reduces our problem to checking whether

$$\mathcal{R}[U] := \bigcup \{\varphi_{s,t}\} \overset{?}{=} U,$$

where the union is taken over all evolution families $\{\varphi_{s,t}\} \subset U$.

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Theorem (P. Gum. — work in progress)

Let $\mathcal{F} \subset \mathbb{T}$ be a finite set, $n := \mathrm{Card}(\mathcal{F})$, and $\tau \in \overline{\mathbb{D}}$.
The following semigroups U admit
the Loewner-type parametric representation, *i.e.* $\mathcal{R}[U] = U$:

- ✔ $U = \mathcal{U}_\tau(\mathbb{D}, \mathcal{F})$ for $\tau \in \mathbb{D}$ and $n = 1$; [Unkelbach and Goryainov]
- ✔ $U = \mathcal{U}_\tau(\mathbb{D}, \mathcal{F})$ for $\tau \in \mathbb{T}$ and $n \leqslant 2$;
- ✔ $U = \mathcal{U}(\mathbb{D}, \mathcal{F})$ for $n \leqslant 3$.

H. Unkelbach, 1940: an attempt to give
the Loewner-type parametric representation for $\mathcal{U}_0(\mathbb{D}, \{1\})$;
V.V. Goryainov, approx. 2013 (submitted): the complete proofs.

Conjecture and open problem

Conjecture [know how to prove]

If $\tau \in \mathbb{D}$, then the semigroup $\mathcal{U}_\tau(\mathbb{D}, \mathcal{F})$ admits
the Loewner type representation for *any finite set* $\mathcal{F} \subset \mathbb{T}$.

Open problem

Given a finite $\mathcal{F} \subset \mathbb{T}$ with $\mathrm{Card}(\mathcal{F}) = n$,

¿ Does the semigroups $\mathcal{U}_\tau(\mathbb{D}, \mathcal{F})$ admits
the Loewner type representation for $\tau \in \mathbb{T}$ and $n > 2$?

¿ Does the semigroups $\mathcal{U}(\mathbb{D}, \mathcal{F})$ admits
the Loewner type representation for $n > 3$?

My conjecture is that the correct answer for both questions is NO.

¡MUCHAS GRACIAS!