Geometric and analytic properties of generalized Loewner chains

Pavel Gumenyuk

University of Bergen

International Workshop on Complex Analysis
dedicated to the 65th anniversary of
Professor Arne Stray, Bergen 2009

Collaborators

New results in the talk are obtained in collaboration with

Prof. Manuel Contreras and

Prof. Santiago Díaz-Madrigal

form the University of Sevilla, SPAIN.

Outline

Parametric Representation

The essence of Loewner's theory is the Parametric Representation Method for univalent functions.

- K. Loewner, 1923,
  introduced the Parametric Representation Method to attack the famous Bieberbach conjecture on coefficients of univalent holomorphic functions.
- P. P. Kufarev, 1943, and C. Pommerenke, 1965,
  contributed to develop the Parametric Representation in its modern form.
- Developments and Applications:
  - Conformal mapping method in the Hele-Shaw problem
  - Schramm – Loewner Evolution (SLE)
- V. V. Goryainov, 1987, 1991, 1992, 1996: investigation of infinitesimal structure of semigroups of univalent functions

General scheme of Loewner's Theory

Parametric Representation is a special way to *embed a given conformal mapping into a homotopy connecting it to the identity mapping*.

The modern Loewner theory can be represented by the following general scheme, which contains 3 notions:

- Loewner chains
- Evolution families
- Herglotz vector fields

There is a *one-to-one correspondence between them*.

Classical Loewner Chains

Definition

A family $(f_t)_{0\leq t<+\infty}$ of holomorphic maps of the unit disc $\mathbb{D} := \{z : |z| < 1\}$ is called a *(classical) Loewner chain* if

1. each function $f_t : \mathbb{D} \to \mathbb{C}$ is univalent,
2. $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for all $0 \leq s < t < +\infty$,
3. for each $t \geq 0$,

$$f_t(z) = e^t z + a_2(t)z^2 + \dots \tag{1}$$

An example

An example of a (classical) Loewner chain can be constructed as follows.

- Consider a Jordan curve $\Gamma \subset \overline{\mathbb{C}} \setminus \{0\}$ ending at ∞.
- Choose a parameterization $\gamma : [0, +\infty] \to \Gamma$, $\Gamma(+\infty) = \infty$, and consider the domains $\Omega_t := \mathbb{C} \setminus \gamma([t, +\infty])$, $t \geq 0$.
- Define f_t to be the conformal mapping of $\mathbb{D}$ onto Ω_t normalized by $f_t(0) = 0$, $f_t'(0) > 0$.
- Using rescaling in the complex plane and reparameterization of Γ one can assume that $f_t'(0) = e^t$. Then (f_t) is a (classical) Loewner chain.

An example

Remark

This example reveals the case originally considered by Loewner. The boundary of Ω_t is being changed at each moment only locally. Loewner proved that the Loewner chain constructed above satisfies the following PDE

$$\frac{\partial f_t(z)}{\partial t} = z\frac{\partial f_t(z)}{\partial z}\frac{e^{iu(t)}+z}{e^{iu(t)}-z}, \quad t \geq 0, \tag{2}$$

where $u : [0,+\infty) \to \mathbb{R}$ is a continuous function. The above equation is known as the Loewner PDE.

Classical Evolution Families

In classical theory,
no independent definition of *Evolution Families* is given.

Definition

The *evolution family* of a Loewner chain (f_t) is the family $(\varphi_{s,t})$, $0 \leq s \leq t < +\infty$, of holomorphic self-mappings of $\mathbb{D}$ defined by the relation

$$\varphi_{s,t} := f_t^{-1} \circ f_s. \tag{3}$$

Properties:

1. $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$, for all $s \geq 0$,
2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \leq s \leq u \leq t < +\infty$,
3. $\varphi_{s,t}(z) = e^{s-t} z + a_2(s,t) z^2 + \dots$

Herglotz vector fields in classical Loewner's theory

In classical setting,
Herglotz vector fields have the form

$$G(z,t) := -z\,p(z,t), \quad z \in \mathbb{D},\ t \geq 0, \tag{4}$$

where the function $p(z,t)$, known also as *driving term*, satisfies the following conditions:

1. $p(\cdot,t)$ is a Carathéodory function for a.e. $t \geq 0$, i.e., $p(\cdot,t)$ is holomorphic in $\mathbb{D}$ and

$$\operatorname{Re} p(z,t) > 0, \quad p(0,t) = 1. \tag{5}$$

2. $p(z,\cdot)$ is measurable on $[0,+\infty)$ for each $z \in \mathbb{D}$.

One-to-one correspondense

- For each driving term $p(z,t)$ there exist a unique (classical) Loewner chain (f_t) satisfying the following *Loewner – Kufarev PDE*

$$\frac{\partial f_t(z)}{\partial t} = z\frac{\partial f_t(z)}{\partial z}p(z,t), \quad t \geq 0. \tag{6}$$

- Moreover, the characteristic equation for (6), the *Loewner – Kufarev ODE*

$$\frac{dw}{dt} = -w\,p(w,t), \quad t \geq s, \;\; w|_{t=s} = z, \tag{7}$$

has a unique solution $w = \varphi_{s,t} = f_t^{-1} \circ f_s$.

One-to-one correspondense

- If we know the evolution family $(\varphi_{s,t})$, then we can reconstruct the Loewner chain (f_t) by means of the formula

$$f_s(z) = \lim_{t\to+\infty} e^t \varphi_{s,t}(z), \quad z \in \mathbb{D}, \; s \geq 0. \tag{8}$$

- Every Loewner chain (f_t) is almost everywhere differentiable w.r.t. the parameter t and there exists an essentially unique driving term $p(z,t)$ such that Loewner – Kufarev PDE (6) holds,

$$\frac{\partial f_t(z)}{\partial t} = z\frac{\partial f_t(z)}{\partial z} p(z,t), \quad t \geq 0. \tag{6}$$

Chordal Loewner equation

The classical setting described above is referred to in modern literature as *radial Loewner evolution*.There is also a so-called *"chordal"* variant of the Loewner – Kufarev ODE

$$\frac{dw}{dt} = p(w,t), \quad p(z,t) := \int_{\mathbb{R}} \frac{d\mu_t(x)}{x-z}, \quad \operatorname{Im} z > 0, t \geq 0, \tag{9}$$

where μ_t is a finite Borel measure on $\mathbb{R}$.
For slit mappings, μ_t is supported at one point, and the equation takes the form

$$\frac{dw}{dt} = \frac{1}{\xi(t) - w}. \tag{10}$$

- P. P. Kufarev, V. V. Sobolev and L. V. Sporysheva, 1968
- N. V. Popova, 1949
- O. Schramm, 2000
- V. V. Goryainov and I. Ba, 1992; R. Bauer, 2005

New approach: authors and motivation

Authors: F. Bracci, M. D. Contreras and S. Díaz-Madrigal (BCM), 2008

Motivation: to find a general construction which contains, as particular cases:

- radial Loewner evolution,
- chordal Loewner evolution,
- one-parametric semigroups.

(Generalized) Evolution Families

Definition

A family $(\varphi_{s,t})$, $0 \le s \le t < +\infty$ of holomorphic self-maps of $\mathbb{D}$ is a *(generalized) evolution family* of order $d \in [1, +\infty]$ if

EF1 $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$,

EF2 $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \le s \le u \le t < +\infty$,

EF3 for any $z \in \mathbb{D}$ and $T > 0$ there exists $k_{z,T} \in L^d([0,T], \mathbb{R})$ such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \le \int_u^t k_{z,T}(\xi)d\xi, \quad 0 \le s \le u \le t \le T. \qquad (11)$$

(Generalized) Herglotz vector fields

Definition

A *(generalized) Herglotz vector field* of order $d \in [1,+\infty]$ is a function $G(z,t)$ satisfying

VF1 $G(z,\cdot)$ is measurable on $[0,+\infty)$ for each $z \in \mathbb{D}$;

VF2 $G(\cdot,t)$ is holomorphic in $\mathbb{D}$ for a.e. $t \geq 0$;

VF3 For any compact set $K \subset \mathbb{D}$ and $T > 0$ there is $k_{K,T} \in L^d([0,T],\mathbb{R})$ such that

$$|G(z,t)| \leq k_{K,T}(t), \quad z \in K, \quad t \in [0,T]. \tag{12}$$

HVF $G(\cdot,t)$ for a.e. fixed $t \geq 0$ is an infinitesimal generator of a one-parametric semigroup, i.e.,

$$G(z,t) = (\tau_t - z)(1 - z\overline{\tau_t})p_t(z), \ \ \tau_t \in \overline{\mathbb{D}}, \ \mathrm{Re}\, p_t(z) \geq 0. \tag{13}$$

For shortness

In what follows,

the word "generalized" will be ***omitted****.*

Connection between Evolution Families and Herglotz Vector Fields

F. Bracci, M. D. Contreras and S. Díaz-Madrigal proved that:

- For each Herglotz vector field $G(z,t)$ of order d there exists a unique solution $w = w(t; s, z)$ to the initial value problem

$$\frac{dw}{dt} = G(w,t), \quad t \geq s \geq 0, \;\; w|_{t=s} = z \in \mathbb{D}. \tag{14}$$

  Moreover the family $(\varphi_{s,t})$ defined by

$$\varphi_{s,t}(z) := w(t; s, z), \quad t \geq s \geq 0, \; z \in \mathbb{D}, \tag{15}$$

  is an *evolution family* of order d.

- For each evolution family $(\varphi_{s,t})$ there exists an essentially unique Herglotz vector field $G(z,t)$ of the same order such that the solution to (14) is given by

$$w = \varphi_{s,t}(z). \tag{16}$$

Connection between Evolution Families and Herglotz Vector Fields

In other words,

Evolution families are exactly the families of evolution operators for the non-autonomous flows generated by Herglotz vector fields.

Problem definition

Loewner chains is a **missing element** in the scheme by F. Bracci, M. D. Contreras and S. Díaz-Madrigal

The problem was:

- to formulate an appropriate definition of what a generalized Loewner chain is;
- to study the connection between Loewner chains and evolution families.

(Generalized) Loewner chains

We introduced

Definition

A family $(f_t)_{0\leq t<+\infty}$ of holomorphic maps of the unit disc will be called a *(generalized) Loewner chain* of order $d \in [1,+\infty]$ if

LC1 each function $f_t : \mathbb{D} \to \mathbb{C}$ is univalent,

LC2 $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for all $0 \leq s < t < +\infty$,

LC3 for any compact set $K \subset \mathbb{D}$ and all $T > 0$ there exists $k_{K,T} \in L^d([0,T],\mathbb{R})$ such that

$$|f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\xi)d\xi, \quad z \in K,\ 0 \leq s \leq t \leq T. \tag{17}$$

Loewner Chains → Evolution Families

Within the above definition, a Loewner chain generates an evolution family in the same manner as in the classical setting:

Theorem

Let (f_t) be a Loewner chain of order $d \in [1, +\infty]$, then the family $(\varphi_{s,t})$, $t \geq s \geq 0$, defined by the formula

$$\varphi_{s,t} := f_t^{-1} \circ f_s, \quad t \geq s \geq 0, \tag{18}$$

is an evolution family of the same order.

Definition

The evolution family $(\varphi_{s,t})$ given by (18) will be called the *evolution family of a Loewner chain (f_t)*.

Evolution Families → Loewner Chains

Theorem

For any evolution family $(\varphi_{s,t})$ *there exists a unique Loewner chain* (f_t) *such that*

(i) $\varphi_{s,t} := f_t^{-1} \circ f_s,\ t \geq s \geq 0,$

(ii) $f_0(0) = 0$ *and* $f_0'(0) = 1,$

(iii) $\Omega := \bigcup_{t \geq 0} f_t(\mathbb{D})$ *is either a Euclidean disk centered at the origin, or coincides with* $\mathbb{C}$.

Moreover, a family (g_t) *is a Loewner chain satisfying (i) if and only if*

$$g_t := h \circ f_t, \quad t \geq 0, \tag{19}$$

where $h : \Omega \to \mathbb{C}$ *is an arbitrary holomorphic univalent function.*

Evolution Families $\to$ Loewner Chains

We can distinguish the cases $\Omega = \mathbb{C}$ and $\Omega = D(0, r) \neq \mathbb{C}$. Consider the following function

$$\beta(z) := \lim_{t \to +\infty} \frac{|\varphi'_{0,t}(z)|}{1 - |\varphi_{0,t}(z)|^2} = 0. \tag{20}$$

The limit (20) exists for all $z \in \mathbb{D}$ and the function β is either zero identically, or never vanishes. We have:

$$\Omega = \mathbb{C} \Longleftrightarrow \beta \equiv 0,$$

$$r = 1/\beta(0).$$

General problem

Another interesting problem is

to characterize geometrically those Loewner chains (f_t) *whose evolution families* $(\varphi_{s,t})$ *satisfy a given normalization and, may be, some additional regularity conditions.*

A trivial variant of this general problem is as follows:

Given a collection $\mathcal{D} = \{\Omega_t : t \geq 0\}$ *of simply connected domains in* $\mathbb{C}$*, does there exists a Loewner chain* (f_t) *such that*

$$\left\{f_t(\mathbb{D}) : t \geq 0\right\} = \mathcal{D}?$$

The answer is "YES" if and only if $\mathcal{D}$ is an *inclusion chain*.

Inclusion chains

Definition

A one-parametric family $\mathcal{D} = (\Omega_t)_{t\geq 0}$ of simply connected domains in $\mathbb{C}$ is called an *inclusion chain* if

IC1 $\Omega_s \subset \Omega_t$ whenever $0 \leq s \leq t < +\infty$;

IC2 Each Ω_s is a connected component of $\mathrm{int}(\Omega_s^+)$, where $\Omega_s^+ := \bigcap_{t>s} \Omega_t$;

IC3 Each Ω_t, $t > 0$, coincides with $\Omega_t^- := \bigcup_{s<t} \Omega_s$.

Problem definition

Simple fact:

For each inclusion chain (Ω_t) there exists a Loewner chain (f_t) such that

(i) $\{f_t : t \geq 0\} = \{\Omega_t : t \geq 0\}$,

(ii) $\varphi_{s,t}(0) = 0$, $\varphi'_{s,t}(0) > 0$, $t \geq s \geq 0$, *where* $\varphi_{s,t} := f_t^{-1} \circ f_s$.

Problem definition

The situation with *boundary Denjoy – Wolff point* is more complicated.
Consider the following classes of holomorphic self-maps of $\mathbb{D}$:

$$\mathcal{C} := \{\varphi \in \mathrm{Hol}(\mathbb{D},\mathbb{D}) : \exists\angle \lim_{z\to 1} \varphi(z) = 1,\ \varphi'(1) \neq \infty\},$$

$$\mathcal{P} := \{\varphi \in \mathcal{C} : \varphi'(1) = 1\},$$

$$\mathcal{P}_0 := \{\varphi \in \mathcal{P} : \varphi(z) = 1 + (z-1) - \frac{\ell(\varphi)\,(z-1)^3}{4} + \gamma(z),$$
$$\exists\angle \lim\nolimits_{z\to 1} \frac{\gamma(z)}{(z-1)^3} = 0\}$$

In the framework of the upper half-plane, the definition of $\mathcal{P}_0$ takes the form:

$$\Phi(z) = z - \frac{\ell(\Phi)}{z} + \gamma(z), \quad \angle \lim_{z\to\infty} z\gamma(z) = 0. \qquad (21)$$

Chordal Evolution Families

Definition

A *chordal evolution family* is an evolution family $(\varphi_{s,t}) \subset \mathcal{C}$ with $\varphi'_{s,t}(1) \leq 1$, $0 \leq s \leq t < +\infty$.

Definition

A *parabolic chordal evolution family* is an evolution family $(\varphi_{s,t}) \subset \mathcal{P}$.

Definition

A *Goryainov – Ba evolution family* is an evolution family $(\varphi_{s,t}) \subset \mathcal{P}_0$ such that the function $t \mapsto \ell(\varphi_{0,t})$ is absolutely continuous.

Results

Theorem (1)

Let (Ω_t) *be an inclusion chain. Suppose there exists a family* $(F_t)_{t\geq 0}$ *of conformal mappings of* $\mathbb{D}$ *such that* $F_t(\mathbb{D}) = \Omega_t$ *and*

$$\Phi_t := F_t^{-1} \circ F_0 \tag{22}$$

belongs to the class $\mathcal{C}$ *for each* $t > 0$. *Then there exists a Loewner chain* (f_t) *such that*

(i) *the evolution family of* (f_t) *is a parabolic chordal evolution family and*

(ii) $\{f_t(\mathbb{D}) : t \geq 0\} = \{\Omega_t : t \geq 0\}$.

Remark

Theorem 1 also holds with

- *$\mathcal{C}$ replaced by $\mathcal{P}_0$ and*
- *parabolic chordal evolution families replaced by Goryainov – Ba evolution families.*

Definition

An inclusion chain (Ω_t) is said to be *chordally admissible* if it satisfies the conditions of Theorem 1, and *GB-admissible* if it satisfies the conditions of Theorem 1 with $\mathcal{C}$ replaced by $\mathcal{P}_0$.

Results

Definition

A C^1-smooth curve $\gamma : [a, b] \to \overline{\mathbb{C}}$ is said to be *Dini-smooth* is the derivative $d\gamma(t)/dt$ is Dini-continuous.

Theorem

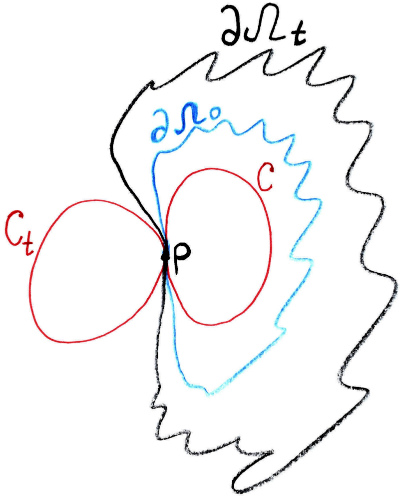
Let (Ω_t) be an inclusion chain and $p \in \partial\Omega_0$. Suppose that the following conditions hold:

(i) *there exists a Dini-smooth closed Jordan curve $C \subset \overline{\mathbb{C}}$ such that $p \in C$ and one of the two connected components of $\overline{\mathbb{C}} \setminus C$ is contained in Ω_0.*

(ii) *for each $t > 0$ there exists a Dini-smooth closed Jordan curve $C_t \subset \overline{\mathbb{C}}$ such that $p \in C_t$, and $\Omega_t \cap C_t = \emptyset$.*

Then the inclusion chain (Ω_t) is chordally admissible.

Results



Results

Definition

A C^n-smooth curve $\gamma : [a, b] \to \overline{\mathbb{C}}$ is said to be *$C^{n,+0}$-smooth* is the derivative $d^n\gamma(t)/dt^n$ is of class Lip(α) with some $\alpha > 0$.

Theorem

Let (Ω_t) be an inclusion chain and $p \in \partial\Omega_0$. Suppose that the following conditions hold:

- **(i)** *there exists a $C^{3,+0}$-smooth closed Jordan curve $C \subset \overline{\mathbb{C}}$ such that $p \in C$ and one of the two connected components of $\overline{\mathbb{C}} \setminus C$ is contained in Ω_0.*
- **(ii)** *for each $t > 0$ there exists a $C^{3,+0}$-smooth closed Jordan curve $C_t \subset \overline{\mathbb{C}}$ such that $p \in C_t$, $\Omega_t \cap C_t = \emptyset$, and C_t has second order contact with C at the point p.*

Then the inclusion chain (Ω_t) is GB-admissible.

Results

Theorem

Let (Ω_t) be an inclusion chain and P_0 any prime end of the domain Ω_0. Suppose that P_0 is degenerate, i.e., the impression of P_0 consists of one point p_0, and for each $t \geq 0$ there exists $\varepsilon > 0$ such that

$$\partial\Omega_t \cap D(p_0, \varepsilon) = \partial\Omega_0 \cap D(p_0, \varepsilon), \tag{23}$$

where $D(p_0, \varepsilon)$ stands for the disk of radius ε centered at p_0. Then the inclusion chain (Ω_t) is GB-admissible.