«Univalent functions and control»

WORKSHOP DEDICATED TO **65TH ANNIVERSARY OF PROFESSOR DMITRI VALENTINOVICH PROKHOROV**

Loewner Theory in Annulus: history and recent developments

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Institut Mittag-Leffler – Djursholm, Sweden, September 12 - 13, 2011

Outline

Collaborators

New results in the talk are obtained in collaboration with

Prof. Manuel D. Contreras and

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from Universidad de Sevilla, SPAIN.

The classical Loewner Theory in the unit disk is due to:

- K. Löwner (C. Loewner), 1923
- P. P. Kufarev, 1943
- C. Pommerenke, 1965

Modern viewpoint —

three fundamental notions of Loewner Theory:

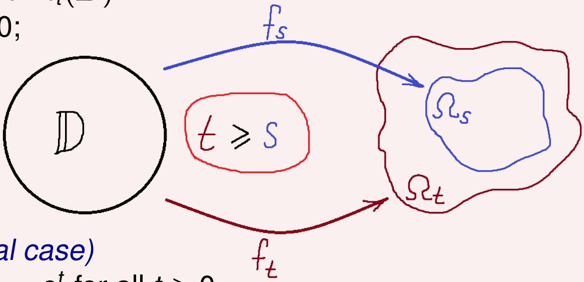
- Loewner chains (f_t)
- Evolution families $(\varphi_{s,t})$
- Herglotz vector fields $G(w,t)$

Definition

A Loewner chain is a one-parametric family of functions (f_t), $t \geqslant 0$, such that:

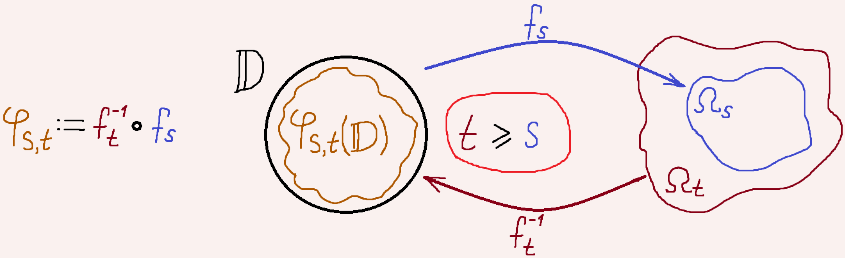
LC1. each $f_t : \mathbb{D} \to \mathbb{C}$, $\mathbb{D} := \{z : |z| < 1\}$, is *holomorphic* and *univalent*;

LC2. $\Omega_s := f_s(\mathbb{D}) \subset \Omega_t := f_t(\mathbb{D})$ whenever $t \geqslant s \geqslant 0$;



LC3. *(the very classical case)* $f_t(0) = 0$ and $f_t'(0) = e^t$ for all $t \geqslant 0$.

Loewner Theory in the disk



Definition

A family $(\varphi_{s,t})$, $t \geqslant s \geqslant 0$, of holomorphic functions $\varphi_{s,t} : \mathbb{D} \to \mathbb{D}$ is an evolution family if:

EF1. $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$; EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $t \geqslant u \geqslant s \geqslant 0$;

EF3. *(the very classical case)*

$\varphi_{s,t}(0) = 0$ and $\varphi'_{s,t}(0) = e^{s-t}$ whenever $t \geqslant s \geqslant 0$.

One definition form the theory of Carathéodory ODE:

Definition

Let $d \in [1, +\infty]$. A function $G : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is a weak holomorphic vector field of order d if:

VF1. $G(z, t)$ is holomorphic in $z \in \mathbb{D}$ for a.e. $t \geqslant 0$;

VF2. $G(z, t)$ is measurable in $t \in [0, +\infty)$ for all $z \in \mathbb{D}$;

VF3. For any compact set $K \subset \mathbb{D}$ and any $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ such that

$$\left|G(z, t)\right| \leqslant k_{K,T}(t), \quad \text{for any } z \in K \text{ and a.e. } t \in [0, T]. \tag{1}$$

Under the above conditions $\exists!$ solution to the Cauchy problem

$$\dot{w} = G(w, t), \tag{2}$$

$$w(s) = z, \quad s \geqslant 0, z \in \mathbb{D}. \tag{3}$$

Definition (general case)

Let $d \in [1, +\infty]$. A function $G : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is a Herglotz vector field of order d if:

HVF1. G is a weak holomorphic vector field of order d;

HVF2. For a.e. $t \geqslant 0$, $G(\cdot, t)$ is an infinitesimal generator.

Berkson – Porta, 1978

$H \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ is an infinitesimal generator if and only if

$$H(z) = (\tau - z)(1 - \overline{\tau} z) p(z), \quad \tau \in \overline{\mathbb{D}}, \quad p \in \mathrm{Hol}(\mathbb{D}, \mathbb{C}), \; \mathrm{Re}\, p \geqslant 0. \tag{4}$$

Berkson – Porta, 1978

$H \in \text{Hol}(\mathbb{D}, \mathbb{C})$ is an infinitesimal generator if and only if

$$H(z) = (\tau - z)(1 - \overline{\tau}z)p(z), \quad \tau \in \overline{\mathbb{D}}, \quad p \in \text{Hol}(\mathbb{D}, \mathbb{C}), \ \text{Re}\, p \geqslant 0. \quad (4)$$

Fixing $\tau = 0$ and normalizing $p(0) = 1$ in (4), we get

Definition (the very classical case)

A classical Herglotz vector field is

$$G(z, t) = -zp(z, t), \quad z \in \mathbb{D}, \quad \text{a.e. } t \geqslant 0, \quad (5)$$

where $p(z, t)$ is holomorphic in z, measurable in t,
$\text{Re}\, p \geqslant 0$, and $p(0, t) = 1$ for a.e. $t \geqslant 0$.

L.Th. in $\mathbb{D}$: main results in classical case

There is 1-to-1 correspondence between classical Loewner chains (f_t), evolution families $(\varphi_{s,t})$ and Herglotz vector fields $G(z,t)$, given via:

$$\varphi_{s,t} = f_t^{-1} \circ f_s, \quad f_s = \lim_{t\to+\infty} e^t \varphi_{s,t}, \tag{6}$$

Loewner – Kufarev ODE

$$\frac{d}{dt}\varphi_{s,t}(z) = G\big(\varphi_{s,t}(z), t\big) = -\varphi_{s,t}(z)p\big(\varphi_{s,t}(z), t\big), \quad t \geqslant s,$$
$$\varphi_{s,t}(z)|_{t=s} = z, \ z \in \mathbb{D}, \tag{7}$$

Loewner – Kufarev PDE

$$\frac{\partial}{\partial t} f_t(z) = -f_t'(z)G(z,t) = zf_t'(z)p(z,t), \quad z \in \mathbb{D}, \quad t \geqslant 0. \tag{8}$$

L.Th. in $\mathbb{D}$: main results in classical case

Theorem (Gutljanskiĭ, 1970; Pommerenke, 1973)

For any $f \in \mathcal{S} := \left\{ f \in \mathrm{Hol}(\mathbb{D},\mathbb{C}) : f(0) = 0,\ f'(0) = 1, \text{ and } f \text{ is 1-to-1} \right\}$ there exists a classical Loewner chain (f_t) s.t. $f_0 = f$.

Parametric Representation

This theorem provides a Parametric Representation of the class $\mathcal{S}$ and therefore has important applications in the theory of univalent functions, especially in Extremal Problems.

$$p(w,t) \quad \mapsto \quad \varphi_{s,t} \quad \mapsto \quad \{f_t\} \quad \mapsto \quad f_0 \in \mathcal{S}$$

convex cone of driving terms $p(w,t) \quad \xrightarrow{\text{onto}} \quad$ the class $\mathcal{S}$

EXTREMAL PROBLEM $\mapsto$ PROBLEM OF OPTIMAL CONTROL

Extremal problems in $\mathcal{S}$ and $\mathcal{S}^M$

New and classical extremal problems for coefficient functionals for normalized univalent functions (class $\mathcal{S}$) and bounded normalized univalent functions ($\mathcal{S}^M := \{f \in \mathcal{S} : |f(z)| < M$ for all $z \in \mathbb{D}\}$):

Dmitri Valentinovich Prokhorov and his students
1984, 1986, 1990, 1991, 1992, 1993, 1994, 1995, 1997, ...

- Parametric Representation
- Pontryagin's Maximum Principle
- Variational technique

- Classical L. Th. also gives a representation of the semigroup $\mathcal{U}_0 := \{\varphi \in \mathsf{Hol}(\mathbb{D}, \mathbb{D}) : \varphi$ is 1-to-1, $\varphi(0) = 0$, $\varphi'(0) > 0\}$.
- Other sub-semigroups of $\mathcal{U} := \{\varphi \in \mathsf{Hol}(\mathbb{D}, \mathbb{D}) : \varphi$ is 1-to-1$\}$ can be represented by constructing corresponding versions of Loewner Evolution (V. V. Goryainov, 1987, 1991, 1992, 1996).

- *Chordal Loewner Evolution*

  (P. P. Kufarev, V. V. Sobolev and L. V. Sporysheva, 1968) — the semigroup $\mathcal{U}_1 \subset \mathcal{U} := \{\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D}) : \varphi \text{ is 1-to-1}\}$ of self-mappings with *hydrodynamic normalization* (parabolic DW-point on the boundary + extra regularity).

  $$\frac{dw}{dt} = p(w,t), \quad w \in \mathbb{U} := \{w : \operatorname{Im} w > 0\},$$
  $$p(w,t) := \int_{\mathbb{R}} \frac{1}{x-w}\, d\mu_t(x),$$

  where μ_t is a finite positive Borel measure.
- Chordal Loewner Evolution → *SLE* (O. Schramm, 2000):

  $$d\mu_t(x) := \delta(x - \sqrt{\kappa}\mathcal{B}_t)\, dx,$$

  where $\kappa > 0$ and $(\mathcal{B}_t)$ is a *standard Brownian motion*.
- SLE: applications in lattice models of Statistical Physics.

New approach

- F. Bracci, M. D. Contreras and S. Díaz-Madrigal, 2008
  a general construction unifying all versions of Loewner Evolution.
- In contrast to the classical theory the whole semigroup
  $\mathcal{U} := \{\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D}) : \varphi$ is 1-to-1$\}$ is involved (no normalization).
- Arbitrary Hergltoz vector fields are considered.
- M. D. Contreras and S. Díaz-Madrigal, and P.G., 2010
  general Loewner chains.

Definition

A (time-dependent) vector field G defined in a set $\mathcal{D} \subset \mathbb{C} \times \mathbb{R}$ is said to be semicomplete if any solution to the equation

$$\dot{w} = G(w, t) \tag{9}$$

can be extended unrestrictedly to the right (to the future).

Theorem (F. Bracci, M. D. Contreras, S. Díaz-Madrigal)

A weak holomorphic vector field G is semicomplete

if and only if

G is a Herglotz vector field,

i.e. if for a.e. $t \geqslant 0$, $G(\cdot, t)$ is an infinitesimal generator.

This allows us to regard the approach proposed by Bracci et al as the most general type of Loewner Evolution in $\mathbb{D}$.

Our aim

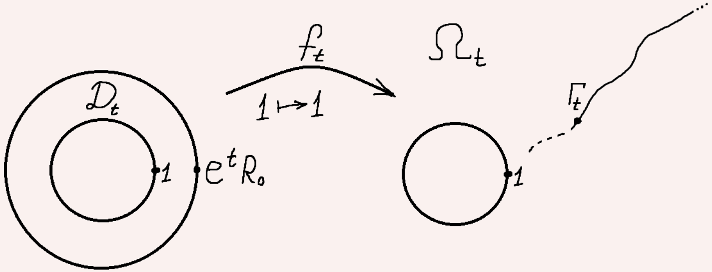
is to construct analogous general Loewner Theory for doubly connected domains.

New feature of Loewner Evolution in the doubly setting

is that instead of static canonical domain (the unit disk $\mathbb{D}$) one has to consider an extending family (D_t) of canonical domains (annuli).

Indeed, a continuous monotonic family (Ω_t) of doubly connected domains cannot consist of conformally equivalent domains.

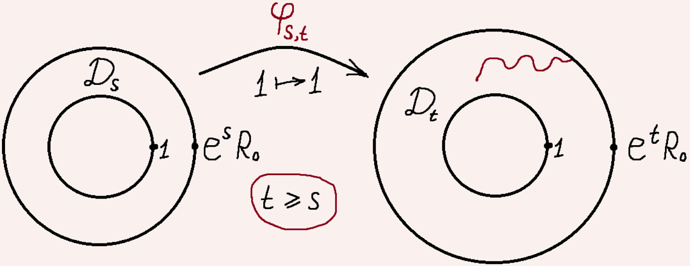
Y. Komatu, 1943; G. M. Goluzin, 1950



Evolution families in the Komatu–Goluzin case

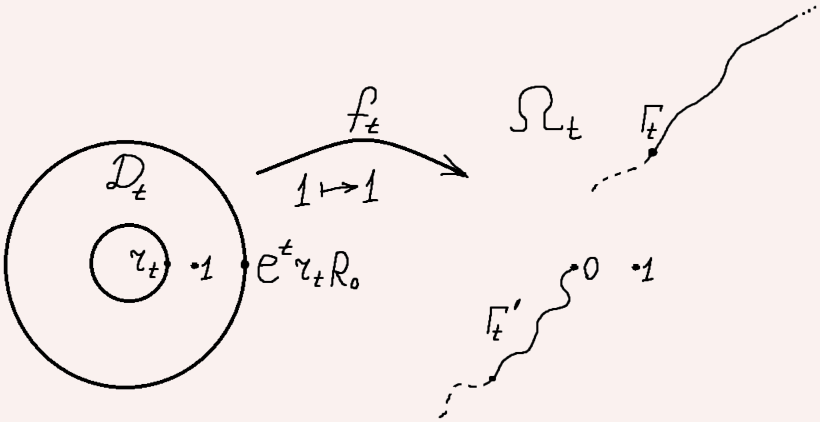
EF1. $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}_s}$; EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $t \geqslant u \geqslant s \geqslant 0$;

EF3. $\varphi_{s,t}(D_s)$ is D_t minus a slit landing on $|w| = e^t R_0$ and $\varphi_{s,t}(1) = 1$ whenever $t \geqslant s \geqslant 0$.



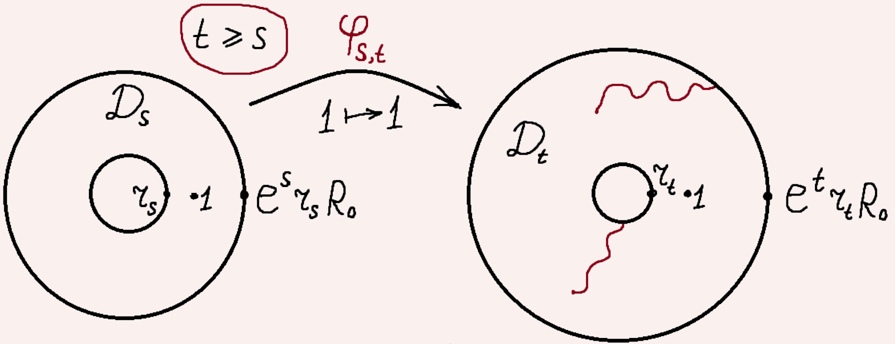
Loewner Theory in annulus: history

Li En Pir, 1953; N. A. Lebedev, 1955



The function $t \mapsto r_t$ is defined by a differential equation.

Evolution families in the Li–Lebedev case



V. Ja. Gutljanskiĭ, 1972

considered a generalization of the Komatu – Goluzin case, when (what can be called) the Loewner chain (f_t) consists of mappings

$$f_t : \{z : 1 < |z| < R_0 e^t\} \xrightarrow{\text{into}} \{w : |w| > 1\}$$

$$\text{with } \left|f_t(z)\right| = 1 \text{ when } |z| = 1 \text{ and } f_t(1) = 1$$

(but the other boundary component is not necessary a slit).

General notion of evolution family

We fix form the very beginning

$$d \in [1, +\infty] \quad \text{— the order.}$$

Notation

- $\mathbb{A}_r := \{z : r < |z| < 1\}$, $r \in [0, 1)$,
- $AC^d(X, Y) := \Big\{ f : X \to Y \,\Big|\, f \text{ is locally absolutely continuous, } f' \in L^d_{\mathrm{loc}}(X, Y) \Big\}$.

Definition (canonical domains (D_t))

$(D_t)_{t \geqslant 0} = (\mathbb{A}_{r(t)})_{t \geqslant 0}$ is a canonical domain system of order d, if

(i) $0 \leqslant r(t) < 1$ for any $t \geqslant 0$; (ii) $t \mapsto r(t)$ is non-increasing;

(iii) $\omega(t) := \begin{cases} -\pi / \log r(t), & \text{if } r(t) \in (0,1), \\ 0, & \text{if } r(t) = 0. \end{cases}$ is of class AC^d.

General notion of evolution family

Definition (Evolution family)

Let (D_t) be a canonical domain system of order d. A family $(\varphi_{s,t})$, $0 \leqslant s \leqslant t$, of holomorphic mappings $\varphi_{s,t} : D_s \to D_t$ is said to be an evolution family of order d over (D_t), if

EF1. $\varphi_{s,s} = \mathrm{id}_{D_s}$; EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $t \geqslant u \geqslant s \geqslant 0$;

EF3 $\big($For all $I := [S, T] \subset [0, +\infty)$, $z \in D_S\big)$ $\exists\, k_{z,I} \in L^d\big(I, \mathbb{R}\big)$ s. t.

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,I}(\xi)d\xi, \quad S \leq s \leq u \leq t \leq T. \qquad (10)$$

Theorem (M.D. Contreras, S. Díaz-Madrigal, P.G.)

Under EF1 and EF2,

$$\text{EF3} \quad \Leftrightarrow \quad \exists z_0 \in D_0 \ \big(t \mapsto \varphi_{0,t}(z_0)\big) \in AC^d\big([0, +\infty), \mathbb{C}\big).$$

General notion of Loewner chain

Definition (Loewner chain)

Let (D_t) be a canonical domain system of order d.
A family $(f_t)_{t\geqslant 0}$ of holomorphic functions $f_t : D_t \to \mathbb{C}$ is called a Loewner chain of order d over (D_t) if:

LC1. each function $f_t : D_t \to \mathbb{C}$ is univalent;

LC2. $f_s(D_s) \subset f_t(D_t)$ whenever $t \geqslant s \geqslant 0$;

LC3. (for any $I := [S, T] \subset [0, +\infty)$, $K \Subset D_S$) $\quad \exists \; k_{K,I} \in L^d\big(I, \mathbb{R}\big)$ s.t.

$$|f_s(z) - f_t(z)| \leq \int_s^t k_{K,I}(\xi)d\xi, \quad z \in K, \;\; S \leq s \leq t \leq T. \qquad (11)$$

Relation between (f_t) and $(\varphi_{s,t})$

Similar to simply connected case there exists essentially 1-to-1 correspondence between evolution families and Loewner chains.
We fix now some $d \in [1, +\infty]$ and some canonical domain system $(D_t) = (\mathbb{A}_{r(t)})$ of order d.

Theorem (M.D. Contreras, S. Díaz-Madrigal, P.G.)

If (f_t) is a Loewner chain of order d over (D_t), then

$$\varphi_{s,t} := f_t^{-1} \circ f_s, \quad t \geqslant s \geqslant 0, \tag{12}$$

is an evolution family of order d over (D_t).

Definition

If (12) holds we will say that (f_t) and $(\varphi_{s,t})$ are associated with each other.

Relation between (f_t) and $(\varphi_{s,t})$

I general,

there are infinitely many (f_t)'s associated with a given $(\varphi_{s,t})$.

To choose one of them we introduce:

Definition (standard Loewner chain)

A Loewner chain (f_t) over (D_t) is called standard if:

(i) for any $t \geqslant 0$ and closed curve $\gamma \subset D_t$, $\operatorname{ind}(f_t \circ \gamma, 0) = \operatorname{ind}(\gamma, 0)$;

(ii) the union of images

$$\Omega := \bigcup_{t\in[0,+\infty)} f_t(D_t)$$

is either $\mathbb{A}_r$ for some $r \in (0,1)$, or $\mathbb{D}^*$, or $\mathbb{C} \setminus \overline{\mathbb{D}}$, or $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

Relation between (f_t) and $(\varphi_{s,t})$

Theorem (M.D. Contreras, S. Díaz-Madrigal, P.G.)

Up to rotation, for each evolution family $(\varphi_{s,t})$ of order d over (D_t) there exists a unique standard Loewner chain (f_t) of order d over (D_t) associated with $(\varphi_{s,t})$.
The set of all Loewner chains of order d associated with $(\varphi_{s,t})$ is given by the formula

$$g_t = F \circ f_t, \quad t \geqslant 0, \tag{13}$$

where $F : \Omega \to \mathbb{C}$ is a univalent function.

$$\Omega := \bigcup_{t \in [0,+\infty)} f_t(D_t).$$

- Fix $d \in [1, +\infty]$ and a canonical domain system $(D_t) = (\mathbb{A}_{r(t)})$ of order d;
- Denote $\mathcal{D} := \big\{(z,t) : t \geqslant 0, z \in D_t\big\} \subset \mathbb{C} \times [0, +\infty)$.

Definition

A function $G : \mathcal{D} \to \mathbb{C}$ is a weak holomorphic vector field of order d if:

VF1. $G(z,t)$ is holomorphic in z;

VF2. $G(z,t)$ is measurable in t;

VF3. (For any $K \Subset \mathcal{D}$) $\exists\ k_K \in L^d(\mathrm{pr}_{\mathbb{R}} K, \mathbb{R} \cup \{+\infty\})$, $\mathrm{pr}_{\mathbb{R}}(z,t) := t$, such that

$$\big|G(z,t)\big| \leqslant k_K(t), \quad (z,t) \in K. \tag{14}$$

Semicomplete = every solution to $\dot{w} = G(w,t)$ is unrestrictedly extendable to the future.

Theorem (M.D. Contreras, S. Díaz-Madrigal, P.G.)

- Let $(\varphi_{s,t})$ be an evolution family of order d over (D_t). Then there exists an (essentially unique) semicomplete weak holomorphic vector field $G : \mathcal{D} \to \mathbb{C}$ of order d s.t. for any $s \geqslant 0$, $z \in D_s$, the function $w(t) := \varphi_{s,t}(z)$ solves the equation
$$\dot{w} = G(w, t). \tag{15}$$
- Let $G : \mathcal{D} \to \mathbb{C}$ be a semicomplete weak holomorphic vector field of order d. Then for any $s \geqslant 0$, $z \in D_s$, there exists a unique solution $w(t) = w_{z,s}(t)$, $t \geqslant s$, to the initial value problem
$$\dot{w} = G(w, t), \quad w(s) = z. \tag{16}$$

The formula
$$\varphi_{s,t}(z) := w_{z,s}(t) \tag{17}$$
defines an evolution family of order d over (D_t).

Assume from now

$D_t := \mathbb{A}_{r(t)}$, where $r(t) > 0$ for all $t \in [0, +\infty)$.

The Villat kernel, $r \in (0, 1)$,

$$\mathcal{K}_r(z) := \frac{1+z}{1-z} + \sum_{\nu=1}^{+\infty}\left(\frac{1+r^{2\nu}z}{1-r^{2\nu}z} - \frac{1+r^{2\nu}/z}{1-r^{2\nu}/z}\right) \tag{18}$$

Notation

$V(r)$ is the class of holomorphic functions $p : \mathbb{A}_r \to \mathbb{C}$ represented by

$$p(z) = \int_{\mathbb{T}} \mathcal{K}_r(\tfrac{z}{\xi})d\mu_1(\xi) + \int_{\mathbb{T}} \left[1 - \mathcal{K}_r(\tfrac{r\xi}{z})\right]d\mu_2(\xi), \quad \mathbb{T} := \{z : |z| = 1\}, \tag{19}$$

where $\mu_1, \mu_2 \geqslant 0$ are Borel measures on $\mathbb{T}$, $\mu_1(\mathbb{T}) + \mu_2(\mathbb{T}) = 1$.

Recall:

- We assumed $D_t := \mathbb{A}_{r(t)}$, where $r(t) > 0$ for all $t \in [0, +\infty)$.
- $\mathcal{D} := \left\{(z, t) : t \geqslant 0, z \in D_t\right\} \subset \mathbb{C} \times [0, +\infty)$.

Theorem (M.D. Contreras, S. Díaz-Madrigal, P.G.)

A function $G : \mathcal{D} \to \mathbb{C}$ is a semicomplete weak holomorphic vector field of order d if and only if it has representation

$$G(w, t) = w\left[iC(t) + \frac{r'(t)}{r(t)}p(w, t)\right] \quad \text{a.e. } t \geqslant 0, \text{ all } w \in D_t, \tag{20}$$

where
(i) for each $t \geqslant 0$, $p(\cdot\,, t) \in V\big(r(t)\big)$;
(ii) p is measurable as a function of t;
(iii) $C \in L^d_{\mathrm{loc}}\big([0, +\infty), \mathbb{R}\big)$.

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For a standard Loewner chain (f_t), denote

- $$\Omega := \bigcup_{t\in[0,+\infty)} f_t(D_t), \quad r_\infty := \lim_{t\to+\infty} r(t),$$
- $\varphi_{s,t} := f_t^{-1} \circ f_s : D_s \to D_t,\ t \geqslant s \geqslant 0,$
- $\tilde{\varphi}_{s,t}(z) := \dfrac{r(t)}{\varphi_{s,t}\big(r(s)/z\big)},\ t \geqslant s \geqslant 0,\ z \in D_s,$ — conjugate of $(\varphi_{s,t})$.

Theorem (M.D. Contreras, S. Díaz-Madrigal, P.G.)

In the above notation:

$$\begin{array}{lcccl}
\Omega = \mathbb{A}_r,\ r \in (0,1) & \Leftrightarrow & r_\infty > 0 & \Leftrightarrow & \varphi_{0,t} \not\to 0 \text{ and } \tilde{\varphi}_{0,t} \not\to 0 \\
\Omega = \mathbb{D}^* & & \Leftrightarrow & & \varphi_{0,t} \not\to 0 \text{ and } \tilde{\varphi}_{0,t} \to 0 \\
\Omega = \mathbb{C} \setminus \overline{\mathbb{D}} & & \Leftrightarrow & & \varphi_{0,t} \to 0 \text{ and } \tilde{\varphi}_{0,t} \not\to 0 \\
\Omega = \mathbb{C}^* & & \Leftrightarrow & & \varphi_{0,t} \to 0 \text{ and } \tilde{\varphi}_{0,t} \to 0
\end{array}$$

Last words ...

- Similar characterization is established in terms of the corresponding weak holomorphic vector field G.
- The results presented in the talk are contained in the preprints:
  - M.D. Contreras, S. Díaz-Madrigal, P. Gumenyuk, *Loewner Theory in annulus I: evolution families and differential equations*. arXiv:1011.4253
  - M.D. Contreras, S. Díaz-Madrigal, P. Gumenyuk, *Loewner Theory in annulus II: Loewner chains.* arXiv:1105.3187

THANK YOU!!!