Loewner Theory in the Unit Disk

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Outline

The starting point of Loewner Theory is the seminal paper by

Czech – German mathematician

Karel Löwner (1893 – 1968) known also as
Charles Loewner

Untersuchungen über schlichte konforme Abbildungen des Einheitskreises,
Math. Ann. **89** (1923), 103–121.

In this paper Loewner introduced a new method to study the famous *Bieberbach Conjecture* concerning the *so-called class* $\mathcal{S}$.

Bieberbach's Conjecture

Ludwig Bieberbach, 1916: analytic properties of conformal mappings

$$f : \mathbb{D} \xrightarrow{\text{into}} \mathbb{C}, \quad \mathbb{D} := \{z : |z| < 1\}, \qquad f(0) = 0, \; f'(0) = 1.$$

Class $\mathcal{S}$

By $\mathcal{S}$ we denote the class of all *holomorphic univalent functions*

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

the famous Bieberbach Conjecture (1916)

$$\forall f \in \mathcal{S} \;\; \forall n = 2, 3, \ldots \qquad |a_n| \leqslant n \tag{2}$$

Bieberbach (1916): $n = 2$; Loewner (1923): $n = 3$; ...

de Branges (1984): all $n \geqslant 2$ — *using Loewner's method*

Elementary properties of class $\mathcal{S}$.

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- – there is no natural linear structure in the class $\mathcal{S}$;
- – the class $\mathcal{S}$ is not even a convex set in $\mathsf{Hol}(\mathbb{D}, \mathbb{C})$;
- + the class $\mathcal{S}$ is compact w.r.t. local uniform convergence in $\mathbb{D}$;
- + $\mathrm{Uni}_0(\mathbb{D}, \mathbb{D}) :=$

  $$\left\{\varphi \in \mathsf{Hol}(\mathbb{D}, \mathbb{D}) : \varphi \text{ is univalent and } \varphi(0) = 0,\ \varphi'(0) > 0\right\}$$

  is a topological semigroup w.r.t. the composition operation $(\varphi, \psi) \mapsto \psi \circ \varphi$ and the topology of locally uniform convergence.

Loewner's construction 1

Loewner considered the dense subclass $\mathcal{S}' \subset \mathcal{S}$ of all *slit mappings*,

$$S' := \Big\{ f \in \mathcal{S} \colon f(\mathbb{D}) = \mathbb{C} \setminus \Gamma, \text{ where } \Gamma \text{ is a Jordan arc extending to } \infty \Big\}.$$

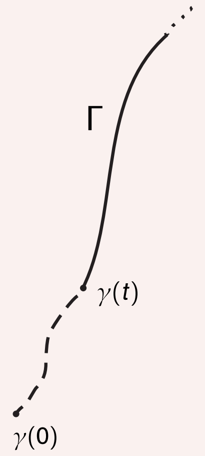
Loewner's construction 2

- Consider $f \in \mathcal{S}'$ and let $\Gamma := \mathbb{C} \setminus f(\mathbb{D})$.
- Choose a parametrization $\gamma : [0, +\infty] \to \Gamma$, $\gamma(+\infty) = \infty$.
- Consider the domains $\Omega_t := \mathbb{C} \setminus \gamma\big([t, +\infty]\big)$, $t \geqslant 0$.
- By Riem. Mapping Th'm $\forall t \geqslant 0$ $\exists!$ *conformal mapping*

$$f_t : \mathbb{D} \xrightarrow{\text{onto}} \Omega_t, \quad f_t(0) = 0, \; f_t'(0) > 0.$$

- Note that $\boxed{f_0 = f}$.
- Reparameterizing Γ: $\quad \forall\, t \geqslant 0 \quad \boxed{f_t'(0) = e^t}$.

Figure 1



Loewner's Theorem

- The family (f_t) is of class C^1 w.r.t. t (even if Γ is NOT smooth!)
- Moreover, $\exists!$ continuous function $\xi : [0, +\infty) \to \mathbb{T} := \partial\mathbb{D}$

(the Loewner PDE) $$\frac{\partial f_t(z)}{\partial t} = z f_t'(z) \frac{1 + \overline{\xi(t)}\, z}{1 - \overline{\xi(t)}\, z}, \qquad z \in \mathbb{D},\ t \geqslant 0. \tag{3}$$

- The following IVP (for the classical Loewner ODE)

$$\frac{dw(t)}{dt} = -w(t) \frac{1 + \overline{\xi(t)}\, w(t)}{1 - \overline{\xi(t)}\, w(t)} \tag{4}$$

$\forall\, s \geqslant 0\ \forall\, z \in \mathbb{D}$ has a unique solution $w = w_{z,s} : [s, +\infty) \to \mathbb{D}$.

- For all $s \geqslant 0$, $$f_s(z) = \lim_{t \to +\infty} e^t w_{z,s}(t). \tag{5}$$

Loewner's construction 3

As a corollary

Every $f \in S'$ is generated by some (uniquely defined) continuous function $\xi : [0, +\infty) \to \mathbb{T}$.

Namely
$$f(z) = \lim_{t \to +\infty} e^t w_{z,0}(t), \tag{6}$$
where $w = w_{z,0}$ is the solution to the IVP
$$\frac{dw(t)}{dt} = -w(t) \frac{1 + \overline{\xi(t)}\, w(t)}{1 - \overline{\xi(t)}\, w(t)}, \quad t \geqslant 0, \quad w(0) = z. \tag{7}$$

Answer (the converse Loewner Theorem)

Yes: for any continuous $\xi : [0, +\infty) \to \mathbb{T}$ relations (6) (7) *define a function $f \in \mathcal{S}$*.

But: $f \in \mathcal{S}'$? — NOT necessarily! [Kufarev 1947]

Loewner's construction 4

Conclusion

A dense subclass of $\mathcal{S}$ is represented by a linear space:

$$C\big([0,+\infty),\mathbb{R}\big) \ni u \quad \mapsto \quad \xi(t) := e^{iu(t)} \quad \xrightarrow[\text{equations}]{\text{Loewner}} \quad f \in \mathcal{S}^0 \supset \mathcal{S}'$$

Remark

For any simply connected domain $0 \in B \subsetneq \mathbb{C}$,
a dense subclass $\mathcal{U}_B^0 \supset \mathcal{U}_B'$ of

$$\mathcal{U}_B := \Big\{ f \in \mathrm{Hol}(\mathbb{D}, B) : \ f \text{ is univalent in } \mathbb{D},\ f(0) = 0,\ f'(0) > 1 \Big\}$$

can be represented in a similar way.

$$f \in \mathcal{U}_B' \quad \overset{\text{def}}{\Longleftrightarrow} \quad f \in \mathcal{U}_B,\ f(\mathbb{D}) = B \setminus [\text{a slit}].$$

Loewner's construction 5

Representation of $\mathcal{U}_B$

A dense subclass $\mathcal{U}_B^0 \subset \mathcal{U}_B$ is represented by the formula

$$f(z) = F\Big(w_{z,0}(T)\Big) \tag{8}$$

where:

- $F : \mathbb{D} \xrightarrow{\text{onto}} B$ conformally with $F(0) = 0$, $F'(0) > 0$;
- $T := \log\Big(F'(0)/f'(0)\Big)$;
- $w_{z,0}$ is the solution to

$$\frac{dw(t)}{dt} = -w(t)\frac{1+\overline{\xi(t)}\,w(t)}{1-\overline{\xi(t)}\,w(t)}, \qquad t \in [0,T], \quad w(0) = z, \tag{9}$$

and $\xi : [0,T] \to \mathbb{T}$ is continuous.

Chordal Loewner Equation 1

Previously we considered the conformal mappings normalized *at the internal point* $z = 0$.
For applications it is important to consider also normalization *at a boundary point*.

$$\mathbb{H} := \{\zeta : \operatorname{Im}\zeta > 0\}$$

P. P. Kufarev, V. V. Sobolev, and L. V. Sporysheva, 1968, considered the following class

$$\mathcal{R} := \Big\{ f \in \operatorname{Hol}(\mathbb{H}, \mathbb{H}) : \ f \text{ is univalent in } \mathbb{H}, \text{ and satisfies (10)} \Big\}.$$

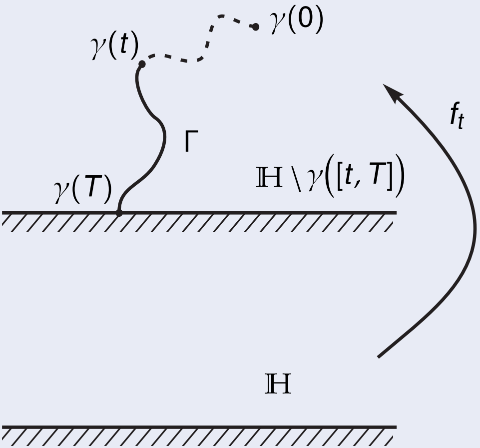
Hydrodynamic normalization: $$\lim_{\mathbb{H} \ni z \to \infty} \Big\{ f(z) - z \Big\} = 0. \tag{10}$$

If $\mathbb{H} \setminus f(\mathbb{H})$ is bounded, then f extends meromorphically to $\mathcal{O}(\infty)$ and the hydrodynamic normalization is equivalent to

$$f(z) = z - \ell(f)/z + c_2/z^2 + c_3/z^3 + \dots \tag{11}$$

Note that $\ell(f) \geqslant 0$, with $\ell(f) = 0 \iff f = \mathrm{id}_{\mathbb{H}}$.

Chordal Loewner Equation 2



$$f(\zeta) = \zeta - \frac{\ell(f)}{\zeta} + o(1/\zeta) \quad (12)$$

as $\mathbb{H} \ni \zeta \to \infty$;

$$f_t(\zeta) = \zeta - \frac{\ell(f) - 2t}{\zeta} + o(1/\zeta) \quad (13)$$

as $\mathbb{H} \ni \zeta \to \infty$.

$$\ell(f_t) = 2(T - t), \; T := \ell(f)/2$$

Chordal Loewner equation 3

The analogue of

classical Loewner ODE — *aka radial Loewner equation*

$$\frac{dw(t)}{dt} = -w(t)\frac{1+\overline{\xi(t)}w(t)}{1-\overline{\xi(t)}w(t)}, \quad w(0) = z \in \mathbb{D},$$

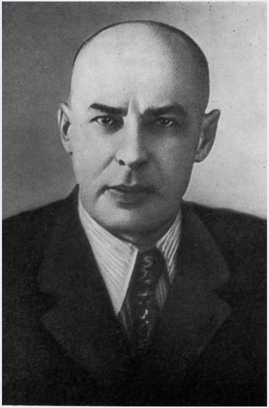
in the case of the class $\mathcal{R}$ considered by Kufarev *et al* is

Kufarev's ODE — *aka chordal Loewner equation*

$$\frac{dw(t)}{dt} = \frac{2}{\lambda(t) - w(t)}, \quad w(0) = \zeta \in \mathbb{H},$$

where $\lambda : [0, T] \to \mathbb{R}$ is a continuous function.

General form of radial Loewner equation 1



Pavel Parfen'evich Kufarev
Tomsk (1909 – 1968)

Christian Pommerenke
(Copenhagen, 17 December 1933)

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The radial Loewner equation can be thought as a special case of a more general equation.

$$\frac{dw(t)}{dt} = -w(t) \underbrace{\boxed{\frac{1+\overline{\xi(t)}w(t)}{1-\overline{\xi(t)}w(t)}}}_{p\left(w(t),t\right)}$$

Note that:

CHF1. $p(\cdot,t) \in \mathrm{Hol}\big(\mathbb{D},\mathbb{C}\big)$ and $\mathrm{Re}\, p(\cdot,t) > 0$ for a.e. $t \geqslant 0$;

CHF2. $p(0,t) = 1$ for a.e. $t \geqslant 0$;

CHF3. $p(z,\cdot)$ is measurable on $[0,+\infty)$ for all $z \in \mathbb{D}$.

Definition

A function $p : \mathbb{D} \times [0,+\infty) \to \mathbb{C}$ is said to be a *classical Herglotz function* if it satisfies CHF1 – CHF3.

Loewner–Kufarev equation

$$\frac{dw(t)}{dt} = -wp\big(w(t), t\big), \quad t \geqslant 0, \qquad w(0) = z \in \mathbb{D}, \tag{14}$$

where p is a classical Herglotz function, *i.e.*

CHF1. $p(\cdot, t) \in \mathrm{Hol}\big(\mathbb{D}, \mathbb{C}\big)$ and $\mathrm{Re}\, p(\cdot, t) > 0$ for a.e. $t \geqslant 0$;
CHF2. $p(0, t) = 1$ for a.e. $t \geqslant 0$;
CHF3. $p(z, \cdot)$ is measurable on $[0, +\infty)$ for all $z \in \mathbb{D}$.

$$\mathcal{S} := \Big\{ f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C}) : \ f \text{ is univalent}, \ f(0) = f'(0) - 1 = 0 \Big\}.$$

Generates the whole class $\mathcal{S}$

$$f(z) = \lim_{t \to +\infty} e^t w_{z,0}(t), \quad z \in \mathbb{D}. \tag{15}$$

Applications to univalent functions

Here we mention some important applications of the classical Loewner Theory to the problems for univalent functions.

The class $\mathcal{S}$:

$f:\mathbb{D}\to\mathbb{C}$ univalent holomorphic normalized by $f(z)=z+\sum_{n=2}^{+\infty}a_nz^n$.

This class is compact, so for any continuous map

$$J:\mathcal{S}\to\mathbb{R} \tag{16}$$

there exists $J_{\max}:=\max_{f\in\mathcal{S}}J(f)$.

Extremal Problem:

is the problem to find $J_{\max}$ and all the functions $f_*\in\mathcal{S}$ such that $J(f_*)=J_{\max}$ (*extremal functions*).

Coefficient functionals: $J(f):=J(a_2,\dots,a_n)$.

Coefficient functionals

$$J(f) := J(a_2, \dots, a_n), \qquad f(z) = \lim_{t\to+\infty} e^t w_{z,0}(t)$$

$$\frac{dw(t)}{dt} = -w(t)\frac{1+\overline{\xi(t)}w(t)}{1-\overline{\xi(t)}w(t)},\ \xi(t) := e^{iu(t)}, \quad w(0) = z \in \mathbb{D}, \ (17)$$

where $u : [0, +\infty) \to \mathbb{R}$ is continuous

except for a finite number of jump discontinuities.

$$e^t w = e^t w_{z,0}(t) = e^t z + \sum_{n=2}^{+\infty} a_n(t) z^n \quad \Rightarrow \quad f(z) = z + \sum_{n=2}^{+\infty} a_n(+\infty) z^n,$$

System of ODE for a_j's

$$\begin{cases} (d/dt)\, a_2(t) &= -2e^{-t}e^{iu(t)}, & a_2(0) = 0, \\ (d/dt)\, a_3(t) &= -2e^{-t}e^{iu(t)}\left(e^{-t}e^{iu(t)} + 2a_2(t)\right), & a_3(0) = 0, \\ & \dots & \end{cases}$$

Examples of Extremal Problems

- ☞ $|a_3| \leqslant 3$ (Loewner, 1923);
- ☞ $|a_n| \leqslant n$, for all $n \geqslant 2$, — the Bieberbach Conjecture (⇐ Milin's Conjecture proved by de Branges, 1984);
- ☞ $|f(z_0)|$, $|f'(z_0)|$, $\left|\dfrac{z_0 f'(z_0)}{f(z_0)}\right|$ ($z_0 \in \mathbb{D} \setminus \{0\}$ arbitrary);
- ☞ $\arg \dfrac{f(z_0)}{z_0}$, $\arg f'(z_0)$, $\arg \dfrac{z_0 f'(z_0)}{f(z_0)}$, $\arg \dfrac{z_0^2 f'(z_0)}{[f(z)]^2}$ (Goluzin, 1936);

(Rotation Theorem)

$$\left|\arg f'(z_0)\right| \leqslant \begin{cases} 4 \arcsin |z_0|, & \text{if } |z_0| \leqslant 1/\sqrt{2}, \\ \pi + \log \dfrac{|z_0|}{1-|z_0|^2}, & \text{if } 1/\sqrt{2} \leqslant |z_0| < 1. \end{cases}$$

- ☞ coefficients of the inverse map

$$f^{-1}(w) = w + \sum_{n=2}^{+\infty} b_n w^n$$ (Loewner, 1923).

"$\Leftrightarrow$"-condition for univalence

Theorem (Pommerenke)

Let $f \in \mathrm{Hol}(\mathbb{D},\mathbb{C})$, $f(0) = f'(0) - 1 = 0$.

Then $f \in \mathcal{S}$ **iff** there exists $(f_t)_{t\geqslant 0} \subset \mathrm{Hol}(\mathbb{D},\mathbb{C})$ with $f_0 = f$ s.t.:

- ➔ $\exists K_0 > 0$ s.t. $|f_t(z)| \leqslant K_0 e^t$ for all $t \geqslant 0$, all $|z| < \varepsilon$;
- ➔ $(z,t) \mapsto f_t(z)$ is locally absolutely continuous solution in $\mathbb{D} \times [0,+\infty)$ to the Loewner – Kufarev PDE

$$\frac{\partial f_t(z)}{\partial t} = z f_t'(z) p(z,t),$$

where $p : \mathbb{D} \times [0,+\infty) \to \mathbb{C}$ is a classical Herglotz function.

CHF1. $p(\cdot,t) \in \mathrm{Hol}\big(\mathbb{D},\mathbb{C}\big)$ and $\mathrm{Re}\, p(\cdot,t) > 0$ for a.e. $t \geqslant 0$;
CHF2. $p(0,t) = 1$ for a.e. $t \geqslant 0$;
CHF3. $p(z,\cdot)$ is measurable on $[0,+\infty)$ for all $z \in \mathbb{D}$.

More applications

- ☞ sufficient conditions for univalence
- ☞ sufficient conditions for quasiconformal extendability

Applications aside Complex Analysis:

☞ Stochastic Loewner Equation (SLE)

Schramm, 2000:

$$\frac{dw(t)}{dt} = -\frac{2}{\sqrt{\kappa}\mathcal{B}_t - w(t)}, \qquad (18)$$

where $\kappa > 0$, and $(\mathcal{B}_t)$ is a (standard 1-dimensional) Brownian motion.

- ! Very IMPORTANT applications in Statistical Physics;
- ! FIELDS MEDALS: W. Werner (2006), S. Smirnov (2010);
- ☹ "stochastic" =(usually)= "more complicated"
- ☺ *in a certain sense,* the equation is still deterministic
- ¿ Why is there a minus?

The whole story here is about **random planar curves.**

Growing slit (chordal case) 1

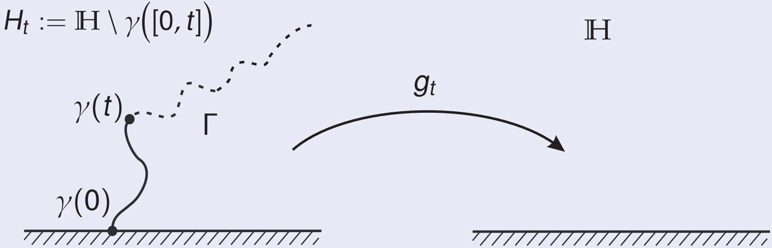
A version of the Kufarev – Sporysheva – Sobolev Theorem

Let:

- Γ be a Jordan arc s.t. one of the end-points $a \in \mathbb{R}$, the other is $b = \infty$, and $\Gamma \setminus \{a, b\} \subset \mathbb{H} := \{\zeta : \operatorname{Im} \zeta > 0\}$;
- $\gamma : [0, +\infty] \to \Gamma$ a parametrization of Γ with $\gamma(0) = a$ and $\gamma(+\infty) = b = \infty$;
- for each $t \geqslant 0$, g_t is the conformal mapping of $H_t := \mathbb{H} \setminus \gamma\big([0, t]\big)$ onto $\mathbb{H}$ with the *hydrodynamic normalization* $g_t(\zeta) - \zeta \to 0$ as $\zeta \to \infty$.
- Under a suitable parametrization γ of the Jordan arc Γ,

$$g_t(\zeta) = \zeta + \frac{2t}{\zeta} + \frac{c_2}{\zeta^2} + \dots \quad (\zeta \to \infty). \tag{19}$$

Growing slit (chordal case) 2



Theorem

There exists a continuous function $\lambda : [0, +\infty) \to \mathbb{R}$ s.t.

$$\frac{dg_s(\zeta)}{ds} = -\frac{2}{\lambda(s) - g_s(\zeta)}, \quad s \geqslant 0, \qquad g_0(\zeta) = \zeta. \tag{20}$$

For each $t \geqslant 0$ the set $H_t := \mathbb{H} \setminus \gamma([0,t])$ coincides with the set of all $\zeta \in \mathbb{H}$ for which the solution to (20) exists on $[0, t + \varepsilon)$ for some $\varepsilon > 0$.

The converse theorem

Let $\lambda : [0, +\infty) \to \mathbb{R}$ continuous. Then the initial value problem

$$\frac{dg_s(\zeta)}{ds} = -\frac{2}{\lambda(s) - g_s(\zeta)}, \quad s \geqslant 0, \qquad g_0(\zeta) = \zeta. \tag{20}$$

defines a family of holomorphic functions

$$g_t(\zeta) = \zeta + \frac{2t}{\zeta} + \frac{c_2}{\zeta^2} + \dots \quad (\zeta \to \infty),$$

each mapping its domain H_t conformally onto $\mathbb{H}$.

Remark

Unfortunately, $\mathbb{H} \setminus H_t$ is NOT always a Jordan curve.

Random planar curves (SLE) 1

Assumption

For simplicity, we will consider the case $0 < \kappa < 4$.

Recall that by definition of a stochastic process

$$\mathcal{B} : \left(\Omega, \mathcal{F}, \mathbb{P}\right) \times [0, +\infty) \longrightarrow \mathbb{R};\ (\omega, t) \mapsto \mathcal{B}_t(\omega).$$

Consider $\lambda(t) := \sqrt{\kappa}\mathcal{B}_t(\omega)$, where $\omega \in \Omega$ is fixed. Then:

- ☞ λ is *almost surely* continuous (by def. of the Brownian motion);
- ☞ moreover, the sets $\mathbb{H} \setminus H_t$ are almost surely Jordan arcs;
- ☞ Hence one gets a random Jordan arc in $\mathbb{H}$

$$\Gamma = \Gamma(\omega) := \bigcup_{t \geqslant 0} \mathbb{H} \setminus H_t$$

joining $a = \mathcal{B}_0 = 0$ and $b = \infty$.

Random planar curves (SLE) 2

O. Schramm, 2000

If a random planar curve Γ satisfies

- ☞ conformal invariance, and
- ☞ the domain Markov property,

then it must be (chordal) SLE, *i.e.*

there exists $\kappa > 0$ s.t. Γ is the set of all $\zeta \in \mathbb{H}$ for which the solution to

$$\frac{dw(t)}{dt} = -\frac{2}{\sqrt{\kappa}\mathcal{B}_t - w(t)}, \quad w(0) = \zeta,$$

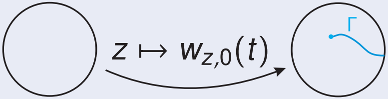
explodes at a finite time $t_0(\zeta) < +\infty$.

Conditions for slit dynamics 1

- P.P. Kufarev, 1946: if $\xi : [0, T] \to \mathbb{T}$ is differentiable and ξ' is bounded, then

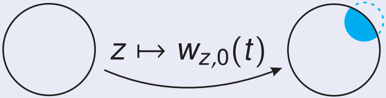
$$\frac{dw(t)}{dt} = -w(t)\frac{1+\overline{\xi(t)}w(t)}{1-\overline{\xi(t)}w(t)}, \quad w(0) = z \in \mathbb{D}, \qquad (21)$$

generates conformal maps of $\mathbb{D}$ onto $\mathbb{D}$ minus a C^1-slit $\Gamma \perp \partial\mathbb{D}$.



- P.P. Kufarev, 1947: example of non-slit maps generated by (21):

$$\xi(t) := \left(e^{-t} + i\sqrt{1-e^{-2t}}\right)^3, \quad \xi'(t) \to \infty \text{ as } t \to +0. \qquad (22)$$



Conditions for slit dynamics 2

- C. Earle and A. Epstein, 2001:
  - if (21) generates a C^n-slit Γ, $n \geqslant 2$, then ξ must be of class C^{n-1}.
  - if Γ is real-analytic, then ξ must be real-analytic.
- D. Marshall and S. Rohde, 2005:
  - if Γ is a *quasislit*, then ξ must be of class $\mathrm{Lip}(\frac{1}{2})$;
  - $\exists\, C_{\mathbb{D}} > 0$ s.t. if $\|\xi\|_{\mathrm{Lip}(\frac{1}{2})} < C_{\mathbb{D}}$, then (21) generates a quasislit.
- the above results by Marshal and Rohde extend to the case of the chordal Loewner equation

$$\frac{dw(t)}{dt} = \frac{2}{\lambda(t) - w(t)}, \quad w(0) = \zeta \in \mathbb{H}. \tag{23}$$

- J. Lind, 2005: the best constant $C_{\mathbb{H}} = 4$.
- D. Prokhorov and A. Vasil'ev, 2009: $C_{\mathbb{D}} = C_{\mathbb{H}}$.
- *Many others ...*

Topics to mention

Modern Loewner Theory turns out to be related to many topics, *e.g.*

- ☞ Hele-Shaw 2D hydrodynamical problem
  P.P. Kufarev, Yu.P. Vinogradov, 1948;
- ☞ DLA (diffusion limited aggregation)
  L. Carleson, N. Makarov, 2001;
- ☞ Integrable Systems
  D. Prokhorov, A. Vasil'ev, 2006;
- ☞ Contour dynamics and image recognition . . .

Semigroup approach 1

Loewner – Kufarev ODE

$$\frac{dw}{dt} = -w(t)\, p\big(w(t), t\big), \quad t \geqslant 0, \qquad w(0) = z \in \mathbb{D}, \qquad (*)$$

where $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is a classical Herglotz function:

CHF1. $p(\cdot, t) \in \mathrm{Hol}\big(\mathbb{D}, \mathbb{C}\big)$ and $\mathrm{Re}\, p(\cdot, t) > 0$ for a.e. $t \geqslant 0$;

CHF2. $p(0, t) = 1$ for a.e. $t \geqslant 0$;

CHF3. $p(z, \cdot)$ is measurable on $[0, +\infty)$ for all $z \in \mathbb{D}$.

$\mathrm{Uni}_0(\mathbb{D}, \mathbb{D}) = \big\{\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D}) : \varphi$ is univalent and $\varphi(0) = 0,\ \varphi'(0) > 0\big\}$

Theorem

$\varphi \in \mathrm{Uni}_0(\mathbb{D}, \mathbb{D})$ if and only if $\varphi(z) = w_{z,0}\big(-\log \varphi'(0)\big)$, where $w = w_{z,0}$ is the solution to $(*)$ with *some* classical Herglotz function p.

Other semigroups of conformal mappings have similar description.

For example:

- ☞ $\mathrm{Uni}_\infty(\mathbb{H},\mathbb{H}) = \Big\{\varphi \in \mathrm{Hol}(\mathbb{H},\mathbb{H}) : \varphi$ is univalent
  and ∞ is its DW-point $(\Leftrightarrow \varphi^{\circ n} \to \infty$ as $n \to +\infty)\Big\}$
  $$dw(t)/dt = ip\big(w(t), t\big), \tag{24}$$
  where $p(\cdot, t) \in \mathrm{Hol}(\mathbb{H},\mathbb{C})$ and $\mathrm{Re}\, p \geqslant 0$.
- ☞ the general version of the chordal Loewner ODE (chordal "Loewner – Kufarev") represents
  a subsemigroup $\mathrm{Uni}_{hydro}(\mathbb{H},\mathbb{H}) \subset \mathrm{Uni}_\infty(\mathbb{H},\mathbb{H})$.
- ☞ V.V. Goryainov, 1986, '89, '91, '93, '96, '98, 2000

Semigroup approach 3

What's about the whole semigroup

$$\mathrm{Uni}(\mathbb{D},\mathbb{D}) := \left\{\varphi \in \mathsf{Hol}(\mathbb{D},\mathbb{D}) : \varphi \text{ is univalent}\right\}?$$

Possible way of representation: *— not intrinsic*

Write $\varphi \in \mathrm{Uni}(\mathbb{D},\mathbb{D})$ as $\varphi = \ell \circ \varphi_0$,

where $\ell \in \mathsf{Aut}(\mathbb{D})$, $\varphi_0 \in \mathrm{Uni}_0(\mathbb{D},\mathbb{D})$.

Intrinsic way to represent $\mathrm{Uni}(\mathbb{D},\mathbb{D})$ comes from a new approach in Loewner Theory by F. Bracci, M. D. Contreras and S. Díaz-Madrigal:

✎ *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, issue **672** (Nov 2012), 1 – 37

✎ *Mathematische Annalen*, **344** (2009), 947 – 962 (generalization to *complex manifolds*)

1-parameter semigroups in $\mathbb{D}$, 1

Definition

A *one-parameter semigroup* in $\mathbb{D}$ is a continuous semigroup homomorphism $[0, +\infty) \ni t \mapsto \phi_t \in \text{Hol}(\mathbb{D}, \mathbb{D})$.
In other words, a family $(\phi_t) \subset \text{Hol}(\mathbb{D}, \mathbb{D})$ is a *one-parameter semigroup* if:

S1. $\phi_0 = \text{id}_{\mathbb{D}}$;

S2. $\phi_t \circ \phi_s = \phi_s \circ \phi_t = \phi_{t+s}$;

S3. $\phi_t(z) \to z$ as $t \to +0$ for any $z \in \mathbb{D}$.

Example

Let $G \in \text{Hol}(\mathbb{D}, \mathbb{C})$. Suppose that for any $z \in \mathbb{D}$ the IVP

$$dw(t)/dt = G\big(w(t)\big), \quad w(0) = z, \tag{25}$$

has a unique solution $w = w_z(t)$ defined for all $t \geqslant 0$.
Then the functions $\phi_t(z) := w_z(t)$ form a one-parameter semigroup.

1-parameter semigroups in $\mathbb{D}$, 2

Theorem

Any one-parameter semigroup (ϕ_t) comes from solution to (25).
In particular, functions ϕ_t are univalent.
The vector field G is uniquely defined by the formula

$$G(z) = \lim_{t \to +0} \frac{\phi_t(z) - z}{t}, \quad z \in \mathbb{D}. \tag{26}$$

The function G is called the *(infinitesimal) generator* of (ϕ_t).

A *naive* analogy with Lie groups would suggest that:

NOT true

For every $\phi \in \mathrm{Uni}(\mathbb{D}, \mathbb{D})$
is contained in some one-parameter semigroup.

Return to the classical Loewner – Kufarev ODE

$$dw/dt = -w(t)\,p\big(w(t),t\big), \quad t \geqslant s \geqslant 0, \quad w(s) = z \in \mathbb{D}. \qquad (27)$$

Let $w = w_{z,s}(t)$ be the unique solution to the above IVP. Denote

$$\boxed{\varphi_{s,t}(z) := w_{z,s}(t).}$$

Then $(\varphi_{s,t})_{s \geqslant t \geqslant 0} \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$ and:

EF1. $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$ for any $s \geqslant 0$;

EF2. if $u \in [s,t]$, then $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{u,s}$;

EF3. stronger version of local absolute continuity for $t \mapsto \varphi_{s,t}(z)$.

Definition (Bracci, Contreras and Díaz-Madrigal)

A family $(\varphi_{s,t})_{t \geqslant s \geqslant 0} \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$ satisfying EF1 – EF3 is called an *evolution family*.

Evolution families 2

- Evolution families generalize one-parameter semigroups:
  if (ϕ_t) is one-parameter semigroup,
  then $(\varphi_{s,t} := \phi_{t-s})$ an evolution family.

- Any $\phi \in \mathrm{Uni}(\mathbb{D},\mathbb{D})$ ($\Leftrightarrow \phi \in \mathrm{Hol}(\mathbb{D},\mathbb{D})$ and injective)
  is contained in some evolution family.

- Each evolution family satisfies a certain ODE.

Again the classical Loewner – Kufarev ODE!!!

$$\frac{dw(t)}{dt} = \underbrace{\boxed{-w(t)\,p\big(w(t),t\big)}}$$

$G(\cdot,t)$ — an infinitesimal generator

Infinitesimal generators

Infinitesimal generator with $G(0) = 0$.

$$G(w) = -wp(w), \quad \operatorname{Re} p \geqslant 0. \tag{28}$$

Arbitrary generators (Berkson and Porta, 1978)

$$G(w) = (\tau - w)(1 - \overline{\tau} w)p(w), \quad \operatorname{Re} p \geqslant 0, \quad \tau \in \overline{\mathbb{D}}. \tag{29}$$

Bracci, Contreras and Díaz-Madrigal suggested:

Equation for evolution families (*generalized Loewner ODE*)

$$\frac{dw(t)}{dt} = G\big(w(t), t\big) = \big(\tau(t) - w(t)\big)\big(1 - \overline{\tau(t)}w(t)\big)p\big(w(t), t\big) \tag{30}$$

Herglotz vector field

Definition (*essentially* from Carathéodory's theory of ODEs)

A function $G : \mathbb{D} \times [0, +\infty)$ is said to be a *weak holomorphic vector field*, if:

WHVF1. $G(\cdot, t)$ is holomorphic in $\mathbb{D}$ for a.e. $t \geqslant 0$;

WHVF2. $G(z, \cdot)$ is measurable on $[0, +\infty)$ for all $z \in \mathbb{D}$;

WHVF3. given $K \Subset \mathbb{D}$, there exists k_K of class L^1_{loc} s.t.

$$\sup_{z \in K} \left| G(z,t) \right| \leqslant k_K(t), \quad \text{a.e. } t \geqslant 0. \tag{31}$$

$\Rightarrow$ local existence and uniqueness for $dw/dt = G(w, t)$.

Definition (Bracci, Contreras and Díaz-Madrigal)

A weak holomorphic vector field $G : \mathbb{D} \times [0, +\infty)$ is said to be a *Herglotz vector field* if for a.e. $t \geqslant 0$, $G(\cdot, t)$ is an infinitesimal generator.

EF and Herglotz VF

Theorem (Bracci, Contreras and Díaz-Madrigal)

A family $(\varphi_{s,t})_{t\geqslant s\geqslant 0} \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$ is an *evolution family* **iff** there exists a Herglotz vector field G s.t. for any $s \geqslant 0$ and any $z \in \mathbb{D}$ the function $w = w_{z,s}(t) := \varphi_{s,t}(z)$ solves the IVP

$$dw(t)/dt = G\big(w(t),t\big), \quad t \geqslant s, \qquad w(s) = z. \tag{32}$$

"General recipe": suppose we wish to obtain the representation

for a subsemigroup $U \subset \mathrm{Uni}(\mathbb{D},\mathbb{D})$.

- Consider all one-parameter semigroups $(\phi_t) \subset U$.
- Characterize their infinitesimal generators — $\mathrm{Gen}(U)$.
- $\mathrm{HVF}(U) := \big\{G : \ G(\cdot,t) \in \mathrm{Gen}(U) \text{ a.e. } t \geqslant 0\big\}$.
- Now equation (32) gives a 1-to-1 correspondence between $\mathrm{HVF}(U)$ and evolution families $(\varphi_{s,t}) \subset U$.
- NB: every $\phi \in U$ is contained in some evolution family $(\varphi_{s,t}) \subset U$.

Examples

Previously known cases

Representations of previously studied subsemigroups are recovered: $\mathrm{Uni}_0(\mathbb{D},\mathbb{D})$, $\mathrm{Uni}_\infty(\mathbb{H},\mathbb{H})$, $\mathrm{Uni}_{hydro}(\mathbb{H},\mathbb{H})$,... in this way.

A new case (Bracci, Contreras, Díaz-Madrigal, Gumenyuk, *in preparation*)

Representation of the semigroup consisting of all injective $\phi \in \mathrm{Hol}(\mathbb{D},\mathbb{D})$ with a regular boundary fixed point at $a = 1$, which means:

$$\exists \angle \lim_{z\to 1} \phi(z) = 1, \qquad \exists \text{ finite } \angle \lim_{z\to 1} \frac{\phi(z)-1}{z-1}.$$

The End *MUCHAS GRACIAS !!!*