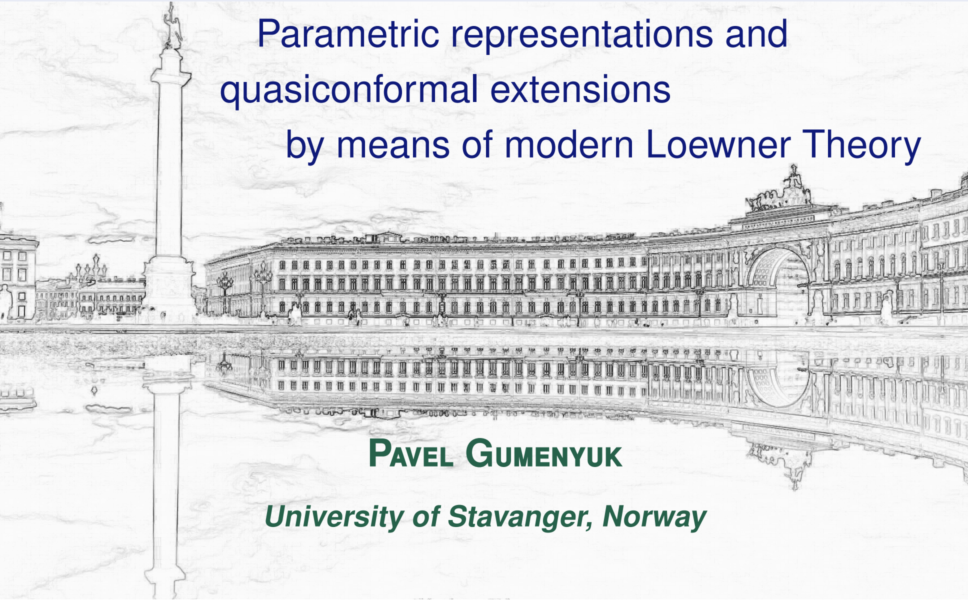

Parametric representations and quasiconformal extensions by means of modern Loewner Theory

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Classical Parametric Representation

Every univalent holomorphic function $f : \mathbb{D} := \left\{z\colon |z| < 1\right\} \to \mathbb{C}$,
$f(0) = 0,\ f'(0) = 1,$
is the initial element, i.e. $f = f_0$,
of some (classical radial) *Loewner chain* $(f_t)_{t\geqslant 0}$.

Definition: $(f_t)_{t\geqslant 0}$ is a *classical radial Loewner chain* if:

(i) for each $t \geqslant 0$, $f_t : \mathbb{D} \to \mathbb{C}$ is univalent in $\mathbb{D}$, $f_t(0) = 0$, $f_t'(0) = e^t$;

(ii) $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leqslant s \leqslant t$.

Loewner – Kufarev PDE

$$\frac{\partial f_t}{\partial t} = -G(z,t)\frac{\partial f_t}{\partial z}, \quad t \geqslant 0, \qquad G(z,t) := -z\,p(z,t),$$

where p is a *classical Herglotz function*.

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Loewner – Kufarev PDE

$$\frac{\partial f_t}{\partial t} = -G(z,t)\,\frac{\partial f_t}{\partial z}, \qquad G(z,t) := -z\,p(z,t), \qquad \text{(L-K PDE)}$$

where p is a *classical Herglotz function*, i.e.

(i) $\forall z \in \mathbb{D}$, $p(z,t)$ is measurable in $t \in [0,+\infty)$;

(ii) $\forall$ a.e. $t \geqslant 0$, $p(\cdot,t)$ is holomorphic in $\mathbb{D}$, $\operatorname{Re} p > 0$, $p(0,t) = 1$.

(L-K PDE) establish a *1-to-1* relation between
class. Herglotz functions p and class. radial Loewner chains (f_t).

Loewner – Kufarev ODE = characteristic eq-n for (L-K PDE)

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{s,t}(z) = G\big(\varphi_{s,t}(z),t\big), \quad t \geqslant s \geqslant 0; \qquad \varphi_{s,s}(z) = z \in \mathbb{D}.$$

$\boxed{\varphi_{s,t} = f_t^{-1} \circ f_s}$: $\mathbb{D} \xrightarrow{\text{holo}} \mathbb{D}$ because $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$, $t \geqslant s \geqslant 0$.

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General question

How are the properties of a classical Herglotz function p reflected in properties of the corresponding Loewner chain (f_t) and evolution family $(\varphi_{s,t})_{t \geqslant s \geqslant 0}$?

Theorem (J. Becker, 1972)

Let $k \in [0, 1)$. SUPPOSE that the class. Herglotz function p satisfies

$$p(\mathbb{D}, t) \subset U(k) := \left\{\zeta\colon \left|\frac{\zeta - 1}{\zeta + 1}\right| \leqslant k\right\} \subset\subset \mathbb{H} \quad \text{a.e. } t \geqslant 0.$$

THEN each function in the corresponding Loewner chain (f_t) *have a k-q.c. extension to $\overline{\mathbb{C}}$.*

NB: Becker also gave an explicit formula for the q.c.-extension of f_t's.

F. Bracci, M.D. Contreras, and S. Díaz-Madrigal, 2008/2012

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{s,t}(z) = G\big(\varphi_{s,t}(z), t\big), \quad t \geqslant s \geqslant 0; \qquad \varphi_{s,s}(z) = z \in \mathbb{D}, \qquad (*)$$

where $G(w,t) := \big(\tau(t) - w\big)\big(1 - \overline{\tau(t)}w\big)\,p(w,t)$ and:

(i) $\tau : [0, +\infty) \to \overline{\mathbb{D}}$ is measurable;

(ii) $\forall z \in \mathbb{D}$, $p(w,t)$ is measurable in $t \in [0, +\infty)$;

(iii) $\forall$ a.e. $t \geqslant 0$, $p(\cdot, t)$ is holomorphic in $\mathbb{D}$ with $\operatorname{Re} p \geqslant 0$;

(iv) $t \mapsto p(0,t)$ is L^1_{loc} on $[0, +\infty)$.

The function G above is referred to as a *Herglotz vector field* and $(\varphi_{s,t})_{t \geqslant s \geqslant 0}$ is the associated *evolution family*.

☞ Classical case: $\tau(t) = 0$ and $p(0,t) = 1$ for a.e. $t \geqslant 0$.

Definition: (f_t) is a (generalized) *Loewner chain* if:

(i) for each $t \geqslant 0$, $f_t : \mathbb{D} \to \mathbb{C}$ is univalent in $\mathbb{D}$;

(ii) $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leqslant s \leqslant t$;

(iii) $\forall\, K \subset\subset \mathbb{D}$, $\sup_{z\in K} |f_t(z) - f_s(z)| \leqslant \displaystyle\int_s^t \alpha_K(\xi)\, \mathrm{d}\xi$ for any $s \leqslant t$ and some L^1_{loc} function $\alpha_K : [0, +\infty) \to [0, +\infty)$.

Theorem (M.D. Contreras, S. Díaz-Madrigal, and P. Gum., 2010)

Let G be a Herglotz vector field with associated evol'n family $(\varphi_{s,t})$. There exists a unique Loewner chain (f_t) such that

(i) $\varphi_{s,t} = f_t^{-1} \circ f_s$ whenever $t \geqslant s \geqslant 0$; (ii) $f_0(0) = 0$, $f_0'(0) = 1$;

(iii) $\bigcup_{t\geqslant 0} f_t(\mathbb{D})$ is $\mathbb{C}$ or a disk centered at 0.

Moreover,

$$\boxed{\partial f_t/\partial t = -G(z,t)\, \partial f_t/\partial z}, \quad t \geqslant 0. \qquad \text{(gL-K PDE)}$$

We call (f_t) the *standard Loewner chain associated* with $(\varphi_{s,t})$ and G.

Corollary

Let $(\varphi_{s,t})$ be an evol'n family with associated st. Loewner chain (f_t).
If $\varphi_{s,t}$ is k-q.c. extendible for any $t \geqslant s \geqslant 0$,
then f_t is also k-q.c. extendible for any $t \geqslant 0$.

P. Gum., I. Prause, 2016

Becker's condition $p(\mathbb{D}, t) \subset U(k)$ is also sufficient
for k-q.c. extendibility of evolution families in the general case.

Notation

Let $a : [0, +\infty) \to \mathbb{H} \cup i\mathbb{R}$ be L^1_{loc} and denote by D_t:

- the (closed) hyperbolic disk in $\mathbb{H}$ of radius $\frac{1}{2} \log \frac{1+k}{1-k}$ centered at $a(t)$ when $a(t) \in \mathbb{H}$;
- the single point $\{a(t)\}$ when $a(t) \in i\mathbb{R}$.

Remark: If $a \equiv 1$, then $D_t \equiv U(k)$.

Theorem (P. Gum., I. Prause, 2016)

If $G(w,t) := \big(\tau(t) - w\big)\big(1 - \overline{\tau(t)}w\big)\,p(w,t)$ is a Herglotz vector field and $p(\mathbb{D},t) \subset D_t$ for a.e. $t \geqslant 0$,

then each $\varphi_{s,t}$ in the assoc'd evolution family is k-q.c. extendible.

[No explicit formula for the q.c.-extensions; we use Slodkowski's λ-Lemma]

SPECIAL CASE $\tau \equiv 1$ (aka "chordal" case): P. Gum., I. Hotta, 2016
[with explicit formula for the extension]

Theorem (P. Gum., I. Hotta, 2016)

SUPPOSE that h is holomorphic and locally univalent in $\mathbb{H}$, and

$$\forall\, z \in \mathbb{H} \quad \alpha\left(\frac{h(z)}{h'(z)} - z\right) + i\beta \;\in\; U(k) = \Big\{\zeta\colon |\zeta - 1| \leqslant k|\zeta + 1|\Big\} \qquad (*),$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are some constants,

THEN h has a k-q.c. extension to $\overline{\mathbb{C}}$ with a fixed point at ∞.

- Loewner chains provide characterization of univalent normalized maps $f : \mathbb{D} \to \mathbb{C}$;
- Similarly, evolution families provide characterization of univalent self-maps $\varphi : \mathbb{D} \to \mathbb{D}$:

$\varphi \in \mathcal{U} := \{\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D}) : \varphi \text{ is univalent in } \mathbb{D}\}$

if and only if φ belongs to some evolution family $(\varphi_{s,t})$.

Question

Given a subclass $\mathcal{U}' \subset \mathcal{U}$,

is it possible to characterize $\varphi \in \mathcal{U}'$ *in a similar way?*

Example: classical case

$\varphi \in \mathcal{U}_0 := \{\varphi \in \mathcal{U} : \varphi(0) = 0,\ \varphi'(0) > 0\}$ **iff** φ belongs to some evolution family $(\varphi_{s,t}) \sim$ a class. Herglotz v.f. $G(z,t) = -z\,p(z,t)$, $\mathrm{Im}\,p(0,t) = 0$.

Definition

We say that $\mathcal{U}' \subset \mathcal{U}$ *admits a Loewner-type param. representation*, if there exists a convex cone $\mathcal{M} \subset \{\text{all Herglotz vector fields}\}$ such that

$$\varphi \in \mathcal{U}' \iff \varphi \in (\varphi_{s,t}) \sim \text{to some } G \in \mathcal{M}.$$

NB: $\mathcal{U}'$ must be closed w.r.t. $(\varphi, \psi) \mapsto \varphi \circ \psi$ and $\mathrm{id}_{\mathbb{D}} \in \mathcal{U}'$.

Denjoy – Wolff point

Let $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\mathrm{id}_{\mathbb{D}}\}$. Then:

☞ either $\exists!\, \tau \in \mathbb{D}\ \varphi(\tau) = \tau$, τ is called the D.W.-point of φ

☞ or $\varphi^{\circ n} \to \tau \in \partial\mathbb{D}$, $\varphi(\tau) := \angle \lim_{z\to\tau} \varphi(z) = \tau$, $\varphi'(\tau) := \angle \lim_{z\to\tau} \varphi'(z) \in (0, 1]$.

Boundary regular fixed points

$$\sigma \in \partial\mathbb{D} \text{ is a } \textit{BRFP} \text{ of } \varphi \quad \overset{\text{def}}{\iff} \quad \begin{cases} \varphi(\sigma) = \angle \lim_{z\to\sigma} \varphi(z) = \sigma \text{ and} \\ \varphi'(\sigma) = \angle \lim_{z\to\sigma} \varphi'(z) \text{ exists finitely.} \end{cases}$$

Let $F = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be a finite subset of $\partial\mathbb{D}$, and let $\tau \in \overline{\mathbb{D}} \setminus F$.

☞ $\mathcal{U}[F] := \Big\{\varphi \in \mathcal{U} \colon$ each $\sigma \in F$ is a BRFP of $\varphi\Big\}$

☞ $\mathcal{U}_\tau[F] := \Big\{\varphi \in \mathcal{U}[F] \setminus \{\mathrm{id}_{\mathbb{D}}\} \colon \tau$ is the DW-point of $\varphi\Big\} \cup \{\mathrm{id}_{\mathbb{D}}\}$

Theorem (P. Gum., arXiv:1603.04043)

The following classes admit a Loewner-type parametric represent'n:

- ✔ $\mathcal{U}[F]$ for $n \leqslant 3$;
- ✔ $\mathcal{U}_\tau[F]$ for $\tau \in \partial\mathbb{D}$ and $n \leqslant 2$;
- ✔ $\mathcal{U}_\tau[F]$ for $\tau \in \mathbb{D}$ and any $n \geqslant 1$. [$n = 1$: Unkelbach and Goryainov]

The corresponding cones $\mathcal{M}$ of Herglotz vector fields are described.

H. Unkelbach, 1940: an attempt to give the Loewner-type parametric representation for $\mathcal{U}_0[\{1\}]$;
V.V. Goryainov, 2015: the complete proof.