Matching univalent functions

PAVEL GUMENYUK

joint research with

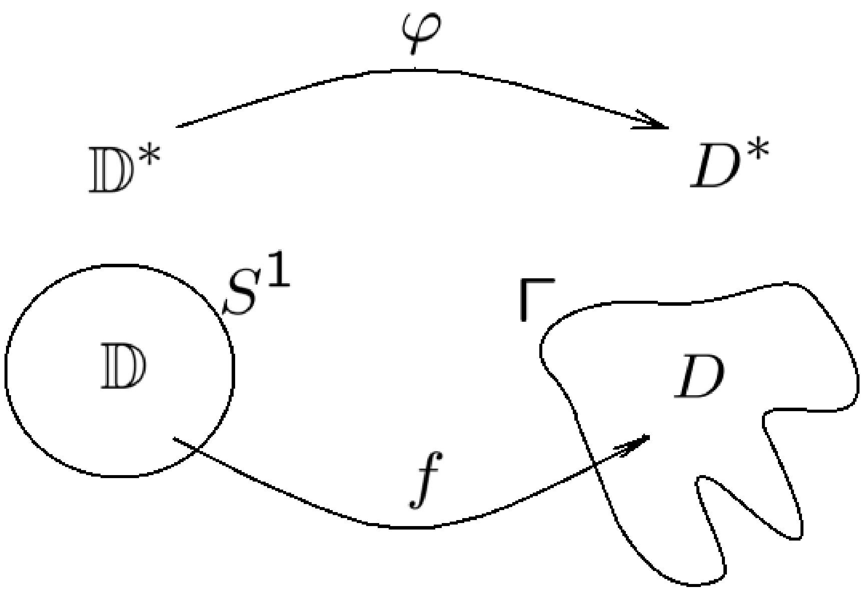
ERLEND GRONG AND ALEXANDER VASIL'EV

1. Matching functions and conformal welding

Definition 1. Suppose that:

- f is a conformal mapping of $\mathbb{D} := \{z : |z| < 1\}$ onto a Jordan domain D;
- φ is a conformal mapping of $\mathbb{D}^* := \overline{\mathbb{C}} \setminus \mathbb{D}$ onto a Jordan domain D^*.

Then the functions f and φ are said to be *matching* if D and D^* are complementary domains, i. e. $D \cap D^* = \emptyset$ and $\Gamma := \partial D = \partial D^*$.



Using fractional-linear change of variables, we can assume that:

(i) $0 \in D$ and $\infty \in D^*$;

(ii) $f(0) = f'(1) - 1 = 0$;

(iii) $\varphi(\infty) = \infty$.

$\mathcal{S} := \{f : \mathbb{D} \to \mathbb{C} : f$ is analytic, univalent, and subject to normalization (ii)$\}$.

Problem 1. Given $f \in \mathcal{S}$ s. t. $f(\mathbb{D})$ is a Jordan domain, find a univalent meromorphic function φ which matches the function f.

A pair of matching functions (f, φ) *defines the homeomorphism of the unit circle* S^1,

$$\gamma = f^{-1} \circ \varphi, \qquad \gamma : S^1 \to S^1. \tag{1}$$

Definition 2. Representation (1) of a homeomorphism $\gamma : S^1 \to S^1$ by means of matching functions is called the *conformal welding*.

Problem 2. Find the conformal welding for a given orientation preserving homeomorphism $\gamma : S^1 \to S^1$, i. e. the pair of matching univalent functions (f, φ) such that $\gamma = f^{-1} \circ \varphi$.

Problem 2 has *a unique solution for all homeomorphisms* γ *that are quasisymmetric, i. e. satisfies*

$$\left| \frac{\gamma\left(e^{i(t+h)}\right) - \gamma\left(e^{it}\right)}{\gamma\left(e^{i(t-h)}\right) - \gamma\left(e^{it}\right)} \right| < C_\gamma < +\infty, \quad \text{for all } t, h \in \mathbb{R},\ 0 < |h| < \pi. \qquad (2)$$

A. Pfluger, 1960;

O. Lehto & K.I. Virtanen, 1960.

Also follows from the Ahlfors – Beurling Extension Theorem,

A. Beurling & L. Ahlfors, 1956

Existence and uniqueness of the conformal welding for the constant C_γ replaced in right-hand side of (2) with $\rho(h) = O(\log h)$,

O. Lehto, 1970; G.L. Jones, 2000.

Denote by:

- $\mathrm{Lip}_\alpha(X,Y)$ the class of all functions $h : X \to Y$ which are Hölder-continuous with exponent α;
- $\mathcal{S}$ the class of all analytic univalent functions $f : \mathbb{D} \to \mathbb{C}$ such that $f(0) = f'(0) - 1 = 0$;
- $\mathcal{S}^{\mathrm{qc}} := \{f \in \mathcal{S} : f$ can be extended to q. c. homeomorphism of $\overline{\mathbb{C}}\}$;
- $\mathcal{S}^{1,\alpha} := \{f \in \mathcal{S} : \partial f(\mathbb{D})$ is a $C^{1,\alpha}$-smooth Jordan curve$\}$, $\alpha \in (0,1)$;
- $\mathcal{S}^{\infty} := \{f \in \mathcal{S} : \partial f(\mathbb{D})$ is a C^{∞}-smooth Jordan curve$\}$;
- $\mathrm{Homeo}^{+}_{\mathrm{qs}}(S^1)$ the group of all orientation preserving q. s. homeomorphisms $\gamma : S^1 \to S^1$.

Remark 1. The conformal welding establishes one-to-one correspondence between $\mathcal{S}^{\mathrm{qc}}$ and $\mathrm{Homeo}^{+}_{\mathrm{qs}}(S^1)/\mathrm{Rot}(S^1)$.

- To calculate $\gamma \in \mathrm{Homeo}^{+}_{\mathrm{qs}}(S^1)$ for given $f \in \mathcal{S}^{\mathrm{qc}}$ one have to solve Problem 1, which is to find the function φ matching f.

- To determine $f \in \mathcal{S}^{\mathrm{qc}}$ for given $\gamma \in \mathrm{Homeo}^{+}_{\mathrm{qs}}(S^1)$ one have to solve the Beltrami equation

$$\bar\partial \tilde f(z) = \mu(z)\,\partial \tilde f(z), \quad \mu(z) := \begin{cases} \bar\partial u(z)/\partial u(z), & \text{if } z \in \mathbb{D}^*, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \bar\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

with the normalization

$$\tilde f(\infty) = \infty \text{ and } \tilde f(0) = \tilde f'(0) - 1 = 0, \tag{4}$$

where u is any q. c. automorphism of $\mathbb{D}^*$ such that

$$u(\infty) = \infty \quad \text{and} \quad u|_{S^1} = \gamma^{-1}.$$

Then

$$f := \tilde f|_{\mathbb{D}}, \quad \varphi := \tilde f|_{\mathbb{D}^*} \circ u^{-1} \tag{5}$$

are matching functions, $f \in \mathcal{S}^{\mathrm{qc}}$, and $\gamma = f^{-1} \circ \varphi$.

2. Main results

The following theorem establishes more explicit relation between f, φ, and γ for the smooth case. For fixed $f \in \mathcal{S}^{1,\alpha}$ define the operator $I_f : \mathrm{Lip}_\alpha(S^1, \mathbb{R}) \to \mathrm{Hol}(\mathbb{D})$ by the formula

$$I_f[v](z) := -\frac{1}{2\pi i} \int_{S^1} \left(\frac{s f'(s)}{f(s)} \right)^2 \frac{v(s)}{f(s) - f(z)} \frac{ds}{s}, \qquad z \in \mathbb{D}. \tag{6}$$

Theorem 1. *Suppose $f \in \mathcal{S}^{1,\alpha}$ and φ, $\varphi(\infty) = \infty$, are matching functions. Then the kernel of the operator $I_f : \mathrm{Lip}_\alpha(S^1, \mathbb{R}) \to \mathrm{Hol}(\mathbb{D})$ is the one-dimensional manifold $\ker I_f = \mathrm{span}\{v_0\}$, where*

$$v_0(z) := \frac{1}{z} \frac{(\psi \circ f)(z)}{f'(z)(\psi' \circ f)(z)}, \qquad \psi := \varphi^{-1}, \qquad z \in S^1. \tag{7}$$

Moreover, the function v_0 is positive on S^1 and satisfies the following condition

$$\int_0^{2\pi} \frac{dt}{v_0(e^{it})} = 2\pi. \tag{8}$$

Remark 2. Theorem 1 reduces Problem 1 to solution of the equation

$$I_f[v] = 0. \tag{9}$$

Indeed, given f and v_0, one can calculate $\psi = \varphi^{-1}$ on the boundary of D^* by solving the following differential equation

$$\begin{aligned} &\psi'(u) = H(u)\psi(u), \quad u \in \partial D^*, \\ &H := \tilde{H} \circ f^{-1}, \quad \tilde{H}(z) := \frac{1}{zf'(z)v_0(z)} \quad \text{for } z \in S^1. \end{aligned} \tag{10}$$

Complex Solutions to $I_f[v] = 0$.

Theorem 2. *Suppose $f \in \mathcal{S}^{1,\alpha}$ and φ, $\varphi(\infty) = \infty$, are matching functions and $\gamma := f^{-1} \circ \varphi$. Then the kernel of the operator $I_f : \mathrm{Lip}_\alpha(S^1, \mathbb{C}) \to \mathrm{Hol}(\mathbb{D})$ is the set of all functions v of the form*

$$v(z) = v_0(z) \cdot (h \circ \gamma^{-1})(z), \quad z \in S^1, \tag{11}$$

where v_0 is defined by (7) and h is an arbitrary holomorphic function in $\mathbb{D}^*$ admitting $\mathrm{Lip}_\alpha(S^1, \mathbb{C})$-extension to S^1.

3. Operator I_f and Kirillov's manifold

By $\mathrm{Diff}^+(S^1)$ denote the Lie – Fréchet group of all orientation preserving C^∞-smooth diffeomorphisms of S^1.

In 1987 A.A. Kirillov proposed to use the 1-to-1 correspondence between $\mathcal{S}^{\mathrm{qc}}$ and $\mathrm{Homeo}^+_{\mathrm{qs}}(S^1)/\mathrm{Rot}(S^1)$ established by conformal welding to represent the homogeneous manifold $\mathcal{M} := \mathrm{Diff}^+(S^1)/\mathrm{Rot}(S^1)$ (Kirillov's manifold) via univalent functions.

The bijection $K : \mathcal{M} \to \mathcal{S}^\infty$ allows to bring the complex structure from $\mathcal{S}^\infty$ to $\mathcal{M}$. A.A. Kirillov proved that *the (left) action of* $\mathrm{Diff}^+(S^1)$ *on* $\mathcal{M}$ *is holomorphic* w.r.t. this complex structure.

The infinitesimal version of $K : \mathcal{M} \to \mathcal{S}^\infty$ is more explicit and expressed by means of $I_f[v]$. Consider the variation of $\gamma \in \mathrm{Diff}^+(S^1)$ given by

$$\gamma_\varepsilon(\zeta) := \gamma(\zeta)\delta\gamma(\zeta), \qquad \delta\gamma := \exp i\varepsilon(v \circ \gamma), \tag{12}$$

where $v\in C^\infty(S^1,\mathbb{R})$ is regarded as an element of $\mathsf{T}_{\mathrm{id}}\mathrm{Diff}^+(S^1)$.

Variation (12) of γ results in the following variation of $f\in\mathcal{S}^\infty$

$$f_\varepsilon := K(\gamma_\varepsilon) = f + \delta f,$$
$$\delta f(z) = \frac{\varepsilon}{2\pi}\int_{S^1}\left(\frac{sf'(s)}{f(s)}\right)^2 \frac{f^2(z)\,v(s)}{f(z)-f(s)}\frac{ds}{s} = i\varepsilon f^2(z) I_f[v](z). \quad (13)$$

Remark 3. A natural consequence of this is that $I_f[v](z)=0$ for all $z\in\mathbb{D}$ if and only if the variation of γ produces no variation of $[\gamma]\in\mathcal{M}$ (up to higher order terms), which can be reformulated as follows: *the element of $\mathsf{T}_\gamma\mathrm{Diff}^+(S^1)$ represented by $v\circ\gamma$ is tangent to the one-dimensional manifold*

$$\gamma\circ\mathrm{Rot}(S^1) = [\gamma] \subset \mathrm{Diff}^+(S^1).$$

The latter is equivalent to

$$v\in \mathrm{Ad}_\gamma\Big(\mathsf{T}_{\mathrm{id}}\mathrm{Rot}(S^1)\Big) = \mathrm{Ad}_\gamma\Big\{\text{constant functions on } S^1\Big\}, \quad (14)$$

where Ad_γ stands for the differential of $\beta \mapsto \gamma \circ \beta \circ \gamma^{-1}$ at $\beta = \mathrm{id}$.

Elementary calculations show that

$$\mathrm{Ad}_\gamma u = \frac{u \circ \gamma^{-1}}{\left(\gamma^{-1}\right)^{\#}}, \quad \beta^{\#} := \left(\pi^{-1} \circ \beta \circ \pi\right)',$$

where $\pi : \mathbb{R} \to S^1$ is the universal covering, $\pi(x) = e^{ix}$.

As a conclusion we get

Proposition 1. *The kernel of* $I_f : C^\infty(S^1, \mathbb{R}) \to \mathrm{Hol}(\mathbb{D})$ *is a one-dimensional manifold and coincides with* $\mathrm{span}\{1/(\gamma^{-1})^{\#}\}$*.*

Remark 4. This Proposition is the special case of Theorem 1 for C^∞-smooth case. It shows that Problem 2 (of finding conformal welding) is reduced by Theorem 1 to finding solution to

$$I_f[1/(\gamma^{-1})^{\#}] = 0, \tag{15}$$

regarded as equation w.r.t. $f \in \mathcal{S}^\infty$.

4. An Example of matching functions

Given an integer $n > 1$, let us consider quadratic differentials

$$\Psi(\zeta)d\zeta^2 := -\frac{d\zeta^2}{\zeta^2};$$

$$W(w)dw^2 := -\frac{w^{n-2}dw^2}{P(w)}, \quad P(w) := \prod_{k=0}^{n-1}(w - w_k), \quad w_k := e^{2\pi ik/n};$$

$$Z(z)dz^2 := -\frac{z^{n-2}dz^2}{Q(z)},$$

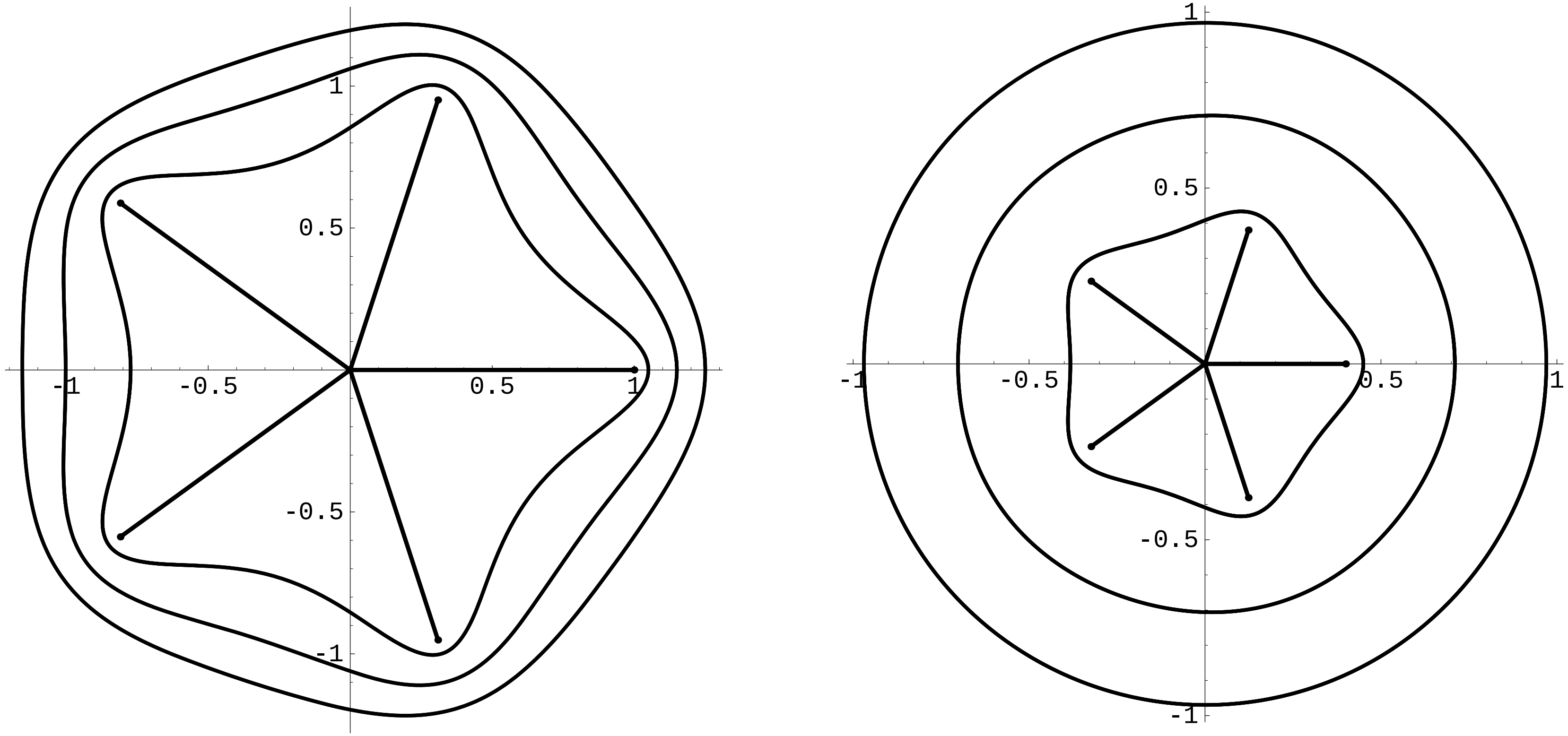
$$Q(z) := \varkappa \prod_{k=0}^{n-1} \frac{|z_k|}{z_k}(z_k - z)(z - 1/\overline{z_k}), \quad z_k := re^{2\pi ik/n}, \quad r \in (0,1).$$

where $\varkappa > 0$ is such that

$$\int_{S^1} \sqrt{Z(z)}dz = 2\pi \tag{16}$$

for the appropriately chosen branch of the square root.

$n = 5$



The structure of trajectories $W(w)dw^2 > 0$ and $Z(z)dz^2 > 0$.

Let Γ be one of the non-singular trajectories of $W(w)dw^2$,
$D \ni 0$ and $D^* \ni \infty$ Jordan domains bounded by Γ.

The corresponding matching $w = f(z)$ and $w = \varphi(\zeta)$ realizing the conformal mappings

$$f : \mathbb{D} \to D, \qquad f(0) = 0, \qquad f'(0) > 0, \qquad \text{and}$$
$$\varphi : \mathbb{D}^* \to D^*, \qquad \varphi(\infty) = \infty, \qquad \varphi'(\infty) > 0,$$

satisfy the following equations (for suitably chosen value of the parameter $r \in (0, 1)$ in quadratic differential $Z(z)dz^2$)

$$W(w)\left(\frac{dw}{dz}\right)^2 = Z(z), \quad W(w)\left(\frac{dw}{d\zeta}\right)^2 = \Psi(\zeta). \tag{17}$$

It follows that

$$v_0(z) = \left(-z^2 Z(z)\right)^{-1/2} = \sqrt{\frac{\varkappa}{r^n}} \prod_{k=0}^{n-1} |z - re^{ikt/n}|, \quad z \in S^1. \tag{18}$$

5. Conformal welding for a class of diffeomorphisms of the unit circle

Consider a diffeomorphism $\gamma : S^1 \to S^1$ such that the function

$$v_0 := (\gamma^{-1})^{\#}, \quad \text{i. e.,} \quad v_0(e^{it}) = \frac{d\gamma^{-1}(e^{it})/dt}{i\gamma^{-1}(e^{it})}, \tag{19}$$

is a Fourier polynomial $v_0(z) := a_0 + \sum_{k=1}^{n}(a_k z^k + \overline{a_k} z^{-k})$.

One can express this Fourier polynomial in the following form

$$v_0(z) = \varkappa \prod_{k=1}^{n} \frac{e^{-it_k}}{z}(r_k e^{it_k} - z)(z - e^{it_k}/r_k), \quad r_k \in (0,1), \quad t_k \in \mathbb{R}, \tag{20}$$

where the coefficient $\varkappa > 0$ is subject to the conditions

$$v_0 > 0, \qquad \int_0^{2\pi} \frac{dt}{v_0(e^{it})} = 2\pi. \tag{21}$$

Proposition 2. *The function $f \in \mathcal{S}^\infty$ that corresponds to γ via conformal welding, satisfies differential equation*

$$\frac{w^{n-1}dw}{P(w)} = \frac{z^{n-1}dz}{Q(z)}, \tag{22}$$

where $P(w) := \prod_{k=1}^{n} (w - w_k),$

$$Q(z) := z^n v_0(z) = \varkappa \prod_{k=1}^{n} \frac{|z_k|}{z_k}(z_k - z)(z - 1/\overline{z_k}), \quad z_k := r_k e^{it_k},$$

and $w_k := f(z_k)$. Moreover, the vector $(w_1, \dots, w_n)$ satisfies system

$$\frac{w_k^{n-1}}{P_k(w_k)} = A_k, \quad P_k(w) := \frac{P(w)}{w - w_k}, \quad A_k := \operatorname*{Res}_{z=z_k} \frac{z^{n-1}}{Q(z)}, \tag{23}$$

$$\prod_{k=1}^{n} w_k = (-1)^n Q(0) = \varkappa \prod_{k=1}^{n} \frac{z_k}{|z_k|}, \tag{24}$$

provided all the roots z_k of Q are simple.