Matching univalent functions

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joint research with

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1. Matching functions and conformal welding

**Definition 1.** Suppose that:

- $f$ is a conformal mapping of $D := \{z : |z| < 1\}$ onto a Jordan domain $D$;
- $\varphi$ is a conformal mapping of $D^* := \mathbb{C} \setminus D$ onto a Jordan domain $D^*$.

Then the functions $f$ and $\varphi$ are said to be **matching** if $D$ and $D^*$ are complementary domains, i.e. $D \cap D^* = \emptyset$ and $\Gamma := \partial D = \partial D^*$. 

![Diagram](image.png)
Using fractional-linear change of variables, we can assume that:

(i) $0 \in D$ and $\infty \in D^*$;
(ii) $f(0) = f'(1) - 1 = 0$;
(iii) $\varphi(\infty) = \infty$.

\[ S := \{ f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic, univalent, and subject to normalization (ii)} \} . \]

**Problem 1.** Given $f \in S$ s. t. $f(\mathbb{D})$ is a Jordan domain, find a univalent meromorphic function $\varphi$ which matches the function $f$.

A pair of matching functions $(f, \varphi)$ defines the homeomorphism of the unit circle $S^1$,

\[ \gamma = f^{-1} \circ \varphi, \quad \gamma : S^1 \to S^1. \]  

**Definition 2.** Representation (1) of a homeomorphism $\gamma : S^1 \to S^1$ by means of matching functions is called the *conformal welding*.
Problem 2. Find the conformal welding for a given orientation preserving homeomorphism $\gamma : S^1 \rightarrow S^1$, i.e. the pair of matching univalent functions $(f, \varphi)$ such that $\gamma = f^{-1} \circ \varphi$.

Problem 2 has a unique solution for all homeomorphisms $\gamma$ that are quasisymmetric, i.e. satisfies

$$\left| \frac{\gamma(e^{i(t+h)}) - \gamma(e^{it})}{\gamma(e^{i(t-h)}) - \gamma(e^{it})} \right| < C_\gamma < +\infty,$$

for all $t, h \in \mathbb{R}$, $0 < |h| < \pi$. (2)

A. Pfluger, 1960;

Also follows from the Ahlfors–Beurling Extension Theorem,
A. Beurling & L. Ahlfors, 1956

Existence and uniqueness of the conformal welding for the constant $C_\gamma$ replaced in right-hand side of (2) with $\rho(h) = O(\log h)$,
Denote by:

- $\text{Lip}_\alpha(X,Y)$ the class of all functions $h : X \to Y$ which are Hölder-continuous with exponent $\alpha$;
- $\mathcal{S}$ the class of all analytic univalent functions $f : \mathbb{D} \to \mathbb{C}$ such that $f(0) = f'(0) - 1 = 0$;
- $\mathcal{S}^{qc} := \{f \in \mathcal{S} : f\text{ can be extended to q. c. homeomorphism of } \overline{\mathbb{C}}\}$;
- $\mathcal{S}^{1,\alpha} := \{f \in \mathcal{S} : \partial f(\mathbb{D}) \text{ is a } C^{1,\alpha}-\text{smooth Jordan curve}\}, \alpha \in (0,1)$;
- $\mathcal{S}^{\infty} := \{f \in \mathcal{S} : \partial f(\mathbb{D}) \text{ is a } C^{\infty}-\text{smooth Jordan curve}\}$;
- $\text{Homeo}^{+}_{qs}(S^1)$ the group of all orientation preserving q. s. homeomorphisms $\gamma : S^1 \to S^1$.

**Remark 1.** The conformal welding establishes one-to-one correspondence between $\mathcal{S}^{qc}$ and $\text{Homeo}^{+}_{qs}(S^1)/\text{Rot}(S^1)$.

- To calculate $\gamma \in \text{Homeo}^{+}_{qs}(S^1)$ for given $f \in \mathcal{S}^{qc}$ one have to solve Problem 1, which is to find the function $\varphi$ matching $f$. 
To determine $f \in \mathcal{S}_{qc}$ for given $\gamma \in \text{Homeo}_{qs}^+(S^1)$ one have to solve the Beltrami equation

$$\overline{\partial} \tilde{f}(z) = \mu(z) \partial \tilde{f}(z), \quad \mu(z) := \begin{cases} \overline{\partial}u(z)/\partial u(z), & \text{if } z \in \mathbb{D}^*, \\ 0, & \text{otherwise}, \end{cases}$$

(3)

with the normalization

$$\tilde{f}(\infty) = \infty \text{ and } \tilde{f}(0) = \tilde{f}'(0) - 1 = 0,$$

(4)

where $u$ is any q.c. automorphism of $\mathbb{D}^*$ such that

$$u(\infty) = \infty \text{ and } u|_{S^1} = \gamma^{-1}.$$

Then

$$f := \tilde{f}|_{\mathbb{D}}, \quad \varphi := \tilde{f}|_{\mathbb{D}^*} \circ u^{-1}$$

(5)

are matching functions, $f \in \mathcal{S}_{qc}$, and $\gamma = f^{-1} \circ \varphi$. 
2. Main results

The following theorem establishes more explicit relation between $f$, $\varphi$, and $\gamma$ for the smooth case. For fixed $f \in S^{1,\alpha}$ define the operator $I_f : \text{Lip}_\alpha(S^1, \mathbb{R}) \to \text{Hol}(\mathbb{D})$ by the formula

$$I_f[v](z) := -\frac{1}{2\pi i} \int_{S^1} \left( \frac{s f'(s)}{f(s)} \right)^2 \frac{v(s)}{f(s) - f(z)} \frac{ds}{s}, \quad z \in \mathbb{D}. \quad (6)$$

**Theorem 1.** Suppose $f \in S^{1,\alpha}$ and $\varphi$, $\varphi(\infty) = \infty$, are matching functions. Then the kernel of the operator $I_f : \text{Lip}_\alpha(S^1, \mathbb{R}) \to \text{Hol}(\mathbb{D})$ is the one-dimensional manifold $\ker I_f = \text{span}\{v_0\}$, where

$$v_0(z) := \frac{1}{z} \frac{(\psi \circ f)(z)}{f'(z)(\psi' \circ f)(z)} , \quad \psi := \varphi^{-1}, \quad z \in S^1. \quad (7)$$

Moreover, the function $v_0$ is positive on $S^1$ and satisfies the following condition

$$\int_0^{2\pi} \frac{dt}{v_0(e^{it})} = 2\pi. \quad (8)$$
Remark 2. Theorem 1 reduces Problem 1 to solution of the equation

\[ I_f[v] = 0. \]  

(9)

Indeed, given \( f \) and \( v_0 \), one can calculate \( \psi = \varphi^{-1} \) on the boundary of \( D^* \) by solving the following differential equation

\[
\psi'(u) = H(u)\psi(u), \quad u \in \partial D^*,
\]

\[
H := \tilde{H} \circ f^{-1}, \quad \tilde{H}(z) := \frac{1}{zf'(z)v_0(z)} \quad \text{for } z \in S^1.
\]  

(10)

Complex Solutions to \( I_f[v] = 0 \).

Theorem 2. Suppose \( f \in S^{1,\alpha} \) and \( \varphi, \varphi(\infty) = \infty \), are matching functions and \( \gamma := f^{-1} \circ \varphi \). Then the kernel of the operator \( I_f : \text{Lip}_\alpha(S^1, \mathbb{C}) \to \text{Hol}(\mathbb{D}) \) is the set of all functions \( v \) of the form

\[
v(z) = v_0(z) \cdot (h \circ \gamma^{-1})(z), \quad z \in S^1,
\]  

(11)

where \( v_0 \) is defined by (7) and \( h \) is an arbitrary holomorphic function in \( \mathbb{D}^* \) admitting \( \text{Lip}_\alpha(S^1, \mathbb{C}) \)-extension to \( S^1 \).
3. Operator $I_f$ and Kirillov’s manifold

By $\operatorname{Diff}^+(S^1)$ denote the Lie–Fréchet group of all orientation preserving $C^\infty$-smooth diffeomorphisms of $S^1$.

In 1987 A.A. Kirillov proposed to use the 1-to-1 correspondence between $S^{qc}$ and $\operatorname{Homeo}^+_q(S^1)/\operatorname{Rot}(S^1)$ established by conformal welding to represent the homogeneous manifold $\mathcal{M} := \operatorname{Diff}^+(S^1)/\operatorname{Rot}(S^1)$ (Kirillov’s manifold) via univalent functions.

The bijection $K : \mathcal{M} \to S^\infty$ allows to bring the complex structure from $S^\infty$ to $\mathcal{M}$. A.A. Kirillov proved that the (left) action of $\operatorname{Diff}^+(S^1)$ on $\mathcal{M}$ is holomorphic w.r.t. this complex structure.

The infinitesimal version of $K : \mathcal{M} \to S^\infty$ is more explicit and expressed by means of $I_f[v]$. Consider the variation of $\gamma \in \operatorname{Diff}^+(S^1)$ given by

$$
\gamma_\varepsilon(\zeta) := \gamma(\zeta)\delta\gamma(\zeta), \quad \delta\gamma := \exp i\varepsilon(v \circ \gamma),
$$

(12)
where $v \in C^\infty(S^1, \mathbb{R})$ is regarded as an element of $T_{id}\text{Diff}^+(S^1)$.

Variation (12) of $\gamma$ results in the following variation of $f \in S^\infty$

$$f_\varepsilon := K(\gamma_\varepsilon) = f + \delta f,$$

$$\delta f(z) = \frac{\varepsilon}{2\pi} \int_{S^1} \left(\frac{sf'(s)}{f(s)}\right)^2 \frac{f^2(z)v(s)}{f(z) - f(s)} \frac{ds}{s} = i\varepsilon f^2(z)I_f[v](z). \quad (13)$$

Remark 3. A natural consequence of this is that $I_f[v](z) = 0$ for all $z \in \mathbb{D}$ if and only if the variation of $\gamma$ produces no variation of $[\gamma] \in \mathcal{M}$ (up to higher order terms), which can be reformulated as follows: *the element of $T_{\gamma}\text{Diff}^+(S^1)$ represented by $v \circ \gamma$ is tangent to the one-dimensional manifold*

$$\gamma \circ \text{Rot}(S^1) = [\gamma] \subset \text{Diff}^+(S^1).$$

The latter is equivalent to

$$v \in \text{Ad}_{\gamma}\left(T_{id}\text{Rot}(S^1)\right) = \text{Ad}_{\gamma}\{\text{constant functions on } S^1\}, \quad (14)$$
where $\text{Ad}_\gamma$ stands for the differential of $\beta \mapsto \gamma \circ \beta \circ \gamma^{-1}$ at $\beta = \text{id}$.

Elementary calculations show that

$$\text{Ad}_\gamma u = \frac{u \circ \gamma^{-1}}{(\gamma^{-1})'}$$

$$\beta' := (\pi^{-1} \circ \beta \circ \pi)'$$

where $\pi : \mathbb{R} \to S^1$ is the universal covering, $\pi(x) = e^{ix}$.

As a conclusion we get

**Proposition 1.** The kernel of $I_f : C^\infty(S^1, \mathbb{R}) \to \text{Hol}(\mathbb{D})$ is a one-dimensional manifold and coincides with $\text{span}\{1/(\gamma^{-1})'\}$.

**Remark 4.** This Proposition is the special case of Theorem 1 for $C^\infty$-smooth case. It shows that Problem 2 (of finding conformal welding) is reduced by Theorem 1 to finding solution to

$$I_f[1/(\gamma^{-1})'] = 0,$$  \hspace{1cm} (15)

regarded as equation w.r.t. $f \in S^\infty$. 

4. An Example of matching functions

Given an integer $n > 1$, let us consider quadratic differentials

$$\psi(\zeta) d\zeta^2 := - \frac{d\zeta^2}{\zeta^2} ;$$

$$W(w) dw^2 := - \frac{w^{n-2} dw^2}{P(w)} , \quad P(w) := \prod_{k=0}^{n-1} (w - w_k) , \quad w_k := e^{2\pi ik/n};$$

$$Z(z) dz^2 := - \frac{z^{n-2} dz^2}{Q(z)} ,$$

$$Q(z) := \kappa \prod_{k=0}^{n-1} \frac{|z_k|}{z_k} (z_k - z)(z - 1/z_k) , \quad z_k := re^{2\pi ik/n} , \quad r \in (0, 1).$$

where $\kappa > 0$ is such that

$$\int_{S^1} \sqrt{Z(z)} dz = 2\pi \quad (16)$$

for the appropriately chosen branch of the square root.
$n = 5$

The structure of trajectories $W(w)dw^2 > 0$ and $Z(z)dz^2 > 0$. 
Let $\Gamma$ be one of the non-singular trajectories of $W(w)dw^2$, $D \ni 0$ and $D^* \ni \infty$ Jordan domains bounded by $\Gamma$.

The corresponding matching $w = f(z)$ and $w = \varphi(\zeta)$ realizing the conformal mappings

$$f : \mathbb{D} \to D, \quad f(0) = 0, \quad f'(0) > 0, \quad \text{and} \quad \varphi : \mathbb{D}^* \to D^*, \quad \varphi(\infty) = \infty, \quad \varphi'(\infty) > 0,$$

satisfy the following equations (for suitably chosen value of the parameter $r \in (0, 1)$ in quadratic differential $Z(z)dz^2$)

$$W(w) \left( \frac{dw}{dz} \right)^2 = Z(z), \quad W(w) \left( \frac{dw}{d\zeta} \right)^2 = \Psi(\zeta). \quad (17)$$

It follows that

$$v_0(z) = (-z^2Z(z))^{-1/2} = \sqrt{\frac{\pi}{rn}} \prod_{k=0}^{n-1} |z - re^{ikt/n}|, \quad z \in S^1. \quad (18)$$
5. Conformal welding for a class of diffeomorphisms of the unit circle

Consider a diffeomorphism $\gamma : S^1 \to S^1$ such that the function

$$v_0 := (\gamma^{-1})^#, \text{ i.e., } v_0(e^{it}) = \frac{d\gamma^{-1}(e^{it})/dt}{i\gamma^{-1}(e^{it})},$$

(19)

is a Fourier polynomial $v_0(z) := a_0 + \sum_{k=1}^{n} (a_k z^k + \overline{a_k} z^{-k})$.

One can express this Fourier polynomial in the following form

$$v_0(z) = \kappa \prod_{k=1}^{n} \frac{e^{-it_k}}{z}(r_k e^{it_k} - z)(z - e^{it_k}/r_k), \quad r_k \in (0, 1), \quad t_k \in \mathbb{R},$$

(20)

where the coefficient $\kappa > 0$ is subject to the conditions

$$v_0 > 0, \quad \int_{0}^{2\pi} \frac{dt}{v_0(e^{it})} = 2\pi.$$  

(21)
Proposition 2. The function $f \in S^\infty$ that corresponds to $\gamma$ via conformal welding, satisfies differential equation

$$\frac{w^{n-1}dw}{P(w)} = \frac{z^{n-1}dz}{Q(z)},$$

(22)

where $P(w) := \prod_{k=1}^{n} (w - w_k)$,

$$Q(z) := z^n v_0(z) = \chi \prod_{k=1}^{n} \frac{|z_k|}{z_k} (z_k - z)(z - 1/z_k), \quad z_k := r_k e^{it_k},$$

and $w_k := f(z_k)$. Moreover, the vector $(w_1, \ldots, w_n)$ satisfies system

$$\frac{w_k^{n-1}}{P_k(w_k)} = A_k, \quad P_k(w) := \frac{P(w)}{w - w_k}, \quad A_k := \text{Res}_{z=z_k} \frac{z^{n-1}}{Q(z)},$$

(23)

$$\prod_{k=1}^{n} w_k = (-1)^n Q(0) = \chi \prod_{k=1}^{n} \frac{z_k}{|z_k|},$$

(24)

provided all the roots $z_k$ of $Q$ are simple.