Doc-course «Complex Analysis and Related Areas»
Workshop on Complex and Harmonic Analysis

Boundary behaviour of one-parameter semigroups and evolution families

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My talk is devoted to the study of the topological semigroup

$$\mathrm{Hol}(\mathbb{D},\mathbb{D}) := \left\{\varphi : \mathbb{D} \to \mathbb{D} \,\middle|\, \varphi \text{ is holomorphic in } \mathbb{D}\right\},$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk.

- the semigroup operation in $\mathrm{Hol}(\mathbb{D},\mathbb{D})$ is the composition $(\varphi,\psi) \mapsto \psi \circ \varphi$, and
- the topology in $\mathrm{Hol}(\mathbb{D},\mathbb{D})$ is induced by the locally uniform convergence in $\mathbb{D}$.

Boundary fixed points

For any $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\mathrm{id}_{\mathbb{D}}\}$ there exists *at most* one fixed point in $\mathbb{D}$ [which follows from the Schwarz Lemma].

However, there can be much more so-called *boundary fixed points*.

Definition

Let $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ and $\sigma \in \mathbb{T} := \partial\mathbb{D}$.

- σ is called a *boundary fixed point (BFP)* if the angular limit

$$\varphi(\sigma) := \angle \lim_{z \to \sigma} \varphi(z) \tag{1}$$

  exists and $\varphi(\sigma) = \sigma$.
- more generally, if the limit (1) exists and $\varphi(\sigma) \in \mathbb{T}$, then σ is called a *contact point* of φ.

Boundary fixed points

It is known that

If σ is a contact point of $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$, then the angular limit

$$\varphi'(\sigma) := \angle \lim_{z \to \sigma} \frac{\varphi(z) - \varphi(\sigma)}{z - \sigma} \quad (2)$$

exists, *finite or infinite.*

It is called the *angular derivative* of φ at σ.

Definition

A contact (or boundary fixed) point σ is said to be *regular*, if the angular derivative $\varphi'(\sigma) \neq \infty$.

In case of a boundary regular fixed point *(BRFP)*,

it is known that $\varphi'(\sigma) > 0$.

Denjoy – Wolff Theorem

Let $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\mathrm{id}_{\mathbb{D}}\}$. Then there exists *exactly one* (boundary) fixed point $\tau \in \overline{\mathbb{D}}$ whose *multiplier* $\lambda := \varphi'(\tau)$ *does not exceed one* in absolute value: $|\lambda| \leqslant 1$. Moreover,

EITHER: φ is an *elliptic automorphism*, *i.e.* $\tau \in \mathbb{D}$, $|\lambda| = 1$, and

$$\varphi = \ell^{-1} \circ \big(z \mapsto \lambda z\big) \circ \ell, \quad \ell(z) := \frac{z - \tau}{1 - \overline{\tau} z}, \ \ell \in \mathrm{M\ddot{o}b}(\mathbb{D}).$$

OR: iterates $\varphi^{\circ n} \longrightarrow \tau$ locally uniformly in $\mathbb{D}$ as $n \to +\infty$.

Definition

The point τ above is called the *Denjoy – Wolff point* of φ.

One-parameter semigroup in $\mathbb{D}$

Definition

A *one-parameter semigroup in* $\mathbb{D}$ is a continuous homomorphism from $\big(\mathbb{R}_{\geqslant 0}, +\big)$ to $\big(\mathrm{Hol}(\mathbb{D}, \mathbb{D}), \circ\big)$. In other words, a *one-parameter semigroup* is a family $(\phi_t)_{t\geqslant 0} \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ such that

(i) $\phi_0 = \mathrm{id}_{\mathbb{D}}$;

(ii) $\phi_{t+s} = \phi_t \circ \phi_s = \phi_s \circ \phi_t$ for any $t, s \geqslant 0$;

(iii) $\phi_t(z) \to z$ as $t \to +0$ for any $z \in \mathbb{D}$.

One-parameter semigroups appear, *e.g.* in:

- iteration theory in $\mathbb{D}$ as *fractional iterates*;
- operator theory in connection with *composition operators*;
- *embedding problem* for time-homogeneous stochastic branching processes.

Fixed points of 1-param. semigroups

In what follows we will assume that

all one-parameter semigroups (ϕ_t) we consider are *not conjugated to a rotation*, *i.e.*, **not** of the form $\phi_t = \ell^{-1} \circ (z \mapsto e^{i\omega t} z) \circ \ell$ for all $t \geqslant 0$, where $\omega \in \mathbb{R}$ and $\ell \in \text{Möb}(\mathbb{D})$.

Theorem (Contreras, Díaz-Madrigal, Pommerenke, 2004)

Let (ϕ_t) be a one-parameter semigroup in $\mathbb{D}$. Then:

- $\sigma \in \overline{\mathbb{D}}$ is a (boundary) fixed point of ϕ_t for some $t > 0$ $\iff$ it is a (boundary) fixed point of ϕ_t for all $t > 0$;
- $\sigma \in \mathbb{T}$ is a boundary regular fixed point of ϕ_t for some $t > 0$ $\iff$ it is a boundary regular fixed point of ϕ_t for all $t > 0$;
- all ϕ_t's, $t > 0$, share the same Denjoy – Wolff point.

Hence we can define in an obvious way the *DW-point* of a one-parameter semigroup, its *boundary fixed points*, and its *BRFPs*.

Angular and unrestricted limits

Some philosophy...

Not every element of $\mathrm{Hol}(\mathbb{D},\mathbb{D})$ can be embedded into a one-parameter semigroup. Elements of one-parameter semigroups enjoy some very specific nice properties.

For example, these functions are *univalent* (=injective).

But especially brightly this shows up in boundary behaviour.

Theorem 1 (Contreras, Díaz-Madrigal, Pommerenke, 2004; **P. Gum.**, 2012)

Let (ϕ_t) be a one-parameter semigroup in $\mathbb{D}$. Then:

(i) for all $t \geqslant 0$ and **every** $\sigma \in \mathbb{T}$ there exists the angular limit

$$\phi_t(\sigma) := \angle \lim_{z\to\sigma} \phi_t(z).$$

Theorem 1 — continued

(ii) moreover, for each $\sigma \in \mathbb{T}$ and each Stolz angle S at σ the convergence $\phi_t(z) \to \phi_t(\sigma)$ as $S \ni z \to \sigma$ is locally uniform in $t \in [0, +\infty)$;

(iii) the family of functions (*"trajectories"*)

$$\Big\{[0, +\infty) \ni t \mapsto \phi_t(z) : z \in \overline{\mathbb{D}}\Big\}$$

is uniformly equicontinuous.

Remark

However, the unrestricted limits

$$\lim_{\mathbb{D} \ni z \to \sigma} \phi_t(z), \quad \sigma \in \mathbb{T},$$

do NOT need to exist. Hence ϕ_t's can be discontinuous on $\mathbb{T}$.

Angular and unrestricted limits

So unrestricted limits of ϕ_t do not need to exists everywhere on $\mathbb{T}$. BUT they have to exists *at every boundary* *fixed* *point* of (ϕ_t):

Theorem 2 (Contreras, Díaz-Madrigal, Pommerenke, 2004; P. Gum., 2012)

Let (ϕ_t) be a one-parameter semigroup in $\mathbb{D}$ and $\sigma \in \mathbb{T}$ its boundary fixed point. Then:

(UnrLim) for any $t \geqslant 0$ there exists the unrestricted limit

$$\lim_{\mathbb{D} \ni z \to \sigma} \phi_t(z) \quad [\text{clearly} = \sigma],$$

(EqCont) for each $T > 0$ the family of mappings

$$\Phi_T := \left\{ \overline{\mathbb{D}} \ni z \mapsto \phi_t(z) \in \overline{\mathbb{D}} : t \in [0, T] \right\}$$

is equicontinuous at the point σ.

Some remarks on Theorem 2.

- ☞ Contreras, Díaz-Madrigal, and Pommerenke proved (UnrLim) for the case of the DW-point $\tau \in \mathbb{D}$.
- ☞ For the case of $\tau \in \mathbb{T} := \partial\mathbb{D}$:
  - ☺ the method of C. – D.-M. – P. works for boundary fixed points $\sigma \in \mathbb{T} \setminus \{\tau\}$,
  - ☹ but it fails for $\sigma = \tau$.
- ☞ In all the cases the so-called *linearization model* is used.

Linearization model

We restrict ourselves to the case of the DW-point $\tau \in \mathbb{T} := \partial\mathbb{D}$.

Theorem

Let (ϕ_t) be a one-parameter semigroup in $\mathbb{D}$ with the DW-point $\tau \in \mathbb{T}$. Then there exists an essentially unique univalent holomorphic function $h : \mathbb{D} \to \mathbb{C}$, called the *Kœnigs function* of (ϕ_t) such that

$$h \circ \phi_t = h + t, \quad \forall\, t \geqslant 0 \quad \text{(Abel's equation)}.$$

- ☞ at every boundary fixed point $\sigma \in \mathbb{T} \setminus \{\tau\}$, the Kœnigs function h has the *unrestricted limit*;
- ☞ at the DW-point, the Kœnigs function h does *NOT* need to have the unrestricted limit.

Theorem

For any one-parameter semigroup (ϕ_t) the limit

$$G(z) := \lim_{t \to +0} \frac{\phi_t(z) - z}{t}, \quad z \in \mathbb{D}, \tag{3}$$

exists and G is a holomorphic function in $\mathbb{D}$.
Moreover, for each $z \in \mathbb{D}$, the function $[0, \infty) \ni t \mapsto w(t) := \phi_t(z) \in \mathbb{D}$ is the unique solution to the IVP

$$\frac{dw(t)}{dt} = G\big(w(t)\big), \quad t \geqslant 0, \quad w(0) = z. \tag{4}$$

Definition

The function G above is called the *infinitesimal generator* of (ϕ_t).

Herglotz vector fields

There is a non-autonomous analogue of the equation

$$\frac{dw(t)}{dt} = G\big(w(t)\big).$$

Definition (Bracci, Contreras, Díaz-Madrigal, 2008)

A function $G : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is said to be a *Herglotz vector field* of order $d \in [1, +\infty]$, if:

(i) for a.e. $t \geqslant 0$ fixed, the function $G(\cdot, t)$ is an infinitesimal generator of some one-parameter semigroup in $\mathbb{D}$;

(ii) for each $z \in \mathbb{D}$ fixed, the function $G(z, \cdot)$ is measurable on $[0, +\infty)$;

(iii) for each compact set $K \subset \mathbb{D}$ there exists a non-negative function $k_K \in L^d_{\text{loc}}\big([0, +\infty)\big)$ such that

$$\sup_{z \in K} |G(z, t)| \leqslant k_K(t) \quad \text{for a.e. } t \geqslant 0.$$

Theorem (Bracci, Contreras, Díaz-Madrigal, 2008)

Let G be a Herglotz vector field of order d. Then for any initial data $s \geqslant 0$, $z \in \mathbb{D}$, the IVP for the *generalized Loewner equation*

$$\frac{dw(t)}{dt} = G\big(w(t), t\big), \quad t \geqslant s, \quad w(s) = z, \tag{5}$$

has a unique solution $w_{z,s} : [s, +\infty) \to \mathbb{D}$.

Evolution family

Fix any $s \geqslant 0$ and any $t \geqslant s$. Then the map

$$\mathbb{D} \ni z \mapsto \varphi_{s,t}(z) := w_{z,s}(t) \in \mathbb{D}$$

belongs to $\mathrm{Hol}(\mathbb{D}, \mathbb{D})$. The family $(\varphi_{s,t})_{0 \leqslant s \leqslant t}$ is called the *evolution family* (of the Herglotz vector field G.)

This is a non-autonom. generalization of one-parameter semigroups.

Similar to one-parameter semigroups, evolution families can be defined *without appeal to differential equations.*

Definition (Bracci, Contreras, Díaz-Madrigal, 2008)

A family $(\varphi_{s,t})_{0\leqslant s\leqslant t} \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$ is an *evolution family* of order $d \in [1,+\infty]$ if

EF1 $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$ for all $s \geqslant 0$;

EF2 $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $0 \leqslant s \leqslant u \leqslant t$;

EF3 for any $z \in \mathbb{D}$ there exists a non-negative function $k_z \in L^d_{\mathrm{loc}}([0,+\infty))$ such that

$$\left|\varphi_{s,u}(z) - \varphi_{s,t}(z)\right| \leqslant \int_u^t k_z(\xi)d\xi, \quad 0 \leqslant s \leqslant u \leqslant t. \tag{6}$$

The problem

A point $\sigma \in \mathbb{T}$ is said to be a *boundary regular fixed point (BRFP)* of $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$, if

$$\exists\, \varphi(\sigma) := \angle \lim_{z\to\sigma} \varphi(z) = \sigma, \quad \exists\, \varphi'(\sigma) := \angle \lim_{z\to\sigma} \frac{\varphi(z)-\varphi(\sigma)}{z-\sigma} \in \mathbb{C}.$$

Theorem A (Contreras, Díaz-Madrigal, Pommerenke, 2006)

Let (ϕ_t) be a one-parameter semigroup in $\mathbb{D}$, G its infinitesimal generator, and $\sigma \in \mathbb{T}$. Then the following conditions are "$\Longleftrightarrow$":

(i) the point σ is a BRFP of (ϕ_t) for some (and hence all) $t > 0$;

(ii) there exists finite limit $\lambda := \angle \lim_{z\to\sigma} G(z)/(z-\sigma)$. (7)

Moreover, if these conditions hold, then $\lambda \in \mathbb{R}$ and $\phi_t'(\sigma) = e^{\lambda t}$.

Problem

Does a generalization of this theorem holds for evolution families?

There has been known the following result in this direction.

Theorem B (Bracci, Contreras, Díaz-Madrigal, 2008)

Let $(\varphi_{s,t})$ be an evolution family of order d and G its Herglotz vector field. Then the following conditions are "$\Longleftrightarrow$":

(i) all $\varphi_{s,t}$'s that are $\neq \mathrm{id}_{\mathbb{D}}$ share the same DW-point $\tau_0 \in \mathbb{T}$;

(ii) for a.e. $t \geqslant 0$, $G(\cdot, t)$ has a BRNP at τ_0, *i.e.* there exists **finite**

$$G'(\tau_0, t) := \angle \lim_{z \to \tau_0} \frac{G(z,t)}{z - \tau_0} =: \lambda(t) \in (-\infty, 0]; \tag{8}$$

Moreover, if (i) and (ii) hold, then:

(iii) the function λ is of class L^d_{loc} on $[0, +\infty)$;

(iv) $\varphi'_{s,t}(\tau_0) = \exp \int_s^t \lambda(t')\, dt'$, whenever $0 \leqslant s \leqslant t$.

Theorem 3 (Bracci, Contreras, Díaz-Madrigal, P. Gum.)

Let $(\varphi_{s,t})$ be an evolution family, G its Herglotz vector field and $\sigma \in \mathbb{T}$. Then the following two assertions are "$\Longleftrightarrow$":

(i) σ is a BRFP of $\varphi_{s,t}$ for each $s \geqslant 0$ and $t \geqslant s$;

(ii) the following two conditions hold:

(ii.1) for a.e. $t \geqslant 0$, $G(\cdot, t)$ has a BRNP at σ, *i.e.* there exists

$$G'(\sigma, t) := \angle \lim_{z \to \sigma} \frac{G(z,t)}{z - \sigma} =: \lambda(t) \neq \infty; \tag{9}$$

(ii.2) the function λ is of class L^1_{loc} on $[0, +\infty)$.

Moreover, if the assertions above hold, then $\lambda(t) \in \mathbb{R}$ and

$$\varphi'_{s,t}(\sigma) = \exp \int_s^t \lambda(t')\, dt' \quad \text{whenever } 0 \leqslant s \leqslant t. \tag{10}$$

Remarks on Theorem 3

☞ Asymmetry in Theorem 3:

σ is a BRFP of all $\varphi_{s,t}$'s $\Rightarrow$ $\varphi'_{s,t}(\sigma)$ is loc. abs-ly continuous in s and t

σ is a BRNP of $G(\cdot, t)$ for a.e. $t \geqslant 0$ $\not\Rightarrow$ $t \mapsto G'(\sigma, t)$ is loc. integrable

☞ Comparison with Theorem B:

if σ is the DW-point of every $\varphi_{s,t}$, then $t \mapsto G'(\sigma, t)$ is of class $L^d_{\rm loc}$, while for the common BRFP σ, we only have $L^1_{\rm loc}$.

Regular contact points

Definition

A point $\sigma \in \mathbb{T}$ is said to be a *regular contact point* of an evolution family $(\varphi_{s,t})$ if it is a regular contact point of $\varphi_{0,t}$ for all $t \geqslant 0$,

i.e., for all $t \geqslant 0$,

$$\exists\, \varphi_{0,t}(\sigma) := \angle \lim_{z \to \sigma} \varphi_{0,t}(z) \in \mathbb{T} \qquad \text{and}$$

$$\varphi'_{0,t}(\sigma) := \angle \lim_{z \to \sigma} \frac{\varphi_{0,t}(z) - \varphi_{0,t}(\sigma)}{z - \sigma} \in \mathbb{C}.$$

We studied regular contact points of evolution families and obtain a partial analogue of Theorem 3.

Regular contact points

Theorem 4 (Bracci, Contreras, Díaz-Madrigal, P. Gum.)

Let $(\varphi_{s,t})$ be an evolution family, G its Herglotz vector field.
Suppose $\sigma \in \mathbb{T}$ is a regular contact point of $(\varphi_{s,t})$.

Then for any $t \geqslant 0$,

$$\varphi_{0,t}(\sigma) = \sigma + \int_0^t G\big(\varphi_{0,s}(\sigma), s\big)\, ds \qquad \text{and}$$

$$\varphi'_{0,t}(\sigma) = \exp \int_0^t G'\big(\varphi_{0,s}(\sigma), s\big)\, ds.$$


THANK YOU !!!