The 3rd International Conference on Applied Mathematics and Informatics

Dedicated to memory of Alexander Vasil'ev

Value regions of univalent self-maps with two boundary fixed points

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Introduction

Consider a holomorphic self-map $\varphi : \mathbb{D} \to \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

Definition

A point $\sigma \in \partial\mathbb{D}$ is called a *boundary regular fixed point* (BRFP) of φ if $\angle \lim_{z\to\sigma} \varphi(z)$ exists and equals σ, and if the *angular derivative* of φ at σ,

$$\varphi'(\sigma) := \angle \lim_{z\to\sigma} \frac{\varphi(z) - \sigma}{z - \sigma} \quad \text{is finite.}$$

REMARK: for every BRFP σ, $\varphi'(\sigma) > 0$.

Denjoy – Wolff point

- ☞ If $\varphi \neq \mathrm{id}_{\mathbb{D}}$ has a (unique) fixed point $\tau \in \mathbb{D}$, then we call τ the *Denjoy – Wolff point* (DW-point).
- ☞ If φ has no fixed point in $\mathbb{D}$, then by the Denjoy – Wolff Theorem, $\exists!$ BRFP $\tau \in \partial\mathbb{D}$ with $\varphi'(\tau) \leqslant 1$, called the *DW-point* of φ.

Introduction 2

Dynamical meaning of the DW-point

☞ If $\varphi \neq \mathrm{id}_{\mathbb{D}}$ and it is not an elliptic automorphism of $\mathbb{D}$, then τ is the attracting fixed point, i.e. $\varphi^{\circ n} := \varphi \circ \ldots \circ \varphi$ (n times) $\to \tau$ as $n \to +\infty$.

☞ All BRFPs $\sigma \in \partial\mathbb{D} \setminus \{\tau\}$ are repelling, i.e. $\varphi'(\sigma) > 1$.

We mostly will consider **univalent** (= holomorphic + injective) $\varphi : \mathbb{D} \to \mathbb{D}$ with given BRFPs.

- H. Unkelbach, 1938, 1940
- C.C. Cowen and Chr. Pommerenke, 1982
- Chr. Pommerenke and A. Vasil'ev, 2001, 2002
- A. Vasil'ev, 2002
- M.D. Contreras, S. Díaz-Madrigal, and A. Vasil'ev, 2007
- J.M. Anderson and A. Vasil'ev, 2008
- A. Frolova, M. Levenshtein, D. Shoikhet, and A. Vasil'ev, 2014
- V. Goryainov, 1991, 2015, 2017

Main results

Conf. map $\ell:\mathbb{D}\to\mathbb{S}:=\{\zeta\colon\ -\pi/2<\operatorname{Im}\zeta<\pi/2\}$; $z\mapsto\log\dfrac{1+z}{1-z}$; $\pm1\mapsto\pm\infty$.

Joint work with Prof. Dmitri Prokhorov:

Theorem (to appear in *Ann. Acad. Sci. Fenn. Math.* **43** (2018))

Fix $T>0$, $z_0\in\mathbb{D}$. The value region $\mathcal{V}(z_0,T)$ of $\varphi\mapsto\varphi(z_0)$ over the class of all univalent self-maps $\varphi:\mathbb{D}\to\mathbb{D}$ having:

(i) *the DW-point $\tau=1$* and **(ii)** *a BRFP $\sigma=-1$ with $\varphi'(\sigma)=e^T$,*

is a closed Jordan domain with the boundary point z_0 excluded.

✔ *More precisely,* $\ell(\mathcal{V}(z_0,T)\cup\{z_0\})=$

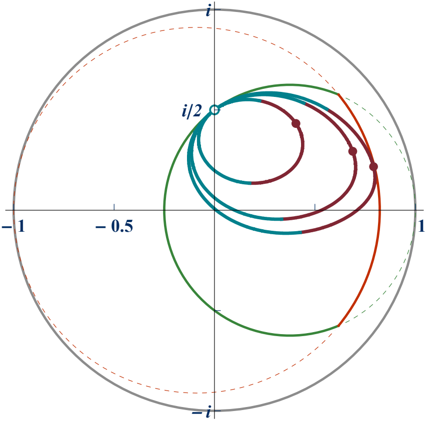
$$\Big\{x+iy\in\mathbb{S}\colon\, y_T^1\leqslant y\leqslant y_T^2,\,\Big|x-\tfrac{T}{2}-\operatorname{Re}\ell(z_0)\Big|\leqslant F_T(y)\Big\},$$

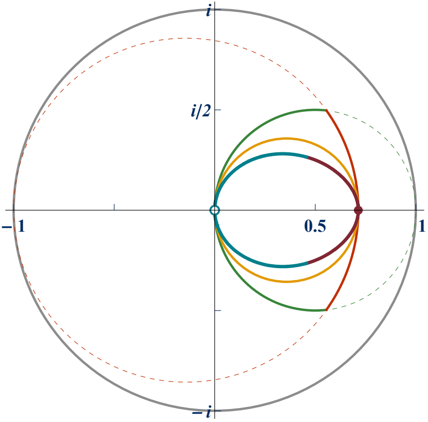
where y_T^1, y_T^2, and $F_T(\cdot)$ are given explicitly.

✔ *Furthermore, every boundary point $\omega\in\partial\mathcal{V}(z_0,T)\setminus\{z_0\}$ is delivered by a unique $\varphi=\varphi_\omega$, which is a hyperbolic automorphism if $\omega=\ell^{-1}(\ell(z_0)+T)$ and a parabolic one-slit map otherwise.*

Main results 2

On the left: $z_0 := i/2$, $T \in \{\log 2, \log 4, \log 6\}$. On the right: $z_0 := 0$, $T := \log 6$,
(and the value region without taking into account univalence).





$\partial\mathbb{D}$ $\partial\mathcal{V}(z_0, T)$ $\circ\ z_0$ $\bullet\ \ell^{-1}(\ell(z_0) + T)$

Main results 3 — Methods

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Corollary (Theorem 6 in [Frolova, Levenshtein, Shoikhet, Vasil'ev, 2014])

For any univalent self-map $\varphi : \mathbb{D} \to \mathbb{D}$ *with BRFPs at* ± 1,

$$\sqrt{\varphi'(-1)\varphi'(1)} \geqslant \Phi(\operatorname{Im} \ell(z_0), \operatorname{Im} \ell(\varphi(z_0))), \tag{1}$$

where $\Phi(\alpha, \beta) := \max\left\{\frac{1+\sin\alpha}{1+\sin\beta}, \frac{1-\sin\alpha}{1-\sin\beta}\right\}$. *Estimate* (1) *is sharp.*

Methods

- ✔ Cowen – Pommerenke inequalities for univalent self-maps: Grunsky-type inequalities;
- ✔ A. Vasil'ev: (a specific form of the) Extremal Length Method;
- ✔ V. Goryanov: a Loewner-type Parametric Representation (for the case of $\tau = 0$ and one BRFP).

Loewner-type Parametric Representation

Embed φ into a family $\varphi_t:\mathbb{D}\to\mathbb{D}$, $t\in[0,1]$, $\varphi_0=\mathrm{id}_{\mathbb{D}}$, $\varphi_1=\varphi$, s.t.

$$\mathrm{d}\varphi_t(z)/\mathrm{d}t = \big(\tau-\varphi_t(z)\big)\big(1-\overline{\tau}\varphi_t(z)\big)\,p(\varphi_t(z),t), \qquad \text{(LK)}$$

where $\operatorname{Re} p\geqslant 0$ plus some other conditions.

PGum, 2017, 2018: Loewner-type Parametric Representation for φ's with *any finite* number of BRFPs and any position of the DW-point.

This gives a new proof of a classical Cowen – Pommerenke inequality [joint work in progress with M.D. Contreras and S. Díaz-Madrigal]:

The value region of $\log\varphi'(0)$ over all univalent $\varphi:\mathbb{D}\to\mathbb{D}$, $\varphi(0)=0$, with BRFPs $\sigma_j,\ldots,\sigma_n$ with prescribed values of $\varphi'(\sigma_j)$'s is described

by the inequality $$\operatorname{Re}\left(-\frac{1}{\log\varphi'(0)}\right)\geqslant\frac{1}{2}\sum_{j=1}^{n}\frac{1}{\log\varphi'(\sigma_j)}. \qquad \text{(CP)}$$