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THE FATOU AND JULIA SETS OF MULTIVALUED ANALYTIC FUNCTIONS

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Abstract: We propose a generalization of some problems of complex dynamics which includes the study of iterations of multivalued functions and compositions of various single-valued functions. We generalize two classical results concerning the Julia set.

Keywords: complex dynamics, iteration, composition, multivalued function, Fatou set, Julia set

Introduction

This article is devoted to studying iterations of multivalued functions. This problem for single-valued functions has been studied in complex dynamics for a long time. We give several traditional definitions. Suppose that $\Delta \in \{\overline{\mathbb{C}}, \mathbb{C}, \mathbb{C}^* = \mathbb{C}\backslash\{0\}\}$ and $g : \Delta \to \Delta$ is a nonconstant single-valued analytic function which is not an automorphism of Δ. Denote by $\mathbb{N}$ the set of positive integers.

Definition 1. The *nth iteration* of g, $n \in \mathbb{N}$, is the function g^n defined recurrently by the relations $g^1 = g$ and $g^n = g \circ g^{n-1}$.

Definition 2. A point $z_0 \in \Delta$ is a *periodic point* of g if $\exists\, n \in \mathbb{N}$ $g^n(z_0) = z_0$. If, in addition, $|(g^n)'(z_0)| > 1$ then z_0 is called a *repelling* periodic point.

Definition 3. The *Fatou set* $\mathscr{F}(g)$ of g is the set of all points $z \in \Delta$ at which the sequence $\{g^n\}_{n\in\mathbb{N}}$ is a normal family. The complement of the Fatou set is called the *Julia set* $\mathscr{J}(g) = \overline{\mathbb{C}}\backslash\mathscr{F}(g)$.

For details we refer to the monographs [1, 2] on complex dynamics.

We give two classical results concerning the Julia set (see, for example, the survey [3, Theorems 26 and 33] and [1, 2]).

Theorem A. *The Julia set has no isolated points and is nonempty.*

Theorem B. *The Julia set coincides with the closure of the set of all repelling periodic points.*

Introduce some notations. Let T be a domain of the complex sphere $\overline{\mathbb{C}}$. Denote by $\mathscr{S}(T)$ the set of all simply connected domains $D \subset T$. Denote the set of all single-valued analytic functions $f : D \to U$ by $\mathscr{H}(D, U)$ and put $\mathscr{H}(D) := \mathscr{H}(D, \overline{\mathbb{C}})$. Denote the closure of a set $E \subset \overline{\mathbb{C}}$ by $\overline{E}$. Also, put notation $B(z, r) = R_z(\{\xi : |\xi| < r\})$, $z \in \overline{\mathbb{C}}$, where $R_z(\xi) = (\xi + z)/(1 - \xi\bar{z})$, $z \neq \infty$, and $R_\infty(\xi) = 1/\xi$. Note that the exterior of $B(z, r)$ coincides with $B(-1/\bar{z}, 1/r)$.

We recall some assertions of the theory of normal families which are important for complex dynamics (see, for example, [4, pp. 67–75]).

Theorem C. *For a family $F \subset \mathscr{H}(D)$ to be normal in D, it is necessary and sufficient that F is normal at each point $z \in D$.*

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Theorem D (Montel's criterion). *For a family $F \subset \mathscr{H}(D)$ of functions to be normal in D, it is sufficient that*

$$\operatorname{Card}\Big(\overline{\mathbb{C}} \setminus \bigcup_{h\in F} h(D)\Big) > 2.$$

REMARK 1. We can restate Montel's criterion in more general form as follows: if there are pairwise disjoint compact sets $K_j \subset \overline{\mathbb{C}}$, $j = 1, 2, 3$, such that $K_j \not\subset h(D)$ for any $j = 1, 2, 3$ and $h \in F$, then the family F is normal in D.

As shows the example of the multivalued function $f(z) = e^{\sqrt{z}}$, iterations are defined nonuniquely: $f^2(z) = \exp(e^{\sqrt{z}/2})$ or $f^2(z) = \exp(-e^{\sqrt{z}/2})$ depending on the choice of the branch of the root. This suggests that we have to extend the class of objects to be iterated. To this end, we introduce the following notion, called an *analytic relation* in this paper:

DEFINITION 4. A set $f \subset \{(D, \varphi) : D \in \mathscr{S}(T),\ \varphi \in \mathscr{H}(D)\}$ is an *analytic relation in a domain* $T \subset \overline{\mathbb{C}}$ if the following conditions are satisfied:

(a) $\operatorname{pr}_1 f = \{D : \exists\varphi\ (D, \varphi) \in f\} = \mathscr{S}(T)$;

(b) for arbitrary $D_1, D_2 \in \mathscr{S}(T)$, φ_1, $(D_1, \varphi_1) \in f$, and every curve $\gamma \subset T$ joining points z_1 and z_2, $z_1 \in D_1$, $z_2 \in D_2$, there is a continuation φ_2 of φ_1 along γ on D_2 such that $(D_2, \varphi_2) \in f$.

A function φ such that $(D, \varphi) \in f$ is called a *branch* of f in D. By the *image of a point* $z \in T$ under f we mean $\varphi(z)$, where φ is one of the branches of f in some simply connected neighborhood $B(z, \varepsilon) \subset T$. By the *image of a set* $A \subset T$ we mean the union

$$f(A) = \bigcup_{(D,\varphi)\in f} \varphi(A \cap D).$$

If $f(T) \subset U$ for some set U then we write $f : T \to U$.

Although complex dynamics deals among other cases with the case in which a function takes values corresponding to its singular points (for example, for transcendental meromorphic functions), in this article we suppose that all values lie in a domain containing no singular points; i.e., $f : T \to T$. Moreover, we adopt the natural assumption that the analytic relations under consideration have no constant branches.

DEFINITION 5. Let $T \subset \overline{\mathbb{C}}$ be a domain and let $f : T \to T$ be an analytic relation without constant branches. Suppose that sequences $\{D_n \in \mathscr{S}(T)\}_{n=0}^{+\infty}$, $\{z_n \in D_n\}_{n=0}^{+\infty}$, and $\{\varphi_n \in \mathscr{H}(D_n)\}_{n=0}^{+\infty}$ are such that $(D_n, \varphi_n) \in f$ and $z_{n+1} = \varphi_n(z_n)$, $n = 0, 1, \dots$. Then, for every $k \in \mathbb{N}$, there is a neighborhood U_k of z_0 in which the composite function $F_k = \varphi_{k-1} \circ \dots \circ \varphi_0$ is well defined. According to the definition of an analytic relation and the monodromy theorem (see, for example, [5, p. 127]), we can extend F_k to the whole domain D_0. The sequence $\{F_k\}_{k\in\mathbb{N}}$ is called an *iteration sequence* of f at z_0 over D_0.

REMARK 2. The definition of the iteration sequence F_k is independent of the choice of D_0 for a given z_0 in the sense that for each $G \in \mathscr{S}(T)$, $z_0 \in G$, the functions F_k, extended from D_0 to G through the connected component of the intersection $D_0 \cap G$ containing z_0, constitute an iteration sequence over G.

On the other hand, for a given D_0 this definition is invariant under the choice of $z_0 \in D_0$.

DEFINITION 6. Let $\mathscr{I}(D)$, $D \in \mathscr{S}(T)$, be the set of all iteration sequences of an analytic relation $f : T \to T$ over D. Then the *nth iteration* of f is the analytic relation f^n whose set of branches in D is $\{F_n : \{F_k\}_{k\in\mathbb{N}} \in \mathscr{I}(D)\}$ for each $D \in \mathscr{S}(T)$.

REMARK 3. Let F be some family of single-valued analytic functions $g : T \to T$. The set $f = A(F) := \{(D, g) : D \in \mathscr{S}(T),\ g \in F\}$ is an analytic relation in T; moreover, $f^n = A(F^n)$, where

$$F^n = \{g_n \circ \dots \circ g_1 : g_k \in F,\ k = 1, \dots, n\},$$

and the iteration sequences are exactly the sequences of the form $F_n = g_n \circ \dots \circ g_1$, where $\{g_k\}_{k\in\mathbb{N}}$ is an arbitrary sequence of functions in F.

Thus, studying infinite compositions of analytic functions taking a domain into itself is a particular instance of studying iterations of analytic relations.

In view of Remark 2, we can henceforth omit the indication of the domain over which an iteration sequence is defined. We denote by $\Phi(f, z_0)$ the set of all terms of all iteration sequences of an analytic relation f at z_0.

Definition 7. We say that the *iterations of an analytic relation $f : T \to T$ are normal at $z_0 \in T$* if there is a neighborhood of z_0 in which each iteration sequence of f at z_0 is a normal family. In this case we also say that z_0 is a *normal iteration point* of f.

Remark 4. The notion of normality can be defined for an arbitrary family F of analytic relations. Say that such an F is normal at z_0 if there is a simply connected domain $D \ni z_0$ in which the family of all branches of all analytic relations $h \in F$ in D is normal.

According to this definition, normality of the family of all iterations $\{f^n : n \in \mathbb{N}\}$ of an analytic relation $f : T \to T$ at a point $z_0 \in T$ means normality of the family $\Phi(f, z_0)$ at z_0.

The following proposition shows that Definition 7 and the definition of Remark 4 are equivalent under rather general additional conditions.

Proposition 1. *Let $f : \Delta \to \Delta$, $\Delta \in \{\overline{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*\}$, be an analytic relation. In the case of $\Delta = \overline{\mathbb{C}}$ we also require that f has a branch which is not a linear-fractional function. If $z_0 \in T$ is a normal iteration point of f then the family $\Phi(f, z_0)$ is normal at z_0.*

Remark 5. In the case when f is a single-valued function the set of all normal iteration points of f is the Fatou set and its complement is the Julia set. Therefore, it is worth giving the following

Definition 8. Let $f : T \to T$, $T \in \{\overline{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*\}$, be an analytic relation. The *Fatou set* $\mathscr{F}(f)$ of f is the set of all normal iteration points of f. The set $\mathscr{J}(f) = \overline{T \backslash \mathscr{F}(f)}$ is called the *Julia set*.

Remark 6. The definition of a normal point implies that the Fatou set is open.

The following proposition is an analog of Theorem A for analytic relations:

Proposition 2. *If* $\operatorname{Card}(\overline{\mathbb{C}} \backslash \mathscr{F}(f)) > 2$ *then $\mathscr{J}(f)$ has no isolated points.*

Definition 9. A point $z_0 \in T$ is a *periodic point* of an analytic relation $f : T \to T$ if there is $g \in \Phi(f, z_0)$ such that $g(z_0) = z_0$. A point z_0 is a *repelling periodic point* if there is $g \in \Phi(f, z_0)$ such that $g(z_0) = z_0$ and $|g'(z_0)| > 1$.

The following theorem generalizes Theorem B:

Theorem 1. *Let $f : T \to T$, $T \in \{\overline{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*\}$, be an analytic relation. In the case of $T = \overline{\mathbb{C}}$ we also require that f has no branches that are linear-fractional functions. Denote by R the set of all repelling periodic points of f. Then $\mathscr{J}(f)$ coincides with $\overline{R}$.*

Basic Lemmas

Lemma 1. *Let h be a nonconstant single-valued analytic function in some neighborhood of a point z_0. For a family of single-valued analytic functions Ψ to be normal at $h(z_0)$, it is necessary and sufficient that the family $\Psi \circ h := \{g \circ h : g \in \Psi\}$ be normal at z_0.*

Proof. Necessity is clear. Prove sufficiency. Since $\Psi \circ h$ is normal at z_0, there is $\varepsilon > 0$ such that each sequence of functions in the family $\Psi \circ h$ has a subsequence uniformly convergent in $W = B(z_0, \varepsilon)$.

By Montel's criterion, it suffices to demonstrate that there is a neighborhood U of $w_0 = h(z_0)$ such that $g(U) \subset B(g(w_0), 2)$ for all $g \in \Psi$. Suppose the contrary. Consider a sequence $\{g_n \in \Psi\}_{n \in \mathbb{N}}$ for which $g_n(U_n) \not\subset B(g_n(w_0), 2)$, $U_n = h(B(z_0, \varepsilon/n))$, $n \in \mathbb{N}$. Extracting from the sequence $v_n = g_n \circ h$ a subsequence that converges uniformly in W, we arrive at a contradiction, since for the limit function v we have $v(B(z_0, \varepsilon/n)) \subset B(v(z_0), 1)$ at a sufficiently large n, whence there is n for which $v_n(B(z_0, \varepsilon/n)) \subset B(v_n(z_0), 2)$.

The following lemma is an immediate consequence of Lemma 1:

Lemma 2. *Let $f : T \to T$ be an analytic relation. For a point z to belong to $\mathscr{F}(f)$, it is necessary that all images of z under f belong to $\mathscr{F}(f)$.*

Lemma 3. *Let $D \subset \mathbb{C}^*$ be a domain and $\Phi \subset \mathscr{H}(D, \mathbb{C}^*)$. If a point $z_0 \in D$ is exterior to the set $P(\Phi) = \{\xi \in D : \exists\, g \in \Phi\ g(\xi) = \xi\}$ then Φ is normal at z_0.*

Proof. By hypothesis, there is a domain $U \subset D$, $z_0 \in U$, such that $U \cap P(\Phi) = \varnothing$. The family $\Xi = \{g(z)/z : g \in \Phi\}$ is normal in U by Montel's criterion, since any $h \in \Xi$ does not take values 0, ∞, and 1 in U. Hence, Φ is normal in U. The lemma is proven.

Lemma 4. *Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be an analytic relation. Then, for each $z_0 \in \mathscr{F}(f)$, the family $\Phi(f, z_0)$ is normal at z_0.*

Proof. Let $W \subset \mathbb{C}^*$, $z_0 \in W$, be a simply connected domain. Take a domain $U \ni z_0$ such that $\overline{U} \subset W$. Put

$$\Xi = \{g \in \Phi(f, z_0) \mid \overline{U} \subset g(U)\}.$$

Suppose that $\Phi(f, z_0)$ is not a normal family in U. Then Ξ is not normal in U either, since the family $\Phi(f, z_0)\backslash\Xi$ is normal in U by Remark 1 with $K_1 = \{0\}$, $K_2 = \{\infty\}$, and $K_3 = \overline{U}$. Therefore, Lemma 3 implies the existence of $g \in \Xi$ and $\xi_0 \in U$ such that $g(\xi_0) = \xi_0$. According to the conditions of the lemma, the iteration sequence $\{g^n\}_{n\in\mathbb{N}}$ is a normal family in W.

Consider two cases.

1. There is a subsequence g^{n_k} of the sequence g^n which converges in W to a constant. Then $U \not\subset g^n(U)$ for some n. However, this contradicts the inclusion $\overline{U} \subset g(U)$.

2. There is a subsequence g^{n_k} of the sequence g^n which converges in W to a nonconstant function. Without loss of generality we may assume that $m_k = n_{k+1} - n_k$ increases. The sequence g^{m_k} converges in W to the identity mapping. However, this contradicts the inclusion $g^n(U) \supset g(U) \supset \overline{U}$, $n > 0$.

The proof of the lemma is over.

Lemma 5. *Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be an analytic relation and let R be the set of all repelling periodic points of f. Then $\mathscr{J}(f)$ coincides with $\overline{R}$.*

Proof. The inclusion $\overline{R} \subset \mathscr{J}(f)$ is obvious.

Prove that $\mathscr{J}(f) \subset \overline{R}$. Take $z_0 \in \mathscr{J}(f) \cap \mathbb{C}^*$ and let $W \subset \mathbb{C}^*$, $z_0 \in W$, be a simply connected domain. Take a domain $U \ni z_0$ so that $\overline{U} \subset W$. Put

$$\Xi = \{g \in \Phi(f, z_0) \mid \overline{U} \subset g(U)\}.$$

The family Ξ is not normal in U. Therefore, by Lemma 3 there are $g \in \Xi$ and $\xi_0 \in U$ such that $g(\xi_0) = \xi_0$.

Consider the entire function $h = \log \circ\, g \circ \exp$. For a proper choice of the branch of the logarithm the point $w_0 = \log \xi_0$ is a fixed point of h. Arguments similar to those in the proof of Lemma 4 show that the family $\{h^n\}_{n\in\mathbb{N}}$ is not normal in $\Omega = \log(W)$. Therefore, Ω contains a point of the set $B = \mathscr{J}(h)\backslash\{\infty\}$. Hence, the assertion of the lemma is obvious in case $h(z)$ has the form $az+b$, while follows from Theorem B otherwise.

Lemma 6. *Suppose that $f : T \to T$, $T \in \{\mathbb{C}, \overline{\mathbb{C}}\}$, is an analytic relation, F is the set of all its branches in T, and G is the set of all finite compositions of functions in F. In the case of $T = \overline{\mathbb{C}}$ we also require that F contains no linear-fractional functions. The set*

$$\bigcup_{g\in G} \mathscr{J}(g)$$

is everywhere dense in $\mathscr{J}(f)$.

Introduce some notations. Given a domain D and $\varphi_n \in \mathscr{H}(D)$, $n = 1, 2, \dots$, put

$$E(\{\varphi_n\}_{n\in\mathbb{N}}, D) = \{\xi \in \overline{\mathbb{C}} : \exists\, z_n(\exists\, n_0\ \forall\, n > n_0\ z_n \notin \varphi_n(D)) \wedge z_n \to \xi\},$$

$$Q(\{\varphi_n\}_{n\in\mathbb{N}}, D) = \bigcup E(\{\varphi_{n_k}\}_{k\in\mathbb{N}}, D),$$

where the union is calculated over all increasing sequences n_k of positive integers.

REMARK 7. The sets $Q(\{\varphi_n\}_{n\in\mathbb{N}}, D)$ and $E(\{\varphi_n\}_{n\in\mathbb{N}}, D)$ are compact.

REMARK 8. For a sequence φ_n to be a normal family in D, it suffices that $\mathrm{Card}(E(\{\varphi_n\}_{n\in\mathbb{N}}, D)) > 2$.

REMARK 9. The following hold for every subsequence $\{\varphi_{n_k}\}_{k\in\mathbb{N}}$ and every subdomain $U \subset D$:

$$E(\{\varphi_n\}_{n\in\mathbb{N}}, D) \subset E(\{\varphi_{n_k}\}_{k\in\mathbb{N}}, D) \subset Q(\{\varphi_{n_k}\}_{k\in\mathbb{N}}, D) \subset Q(\{\varphi_n\}_{n\in\mathbb{N}}, D),$$

$$E(\{\varphi_n\}_{n\in\mathbb{N}}, D) \subset E(\{\varphi_n\}_{n\in\mathbb{N}}, U), \quad Q(\{\varphi_n\}_{n\in\mathbb{N}}, D) \subset Q(\{\varphi_n\}_{n\in\mathbb{N}}, U).$$

The assertions of Remarks 7–9 will be used below without specification.

We prove the following two auxiliary assertions:

Lemma 7. *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of analytic functions in a domain D which is not a normal family in D. Denote by $\mathfrak{N}$ the set of all subsequences $\{\phi_{n_k}\}_{k\in\mathbb{N}}$ which are not normal families in D. Then there is $\{v_k\}_{k\in\mathbb{N}} \in \mathfrak{N}$ such that $Q(\{v_k\}_{k\in\mathbb{N}}, D) = E(\{v_k\}_{k\in\mathbb{N}}, D)$.*

PROOF. Put

$$\ell = \max\{\mathrm{Card}(E(\{\psi_k\}_{k\in\mathbb{N}}, D)) : \{\psi_k\}_{k\in\mathbb{N}} \in \mathfrak{N}\}.$$

Note that $\ell < 3$. Suppose that $\mathrm{Card}(E(\{\chi_{0\,k}\}_{k\in\mathbb{N}}, D)) = \ell$ and $\{\chi_{0\,k}\}_{k\in\mathbb{N}} \in \mathfrak{N}$. Prove the following

Assertion 1. *For every subsequence $\{\eta_k\}_{k\in\mathbb{N}} \in \mathfrak{N}$ of $\{\chi_{0\,k}\}_{k\in\mathbb{N}}$ and every compact set K, $K \cap E(\{\eta_k\}_{k\in\mathbb{N}}, D) = \varnothing$, there is a subsequence $\{\vartheta_k\}_{k\in\mathbb{N}} \in \mathfrak{N}$ of $\{\eta_k\}_{k\in\mathbb{N}}$ such that*

$$K \cap Q(\{\vartheta_k\}_{k\in\mathbb{N}}, D) = \varnothing.$$

PROOF OF ASSERTION 1. Denote by $\{m_k\}_{k\in\mathbb{N}}$ the sequence of all numbers m such that $K \subset \eta_m(D)$ and denote by $\{p_k\}_{k\in\mathbb{N}}$ the sequence of the other numbers. Put $\vartheta_k = \eta_{m_k}$. It suffices to demonstrate that the sequence $\{\eta_{p_k}\}_{k\in\mathbb{N}}$ is a normal family in D. Let $\{\varrho_k\}_{k\in\mathbb{N}}$ be an arbitrary subsequence. Let us show that from this subsequence we can extract a convergent subsequence in D. It suffices to choose a subsequence which is a normal family in D. To this end, take a sequence $z_k \in K\backslash\varrho_k(D)$. Extract a convergent subsequence $\{z_{k_q}\}_{q\in\mathbb{N}}$ from it. The corresponding subsequence $\{\varrho_{k_q}\}_{q\in\mathbb{N}}$ possesses the property $\mathrm{Card}(E(\{\varrho_{k_q}\}_{q\in\mathbb{N}}, D)) > \ell$. By construction, this means that $\{\varrho_{k_q}\}_{q\in\mathbb{N}}$ is a normal family in D.

Now, we consider two cases.

1. $\ell = 0$. The assertion of the lemma is an immediate consequence of Assertion 1 for $\eta_k = \chi_{0\,k}$ and $K = \overline{\mathbb{C}}$.

2. $\ell > 0$. Without loss of generality we may assume that $\{\chi_{0\,k}\}_{k\in\mathbb{N}}$ has no convergent subsequences in D (if necessary we can drop down to a subsequence with this property; there is such a subsequence, since $\{\chi_{0\,k}\}_{k\in\mathbb{N}}$ is not a normal family). There is a sequence $\{\{\chi_{j\,k}\}_{k\in\mathbb{N}} \in \mathfrak{N}\}_{j\in\mathbb{N}}$ such that, for all $j = 1, 2, \dots$, the sequence $\{\chi_{j\,k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{\chi_{j-1\,k}\}_{k\in\mathbb{N}}$ and the inclusion

$$Q(\{\chi_{j\,k}\}_{k\in\mathbb{N}}, D) \subset Q_j$$

holds, where

$$Q_j = \bigcup_{\omega\in E(\{\chi_{0\,k}\}_{k\in\mathbb{N}}, D)} B(\omega, 1/j).$$

Indeed, since each subsequence $\{\eta_k\}_{k\in\mathbb{N}} \in \mathfrak{N}$ of $\{\chi_{0\,k}\}_{k\in\mathbb{N}}$ satisfies the equality

$$E(\{\eta_k\}_{k\in\mathbb{N}}, D) = E(\{\chi_{0\,k}\}_{k\in\mathbb{N}}, D),$$

Assertion 1 with $\eta_k = \chi_{j\,k}$ and $K = \overline{\mathbb{C}}\backslash Q_{j+1}$ implies that each sequence $\{\chi_{j\,k}\}_{k\in\mathbb{N}} \in \mathfrak{N}$ has a subsequence $\{\chi_{j+1\,k}\}_{k\in\mathbb{N}} \in \mathfrak{N}$ such that $Q(\{\chi_{j+1\,k}\}_{k\in\mathbb{N}}, D) \subset Q_{j+1}$. Thus, we can prove the last assertion by induction.

Put $v_k = \chi_{k\,k}$. For each $j \in \mathbb{N}$ the sequence $\{v_{k+j}\}_{k\in\mathbb{N}}$ is a subsequence of $\{\chi_{j\,k}\}_{k\in\mathbb{N}}$; hence,

$$Q(\{v_k\}_{k\in\mathbb{N}}, D) \subset Q(\{\chi_{j\,k}\}_{k\in\mathbb{N}}, D) \subset Q_j.$$

Therefore, $Q(\{v_k\}_{k\in\mathbb{N}}, D) = E(\{\chi_{0\,k}\}_{k\in\mathbb{N}}, D)$. Since $\{\chi_{0\,k}\}_{k\in\mathbb{N}}$ has no subsequences convergent in D, it follows that $\{v_k\}_{k\in\mathbb{N}} \in \mathfrak{N}$.

The lemma is proven.

Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of analytic functions in some neighborhood W of a point ζ_0. Denote by $\mathfrak{N}(\{s_n\}_{n\in\mathbb{N}}, \zeta_0, R)$ the set of all subsequences of $\{s_n\}_{n\in\mathbb{N}}$ that are not normal families in $B(\zeta_0, R) \subset W$, and put

$$\mathfrak{M}(\{s_n\}_{n\in\mathbb{N}}, \zeta_0) = \bigcap_{R>0} \mathfrak{N}(\{s_n\}_{n\in\mathbb{N}}, \zeta_0, R).$$

Introduce the following abbreviations:

$$E(\{\varphi_n\}_{n\in\mathbb{N}}, z, R) := E(\{\varphi_n\}_{n\in\mathbb{N}}, B(z, R)),$$

$$Q(\{\varphi_n\}_{n\in\mathbb{N}}, z, R) := Q(\{\{\varphi_n\}_{n\in\mathbb{N}}, B(z, R)).$$

Lemma 8. *There exist a point $\xi_0 \in W$, a number $\delta^* > 0$, and a set E_0 such that, for every $\delta \in (0; \delta^*]$, there is $\{v_k\}_{k\in\mathbb{N}} \in \mathfrak{N}(\{s_n\}_{n\in\mathbb{N}}, \xi_0, \delta)$ satisfying $Q(\{v_k\}_{k\in\mathbb{N}}, \xi_0, \delta) = E(\{v_k\}_{k\in\mathbb{N}}, \xi_0, \delta) = E_0$.*

Proof. Define the sequences

$$\{\xi_j\}_{j\in\mathbb{N}}, \quad \{\delta_j > 0\}_{j\in\mathbb{N}}, \quad \{\{\psi_{j\,k}\}_{k\in\mathbb{N}}\}_{j\in\mathbb{N}},$$

$$B(\xi_{j+1}, \delta_{j+1}) \subset B(\xi_j, \delta_j), \quad \{\psi_{j+1\,k}\}_{k\in\mathbb{N}} \in \mathfrak{M}(\{\psi_{j\,k}\}_{k\in\mathbb{N}}, \xi_{j+1}), \quad j = 1, 2, \dots,$$

as follows: Put $\psi_{1\,k} = s_k$. By Theorem C, there is a point $\xi_1 \in W$ such that $\{\psi_{1\,k}\}_{k\in\mathbb{N}} \in \mathfrak{M}(\{s_n\}_{n\in\mathbb{N}}, \xi_1)$. Take a number $\delta_1 > 0$ such that $B(\xi_1, \delta_1) \subset W$.

The other terms of the sequences are defined recurrently. Take $j \in \mathbb{N}$. Consider a sequence $\{\psi_{j\,k}\}_{k\in\mathbb{N}}$. Put

$$\ell_j = \max\{\mathrm{Card}(E(\{\varphi_k\}_{k\in\mathbb{N}}, \xi_j, \gamma)) : 0 < \gamma \leqslant \delta_j,\ \{\varphi_k\}_{k\in\mathbb{N}} \in \mathfrak{N}(\{\psi_{j\,k}\}_{k\in\mathbb{N}}, \xi_j, \gamma)\}.$$

Take $\gamma_j \in (0, \delta_j]$ and a sequence $\{\varphi_{j\,k}\}_{k\in\mathbb{N}} \in \mathfrak{N}(\{\psi_{j\,k}\}_{k\in\mathbb{N}}, \xi_j, \gamma_j)$ such that

$$\mathrm{Card}(E(\{\varphi_{j\,k}\}_{k\in\mathbb{N}}, \xi_j, \gamma_j)) = \ell_j.$$

Using Lemma 7, choose $\{\psi_{j+1\,k}\}_{k\in\mathbb{N}} \in \mathfrak{N}(\{\varphi_{j\,k}\}_{k\in\mathbb{N}}, \xi_j, \gamma_j)$ such that

$$E(\{\psi_{j+1\,k}\}_{k\in\mathbb{N}}, \xi_j, \gamma_j) = Q(\{\psi_{j+1\,k}\}_{k\in\mathbb{N}}, \xi_j, \gamma_j).$$

By Theorem C, there is $\xi_{j+1} \in B(\xi_j, \gamma_j)$ such that

$$\{\psi_{j+1\,k}\}_{k\in\mathbb{N}} \in \mathfrak{M}(\{s_n\}_{n\in\mathbb{N}}, \xi_{j+1}).$$

Take a number δ_{j+1} such that $B(\xi_{j+1}, \delta_{j+1}) \subset B(\xi_j, \gamma_j)$. Note that, by construction, the integer-valued sequence $\{\ell_j\}_{j\in\mathbb{N}}$ is bounded and nondecreasing. Moreover,

$$E(\{\varphi_{j\,k}\}_{k\in\mathbb{N}}, \xi_j, \gamma_j) \subset E(\{\psi_{j+1\,k}\}_{k\in\mathbb{N}}, \xi_{j+1}, \delta_{j+1}) \subset E(\{\varphi_{j+1\,k}\}_{k\in\mathbb{N}}, \xi_{j+1}, \gamma_{j+1}).$$

Therefore, there is $m \in \mathbb{N}$ such that the following equality holds for every $0 < \gamma \leqslant \delta_m$ and every sequence $\{\varphi_k\}_{k\in\mathbb{N}} \in \mathfrak{N}(\{\psi_{m\,k}\}_{k\in\mathbb{N}}, \xi_m, \gamma)$:

$$E(\{\varphi_k\}_{k\in\mathbb{N}}, \xi_m, \gamma) = E(\{\psi_{m\,k}\}_{k\in\mathbb{N}}, \xi_m, \delta_m).$$

Denote $\xi_0 := \xi_m$ and $E_0 := E(\{\psi_{m\,k}\}_{k\in\mathbb{N}}, \xi_m, \delta_m)$. Put $\delta^* = \delta_m$. Then the assertion of Lemma 8 follows from Lemma 7 with $\phi_n = \psi_{m\,n}$ and $D = B(\xi_m, \delta)$.

Proof of Lemma 6. Let $z_0 \in T \cap \mathscr{J}(f)$ and let $\{g_n\}_{n\in\mathbb{N}}$, $g_n = h_n \circ g_{n-1}$, $h_n \in F$, $g_0(z) \equiv z$, be some iteration sequence which is not a normal family at z_0.

By arguments similar to those in the proof of Lemma 4, it suffices to demonstrate that, for every $\varepsilon > 0$, there exist a domain U, $\overline{U} \subset B(z_0, \varepsilon)$, and a function $\varphi \in G$ such that $\overline{U} \subset \varphi(U)$. To this end, it suffices to establish the existence of a subsequence $\{\theta_k\}_{k\in\mathbb{N}}$ of $\{g_k\}_{k\in\mathbb{N}}$, a point w_0, and a number $\alpha > 0$ such that

$$Q(\{\theta_k\}_{k\in\mathbb{N}}, w_0, \alpha) \subset B(-1/\overline{w}_0, 1/\alpha) \quad \text{and} \quad B(w_0, \alpha) \subset B(z_0, \varepsilon).$$

We apply Lemma 8 with $s_n = g_n$, $\zeta_0 = z_0$, and $W = B(z_0, \varepsilon)$. Consider the following cases:

1. $\xi_0 \notin E_0$. Choosing $\alpha \in (0, \delta^*]$ so small that

$$\overline{B(\xi_0, \alpha)} \subset B(z_0, \varepsilon)\backslash E_0,$$

putting $\delta = \alpha$ in Lemma 8, and denoting $w_0 := \xi_0$ and $\theta_k := v_k$, we complete the proof of the theorem.

2. $\xi_0 \in E_0$. Take $\delta \in (0; \delta^*]$ such that $B(\xi_0, \delta) \subset B(z_0, \varepsilon)\backslash(E_0\backslash\{\xi_0\})$. Let $\{v_k\}_{k\in\mathbb{N}}$ be the sequence of Lemma 8. Consider the set X of all points $z \in B(\xi_0, \delta)$, $z \neq \xi_0$, such that $\{v_k\}_{k\in\mathbb{N}} \in \mathfrak{M}(\{g_n\}_{n\in\mathbb{N}}, z)$.

First, assume $\operatorname{Card}(E_0) = 2$.

Suppose $X \neq \varnothing$. Take $w_0 \in X$ and let $\alpha > 0$ be so small that $\overline{B(w_0, \alpha)} \subset B(\xi_0, \delta)\backslash\{\xi_0\}$. Since $\{v_k\}_{k\in\mathbb{N}} \in \mathfrak{M}(\{g_n\}_{n\in\mathbb{N}}, w_0)$, there is a sequence $\{\theta_k\}_{k\in\mathbb{N}} \in \mathfrak{N}(\{v_k\}_{k\in\mathbb{N}}, w_0, \alpha)$ having no convergent subsequences in $B(w_0, \alpha)$. Now, the assertion of the theorem follows from the equalities

$$E(\{\theta_k\}_{k\in\mathbb{N}}, w_0, \alpha) = Q(\{\theta_k\}_{k\in\mathbb{N}}, w_0, \alpha) = E_0 \quad \text{and} \quad \overline{B(w_0, \alpha)} \cap E_0 = \varnothing.$$

Show that $X \neq \varnothing$ indeed. Suppose the contrary. Without loss of generality we may assume that $\xi_0 \neq \infty$. Let $\tilde{v}_k := \lambda_k \circ v_k$, where λ_k is a sequence of conformal automorphisms of the sphere which converges to a nondegenerate linear-fractional mapping such that $\{\xi_0, \infty\} \cap \tilde{v}_k(B(\xi_0, \delta)) = \varnothing$ for all sufficiently large numbers k. Such a sequence exists, since $\operatorname{Card}(E_0) = 2$.

Without loss of generality we may assume that $\{\tilde{v}_k\}_{k\in\mathbb{N}}$ has no convergent subsequences in any neighborhood of ξ_0, while converges in its deleted neighborhood $B(\xi_0, \delta)\backslash\{\xi_0\}$ (if necessary we drop down to a subsequence of $\{\tilde{v}_k\}_{k\in\mathbb{N}}$ with this property). Denote the limit by $q(z)$. Since $\tilde{v}_k$ in a neighborhood of ξ_0 is representable as a Cauchy integral, we have $q(z) \equiv \infty$. Put $m_k := \min\{|\tilde{v}_k(z) - \xi_0| : z \in \gamma\}$, where $\gamma \subset B(\xi_0, \delta)$ is a fixed circle $|z - \xi_0| = r$. From the maximum principle applied to the functions $1/(\tilde{v}_k(z) - \xi_0)$ we conclude that $m_k < |\tilde{v}_k(z) - \xi_0|$, $|z - \xi_0| < r$. Since $m_k \to \infty$ as $k \to \infty$, it follows that $\tilde{v}_k$ converges to ∞ in the domain $|z - \xi_0| < r$. This contradiction completes the proof.

Now, assume $\operatorname{Card}(E_0) = 1$.

This case may hold only for $T = \overline{\mathbb{C}}$. From some number on, we then have $v_k(B(\xi_0, \delta)) \cup B(\xi_0, \delta) = \overline{\mathbb{C}}$ and consequently either $B(\xi_0, \delta) \cap \mathscr{J}(v_k) \neq \varnothing$ or $\mathscr{J}(v_k) = \varnothing$. In the first case the theorem has been already proven. The second case is impossible by Theorem A.

The proof is complete.

Lemma 9. *Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic relation and $\mathscr{F}(f) = \mathbb{C}$. Then for each $z_0 \in \mathbb{C}$ the family $\Phi(f, z_0)$ is normal at z_0.*

Proof. Denote the set of all compositions of the branches of f by G. Consider an arbitrary $g \in G$. The set $\mathscr{J}(g)\backslash\{\infty\}$ is empty. Therefore, $g = az + b$, where a and b are constants: otherwise $\mathscr{J}(g)$ would contain infinitely many points by Theorem A. Moreover, $|a| \leqslant 1$: otherwise there would exist a repelling fixed point $b/(1 - a) \in \mathbb{C}$.

Take $\{g_n \in G\}_{n\in\mathbb{N}}$, $g_n = a_n z + b_n$. Choosing a subsequence of positive integers n_k such that a_{n_k} and b_{n_k} converge, we obtain a convergent subsequence g_{n_k}. It follows that G is a normal family, as required.

Proofs of Propositions 1 and 2 and Theorem 1

Proof of Proposition 1. Denote by $\mathscr{F}^*(f)$ the set of all points $z_0 \in \Delta$ for which the family $\Phi(f, z_0)$ is normal at z_0. Obviously, $\mathscr{F}(f) \subset \mathscr{F}^*(f)$.

Prove the reverse inclusion.

Denote $\ell = \mathrm{Card}(\overline{\mathbb{C}} \backslash \mathscr{F}(f))$. If $\ell > 2$ then the proof reduces to appealing to Lemma 2 and Montel's criterion.

Now, suppose that $\ell \leqslant 2$. Without loss of generality we may assume that $\mathscr{F}(f) \in \{\overline{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*\}$. Note that the case $\mathscr{F}(f) = \overline{\mathbb{C}}$ is impossible. Supposing the contrary, we find that $\Delta = \overline{\mathbb{C}}$; in consequence, each branch g of f is a rational function and so $\overline{\mathbb{C}} = \mathscr{F}(f) \subset \mathscr{F}(g)$, which contradicts Theorem A.

Thus, the assertion of the theorem follows from Lemmas 2 and 4 or 9 (depending on whether $\mathscr{F}(f) = \mathbb{C}^*$ or $\mathscr{F}(f) = \mathbb{C}$) applied to $T = \mathscr{F}(f)$.

Proof of Proposition 2. Suppose the contrary; i.e., suppose that there exist a point $z_0 \in T \cap \mathscr{J}(f)$ and its simply connected neighborhood U such that $D = U \backslash \{z_0\} \subset \mathscr{F}(f)$. This means that there is a subsequence $\{g_n\}_{n \in \mathbb{N}}$ of the iteration sequence which has no convergent subsequence in any neighborhood of z_0. Without loss of generality we may assume that $\infty \notin \mathscr{F}(f)$ and g_n converges on D to some function $h : D \to \overline{\mathbb{C}}$. Note that, by Lemma 2, if h is nonconstant then $h(D) \subset \mathscr{F}(f)$.

Take $z_1, z_2 \in (\overline{\mathbb{C}} \backslash \mathscr{F}(f) \backslash \{z_0\})$, $z_1 \neq z_2$. By the Cauchy integral theorem, if $\infty \notin h(D)$ and $g_{n_k}(z_0) \neq \infty$ for all $k \in \mathbb{N}$ then g_{n_k} converges in U.

The following cases exhaust all possibilities.

1. $h(D) \cap \{z_1, z_2\} = \varnothing$. Using conformal automorphisms of the sphere, we can pass to the case of $z_1 = \infty$. Therefore, $\exists\, n_1\ \forall\, n \geqslant n_1\ g_n(z_0) = z_1$. Similarly, $\exists\, n_2\ \forall\, n \geqslant n_2\ g_n(z_0) = z_2$. Thus, we arrive at a contradiction, since $z_1 \neq z_2$.

2. $h \equiv z_j$, $j \in \{1, 2\}$. Without loss of generality we may assume that $j = 1$. Then $\exists\, n_2\ \forall\, n \geqslant n_2$ $g_n(z_0) = z_2$. Thus, the equality $g_{n_2+m} = \varphi_m \circ g_{n_2}$, $m > 0$, holds, where φ_m is some subsequence of the iteration sequence at z_2; moreover, $\varphi_m(z_2) = z_2$ for all m. The point z_2 has a deleted neighborhood $W = g_{n_2}(D) \subset \mathscr{F}(f)$. Placing ∞ at z_0 by means of conformal automorphisms of the sphere, we conclude that $\varphi_m \to z_0$ on W; consequently, $h \equiv z_0$ which contradicts the assumption.

The contradiction proves the proposition.

Proof of Theorem 1. Theorem 1 follows from Lemmas 5 and 6.

References

1. *Beardon A. F.*, Iteration of Rational Functions, Springer-Verlag, New York (1991).
2. *Carleson L. and Gamelin T. W.*, Complex Dynamics, Springer-Verlag, New York (1993).
3. *Bergweiler W.*, "An introduction to complex dynamics," Textos de Matemática, Universidade de Coimbra, Sér. B, **6**, 1–37 (1995).
4. *Goluzin G. M.*, Geometric Theory of Functions of a Complex Variable [in Russian], Nauka, Moscow (1966).
5. *Nevanlinna R.*, Uniformization [Russian translation], Izdat. Inostr. Lit., Moscow (1955).