Analysis and Mathematical Physics
Trends in Mathematics, 165–184

Carathéodory Convergence of Immediate Basins of Attraction to a Siegel Disk

Pavel Gumenyuk

Abstract. Let f_n be a sequence of analytic functions in a domain U with a common attracting fixed point z_0. Suppose that f_n converges to f_0 uniformly on each compact subset of U and that z_0 is a Siegel point of f_0. We establish a sufficient condition for the immediate basins of attraction $\mathcal{A}^*(z_0, f_n, U)$ to form a sequence that converges to the Siegel disk of f_0 as to the kernel w. r. t. z_0. The same condition is shown to imply the convergence of the Kœnigs functions associated with f_n to that of f_0. Our method allows us also to obtain a kind of quantitative result for analytic one-parametric families.

Mathematics Subject Classification (2000). Primary 37F45; Secondary 30D05, 37F50.

Keywords. Iteration of analytic functions, Fatou set, Siegel disk, basin of attraction, convergence as to the kernel.

1. Introduction

1.1. Preliminaries

Let U be a domain on the Riemann sphere $\overline{\mathbb{C}}$ and $f : U \to \overline{\mathbb{C}}$ a meromorphic function. Define f^n, the n-fold *iterate* of f, by the following relations: $f^1 : U \to \overline{\mathbb{C}}$, $f^1 := f$, $f^{n+1} : \big(f^n\big)^{-1}(U) \to \overline{\mathbb{C}}$, $f^{n+1} := f \circ f^n$, $n \in \mathbb{N}$. It is convenient to define f^0 as the identity map of U. Denote

$$E(f, U) := \bigcap_{n\in\mathbb{N}} \big(f^n\big)^{-1}(U).$$

The *Fatou set* $\mathcal{F}(f, U)$ of the function f (w. r. t. the domain U) is the set of all interior points z of $E(f, U)$ such that $\{f^n\}_{n\in\mathbb{N}}$ is a normal family in some neigh-

This work is partially supported by the *Research Council of Norway*, the *Russian Foundation for Basic Research* (grant #07-01-00120), and *ESF Networking Programme* “Harmonic and Complex Analysis and its Applications”.

bourhood of z. Define the *Julia set* $\mathcal{J}(f,U)$ of f (with respect to the domain U) to be the complement $U \setminus \mathcal{F}(f,U)$ of the Fatou set.

Classically iteration of analytic (meromorphic) functions has been studied for the case of $U \in \{\overline{\mathbb{C}}, \mathbb{C}, \mathbb{C}^* := \mathbb{C} \setminus \{0\}\}$ and $f : U \to U$, see survey papers [1, 2] for the details. As an extension the cases of transcendental meromorphic functions and functions meromorphic in $\overline{\mathbb{C}}$ except for a compact totally disconnected set have been also investigated, see, e.g., [3, 4]. (Note that $f(U) \not\subset U$ for these cases.) In this paper we shall restrict ourselves by the following

Assumption. Suppose that $U, f(U) \subset \mathbb{C}$, i.e., f is an analytic function in a subdomain U of $\mathbb{C}$.

One of the basic problems in iteration theory of analytic functions is to study how the limit behaviour of iterates changes as the function f is perturbed. A large part of papers in this direction are devoted to the continuity property for the dependence of the Fatou and Julia sets on the function to be iterated. We mention the work of A. Douady [5], who investigates the mapping $f \mapsto \mathcal{J}(f,\mathbb{C})$ from the class of polynomials of fixed degree to the set of nonempty plane compacta equipped with the Hausdorff metric $d_H(X,Y) := \max\{\partial(X,Y), \partial(Y,X)\}$, $\partial(X,Y) := \sup_{x\in X} \operatorname{dist}(x,Y)$. We also mention subsequent papers [6]–[11] dealing with other classes of functions. Continuity of Julia sets is closely related to behaviour of connected components of the Fatou set containing periodic points. Now we recall necessary definitions.

Let $z_0 \in U$ be a fixed point of f. The number $\lambda := f'(z_0)$ is called the *multiplier* of z_0. According to the value of λ the fixed point z_0 is said to be *attracting* if $|\lambda| < 1$, *neutral* if $|\lambda| = 1$, and *repelling* if $|\lambda| > 1$. An attracting fixed point is *superattracting* if $\lambda = 0$, or *geometrically attracting* otherwise. Suppose z_0 is a neutral fixed point of f and none of f^n, $n \in \mathbb{N}$, turns into the identity map; then the fixed point z_0 is *parabolic* if $\lambda = e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{Q}$, or *irrationally neutral* otherwise. If an irrationally neutral fixed point belongs to $\mathcal{F}(f,U)$, then it is called a *Siegel point*.

The component of the Fatou set $\mathcal{F}(f,U)$ that contains a fixed point z_0 is called the *immediate basin* of z_0 and denoted by $\mathcal{A}^*(z_0,f,U)$. The immediate basin of a Siegel point is called a *Siegel disk*, and the immediate basin of an attracting fixed point is called an *immediate basin of attraction*. It is a reasonable convention to put by definition $\mathcal{A}^*(z_0,f,U) := \{z_0\}$ for fixed points $z_0 \in \mathcal{J}(f,U)$, in particular for repelling and parabolic ones.

By passing to a suitable iterate of f, the above definitions are naturally extended to periodic points.

1.2. Main results

Consider a sequence $\{f_n : U \to \mathbb{C}\}_{n\in\mathbb{N}}$ of analytic functions with a common attracting fixed point $z_0 \in U$. Suppose that f_n converges to f_0 uniformly on each compact subset of U. It is easily follows from arguments of [5] that $\mathcal{A}^*(z_0,f_n,U) \to \mathcal{A}^*(z_0,f_0,U)$ as to the kernel w. r. t. z_0 provided z_0 is an attracting or parabolic fixed point of the limit function f_0. At the same time $\mathcal{A}^*(z_0, f_n, U)$ fails to converge to $\mathcal{A}^*(z_0, f_0, U)$ in general if z_0 is a Siegel point of f_0 (see Example 1 in Section 4). Similarly, the dependence of Julia sets on the function under iteration fails to be continuous at f_0 (with respect to the Hausdorff metric) if f_0 has (generally speaking, periodic) Siegel points. Nevertheless, in the paper [12] devoted to the continuity of Julia sets for one-parametric families of transcendental entire functions H. Kriete established an assertion, which can be stated as follows.

Theorem A. *Suppose $f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$; $(\lambda, z) \mapsto f_\lambda(z)$ is an analytic family of entire functions $f_\lambda(z) = \lambda z + a_2(\lambda)z^2 + \cdots$ and $\lambda_0 := e^{2\pi i\alpha_0}$, where $\alpha_0 \in \mathbb{R} \setminus \mathbb{Q}$ is a Diophantine number. Let Δ be any Stolz angle at the point λ_0 with respect to the unit disk $\{\lambda : |\lambda| < 1\}$. Then $\mathcal{A}^*(0, f_\lambda, \mathbb{C}) \to \mathcal{A}^*(0, f_{\lambda_0}, \mathbb{C})$ as to the kernel w. r. t. z_0 when $\lambda \to \lambda_0$, $\lambda \in \Delta$.*

Remark 1.1. It was proved by C. Siegel [13] that for a fixed point with multiplier $e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, to be a Siegel point, it is sufficient that α be Diophantine. This condition is not necessary even if restricted to the case of quadratic polynomials $f(z) := z^2 + c$, $c \in \mathbb{C}$ (see [14, Th. 6] and [15]). Furthermore, it is easy to construct a nonlinear analytic germ with a Siegel point for any given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

The Diophantine condition on α_0 is substantially employed in [12], and in view of the above remark it is interesting to find out whether this condition is really essential in Theorem A. Another question to consider is the role of analytic dependence of f_λ on λ. A possible answer is the following statement improving Theorem A.

Theorem 1.2. *Let $f_0 : U \to \mathbb{C}$ be an analytic function with a Siegel point $z_0 \in U$ and $\{f_n : U \to \mathbb{C}\}_{n\in\mathbb{N}}$ a sequence of analytic functions with an attracting fixed point at z_0. Suppose that f_n converges to f_0 uniformly on each compact subset of U and the following conditions hold*

(i) $\left|\arg\left(1 - f_n'(z_0)/f_0'(z_0)\right)\right| < \Theta$ *for some $\Theta < \pi/2$ and all $n \in \mathbb{N}$;*

(ii) *the functions $\left(f_n(z) - f_0(z)\right)/\left(f_n'(z_0) - f_0'(z_0)\right)$, $n \in \mathbb{N}$, are uniformly bounded on each compact subset of U.*

Then $\mathcal{A}^*(z_0, f_n, U)$ converges to $\mathcal{A}^*(z_0, f_0, U)$ as to the kernel w. r. t. z_0.

Condition (i) in this theorem requires that $\lambda_n := f_n'(z_0)$ tends to $\lambda_0 := f_0'(z_0)$ within a Stolz angle, condition (ii) appears instead of analytic dependence of f_λ on λ, and the Diophantine condition on α_0 turns out to be unnecessary. Both conditions (i) and (ii) are essential. We discuss this in Section 4.

Dynamics of iterates in the immediate basin of a fixed point can be described by means of so-called Kœnigs function.

Let z_0 be a fixed point of an analytic function f. The *Kœnigs function* φ associated with the pair (z_0, f) is a solution to the *Schröder functional equation*

$$\varphi\big(f(z)\big) = \lambda\varphi(z), \quad \lambda := f'(z_0), \tag{1.1}$$

analytic in a neighbourhood of z_0 and subject to the normalization $\varphi'(z_0) = 1$.

It is known (see, e.g., [16, pp. 73–76, 116], [17]) that the Kœnigs function exists, is unique, and can be analytically continued all over $\mathcal{A}^*(z_0, f, U)$ provided z_0 is a geometrically attracting or Siegel fixed point. If the Kœnigs function is known, then the iterates can be determined by means of the equality

$$\varphi\big(f^n(z)\big) = \lambda^n \varphi(z), \quad \lambda := f'(z_0).$$

By φ_k, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, denote the Kœnigs function associated with the pair (z_0, f_k). We prove the following

Theorem 1.3. *Under the conditions of Theorem* 1.2*, the sequence φ_n converges to φ_0 uniformly on each compact subset of $\mathcal{A}^*(z_0, f_0, U)$.*

The assertion of Theorem 1.3 should be understood in connection with Theorem 1.2, because the uniform convergence of φ_n on a compact set $K \subset \mathcal{A}^*(z_0, f_0, U)$ requires that K were in the range of definition of φ_n, i.e., in $\mathcal{A}^*(z_0, f_n, U)$, for all $n \in \mathbb{N}$ apart from a finite number.

Assumption. Hereinafter it is convenient to assume without loss of generality that $z_0 = 0$, saving symbol z_0 for other purposes.

For any $a \in \mathbb{C}$ and $A \subset \mathbb{C}$ let us use aA as the short variant of $\{az : z \in A\}$. By $\mathrm{D}(\xi_0, \rho)$ denote the disk $\{\xi : |\xi - \xi_0| < \rho\}$, but reserve the notation $\mathbb{D}$ for the unit disk $\mathrm{D}(0, 1)$.

Remark 1.4. The Kœnigs function φ_0 associated with the Siegel point of f_0 admits another description (see, e.g., [16, p. 116], [17]) as the conformal mapping of the Siegel disk $\mathcal{A}^*(0, f_0, U)$ onto a Euclidean disk $D(0, r)$ that satisfies the condition $\varphi_0(0) = \varphi_0'(0) - 1 = 0$. From this viewpoint it will be convenient to consider the conformal mapping φ, $\varphi(0) = 0$, $\varphi'(0) > 0$, of $\mathcal{A}^*(0, f_0, U)$ onto the unit disk $\mathbb{D}$ instead of the Kœnigs function φ_0. Obviously, $\varphi(z)/\varphi_0(z)$ is constant, and consequently, φ satisfies the Schröder equation (1.1) for $f := f_0$. For shortness, S will stand for $\mathcal{A}^*(0, f_0, U)$. By ψ denote the inverse function to φ and let $S_r := \psi(r\mathbb{D})$, $\mathcal{L}_r := \partial S_r$ for $r \in [0, 1]$. One of the consequences of the fact mentioned above is that f_0 is a conformal automorphism of S and S_r, $r \in (0, 1)$.

During the preparation of this paper another proof of Theorems 1.2 and 1.3 given in [18, p. 3] became known to the author. However, our method allows us also to establish an asymptotic estimate for the rate of covering level-lines of the Siegel disk by basins of attraction for one-parametric analytic families. Let $f : W \times U \to \mathbb{C}$; $(\lambda, z) \mapsto f_\lambda(z)$, where $U \ni 0$ and W are domains in $\mathbb{C}$, be a family of functions and α_0 an irrational number satisfying the following conditions:

(i) $f_\lambda(z)$ depends analytically on both the variable $z \in U$ and the parameter $\lambda \in W$;
(ii) $f_\lambda(0) = 0$ and $f_\lambda'(0) = \lambda$ for all $\lambda \in W$;
(iii) $\lambda_0 := \exp(2\pi i \alpha_0) \in W$ and the function f_{λ_0} has a Siegel point at $z_0 = 0$, with $S := \mathcal{A}^*(0, f_{\lambda_0}, U)$ lying in U along with its boundary ∂S.

Consider the continued faction expansion of α_0 and denote the nth convergent by p_n/q_n. (See, e.g., [19, 20] for a detailed exposition on continued factions.) For $x > 0$ we set

$$n_0(x) := \min\left\{n \in \mathbb{N} : \frac{2q_n q_{n+1}}{q_n + q_{n+1}} \geq x\right\}, \quad \ell(x) := q_{n_0(x)}.$$

Notation φ, ψ, S, and S_r will refer to the limit function f_{λ_0}. Lemma 2.2 with a slight modification can be used to prove the following statement.

Theorem 1.5. *For any Stolz angle Δ at the point λ_0 there exist a constant $C > 0$ and a function $\varepsilon : (0,1) \to (0,+\infty)$ such that for any $r \in (0,1)$ the following statements are true:*

(i) $S_r \subset \mathcal{A}^*(0, f_\lambda, U)$ *for all* $\lambda \in W \cap \Delta$ *satisfying* $|\lambda - \lambda_0| < \varepsilon(r)$*;*
(ii) $\varepsilon(r) \geq C(1-r)^3/\ell\big((1-r)^{-\gamma}\big)$,

where $\gamma > \gamma_0 := 1 + \max\big\{\beta_\psi(1), \beta_\psi(-1)\big\}$ and β_ψ stands for the integral means spectrum of the function ψ,

$$\beta_\psi(t) := \limsup_{r\to 1-} \frac{\log \int_0^{2\pi} |\psi'(re^{i\theta})|^t \, d\theta}{-\log(1-r)}. \tag{1.2}$$

It is known [21] that $\beta_\psi(1) \leq 0.46$ and $\beta_\psi(-1) \leq 0.403$ for any function ψ bounded and univalent in $\mathbb{D}$. Consequently, $\gamma_0 \leq 1.46$.

Theorem 1.5 has been published in [22]. We sketch its proof and specify the function $\varepsilon(r)$ explicitly in Section 3.

2. Proof of theorems

2.1. Lemmas

Denote $\lambda_k := f_k'(0)$, $k \in \mathbb{N}_0$. Let us fix arbitrary $n_* \in \mathbb{N}$ and consider the linear family

$$f_\lambda[n_*](z) := (1-t)f_0(z) + tf_{n_*}(z), \; t := \frac{\lambda - \lambda_0}{\lambda_{n_*} - \lambda_0}, \quad z \in U, \; \lambda \in \mathbb{C}. \tag{2.1}$$

The number n_* will be not varied throughout the discussion in the present section. So we shall not indicate dependence on n_* until it is necessary. In particular we shall often write f_λ instead of $f_\lambda[n_*]$.

We need the following elementary statement on approximation of integrals by quadrature sums (see, e.g., [23, pp. 55–62]).

Theorem B. *Suppose ϕ is a continuously differentiable function on $[0,1]$. Then for any $N \in \mathbb{N}$ and any set of points $x_0, x_1, \ldots, x_{N-1} \in [0,1]$ the following inequality holds*

$$\left|\int_0^1 \phi(x)\,dx - \frac{1}{N}\sum_{n=0}^{N-1}\phi(x_n)\right| < Q\big(x_0, x_1, \ldots, x_{N-1}\big)\int_0^1 |\phi'(x)|\,dx, \tag{2.2}$$

where $Q\big(x_0, x_1, \ldots, x_{N-1}\big) := \sup_{x\in[0,1]} \big|F(x; x_0, x_1, \ldots, x_{N-1}) - x\big|$ *and*

$$F(x; x_0, x_1, \ldots, x_{N-1}) := \frac{1}{N}\sum_{n=0}^{N-1} \theta(x - x_n),\ \ \theta(y) := \begin{cases} 1, & \text{if } y > 0, \\ 0, & \text{if } y \le 0. \end{cases}$$

Remark 2.1. Consider the sequence $x_n^\beta := \{\alpha_0 n + \beta\}$, where $\{\,\cdot\,\}$ stands for fractional part, α_0 is given by $\lambda_0 = e^{2\pi i\alpha_0}$, and β is an arbitrary real number. Denote

$$Q_{\beta,N} := Q\big(x_0^\beta, x_1^\beta, \ldots, x_{N-1}^\beta\big).$$

Since $\alpha_0 \in \mathbb{R}\backslash\mathbb{Q}$, we have (see, e.g., [23, pp. 102–108]) $Q_{\beta,N} \to 0$ as $N \to +\infty$.

Fix any $r_0 \in (0,1)$. The following lemma allows us to determine $\varepsilon_* > 0$ such that $S_{r_0} \subset \mathcal{A}^*(0, f_\lambda, U)$ whenever $|\arg(1-\lambda/\lambda_0)| < \Theta$ and $|\lambda - \lambda_0| < \varepsilon_*$. In order to state this assertion we need to introduce some notation.

Denote

$$k_0(z) := \frac{z}{(1-z)^2},\ \ z \in \mathbb{D}, \quad k_\gamma(z) := e^{i\gamma}k_0(e^{-i\gamma}z),\ \ \gamma \in \mathbb{R},$$

$$u(z) := \frac{f_{n_*}(z) - f_0(z)}{\lambda_{n_*} - \lambda_0}, \quad H(\xi) := 1 + \frac{\xi\psi''(\xi)}{\psi'(\xi)},$$

$$J(t) := \frac{\xi u'(\psi(\xi))\psi'(\xi) - u(\psi(\xi))H(\lambda_0\xi)}{\lambda_0\xi\psi'(\lambda_0\xi)},\ \ \xi := r_0 e^{2\pi i t}.$$

For $\tau \in (0, -\log r_0)$ and $N \in \mathbb{N}$ we put

$$Q_N := \inf_{\beta\in\mathbb{R}} Q_{\beta,N}, \quad a_N := 2\pi Q_N \int_0^1 |J(t)|\, dt,$$

$$\Lambda_N(\tau, \varepsilon) := \frac{\sqrt{1 + 2b^2\cos 2\vartheta + b^4} - 1 + b^2}{2b\cos\vartheta}, \quad \varepsilon > 0,$$

where $\vartheta := \Theta + \arcsin a_N$, $b := \pi\varepsilon N(1-a_N)/(4\tau)$,

$$\varepsilon_N(\tau) := \frac{1 - k_\pi(r_*)/k_\pi(r^*)}{\sup\limits_{z\in S_{r_*}} |1 - f_{n_*}(z)/f_0(z)|}\, |\lambda_{n_*} - \lambda_0|, \quad r_* := r_0 e^{\tau(1-1/N)},\ \ r^* := r_0 e^{\tau}.$$

Lemma 2.2. *Let* $N \in \mathbb{N}$ *and* $\tau \in (0, -\log r_0)$. *If* $a_N < \sin(\pi/2 - \Theta)$, *then* $f_\lambda^N\big(S_{r_0}\big) \subset S_{r_0}$ *for all* λ *such that* $\big|\arg(1-\lambda/\lambda_0)\big| < \Theta$ *and* $|\lambda - \lambda_0| < \varepsilon_*$, *where* $\varepsilon_* := \varepsilon_N(\tau)\,\Lambda_N\big(\tau, \varepsilon_N(\tau)\big)$.

Remark 2.3. In view of Montel's criterion the inclusion $f_\lambda^N\big(S_{r_0}\big) \subset S_{r_0}$ in Lemma 2.2 implies that $S_{r_0} \subset \mathcal{A}^*(0, f_\lambda, U)$. We will use this simple fact without reference.

Lemma 2.2 in a slightly different form has been proved in [22]. We state its proof here for completeness of the discussion. The scheme of the proof is the following. The main idea is to fix arbitrary $z_0 \in \mathcal{L}_{r_0}$ and consider the function $s_N(\lambda) = s_N(z_0, \lambda) := \varphi(f_\lambda^N(z_0))$. The first step (Lemma 2.4) is to determine a neighbourhood of λ_0 where s_N is well defined, analytic and takes values from a prescribed domain of the form $\{\xi : \rho_1 < |\xi| < \rho_2\}$. The next step (Lemma 2.5) is to calculate the value of $(\partial/\partial\lambda)\log s_N(\lambda)$ at $\lambda = \lambda_0$, which turns out to be equal to

$$A_N(z_0) := \frac{s_N'(\lambda_0)}{s_N(\lambda_0)} = \sum_{k=0}^{N-1} G\bigl(\lambda_0^k \varphi(z_0)\bigr),$$

where G is an analytic function in $\mathbb{D}$. The concluding step is to use the equality $\int_0^1 G\bigl(e^{2\pi i t}\varphi(z_0)\bigr)\,dt = 1/\lambda_0$ and Theorem B in order to estimate $|A_N(z_0)|$ and $|\arg A_N(z_0)|$. This allows us to employ a consequence of the Schwarz lemma (Proposition 2.6) for proving that $|s_N(\lambda)| \le |\varphi(z_0)|$ for any λ satisfying $|\arg(1 - \lambda/\lambda_0)| < \Theta$ and $|\lambda - \lambda_0| < \varepsilon_*$. Since $z_0 \in \mathcal{L}_{r_0}$ is arbitrary, this means that $f_\lambda^N\bigl(S_{r_0}\bigr) \subset S_{r_0}$ for all such values of λ.

Lemma 2.4. *Under the conditions of Lemma 2.2, $s_N(z, \lambda) := \varphi\bigl(f_\lambda^N(z)\bigr)$ is a well-defined and analytic function for all $z \in \overline{S_{r_0}}$ and $\lambda \in \mathrm{D}(\lambda_0, \varepsilon_N(\tau))$. Moreover, the following inequality holds*

$$r_0 e^{-\tau} < |s_N(z, \lambda)| < r_0 e^{\tau}, \quad z \in \mathcal{L}_{r_0},\ \lambda \in \mathrm{D}(\lambda_0, \varepsilon_N(\tau)). \tag{2.3}$$

Proof. Let us show that for any $r_1 \in (0, 1)$, $r_2 \in (r_1, 1)$ the following inclusion holds

$$B(z_0, r_1, r_2) := \bigl\{z : |z - z_0| < |z_0|\bigl(1 - k_\pi(r_1)/k_\pi(r_2)\bigr)\bigr\} \subset S_{r_2} \setminus \overline{S_{r_3}}, \tag{2.4}$$

where $z_0 \in \mathcal{L}_{r_1}$ and $r_3 := r_1^2/r_2$. To this end we remark that for any $z_0 \in \mathcal{L}_{r_1}$ the domain $S_{r_2} \setminus \overline{S_{r_3}}$ contains all points z such that

$$\bigl|\log(z/z_0)\bigr| < \log\bigl(k_\pi(r_2)/k_\pi(r_1)\bigr) \tag{2.5}$$

for some of the branches of log. To make sure this statement is true it is sufficient to employ the following estimate, see, e.g., [24, p. 117, inequal. (18)],

$$\left|\log\frac{z\psi'(z)}{\psi(z)}\right| \le \log\frac{1 + |z|}{1 - |z|}, \qquad z \in \mathbb{D}, \tag{2.6}$$

Owing to (2.6), for any rectifiable curve $\Gamma \subset \overline{S_{r_2}} \setminus S_{r_3}$ that joins z_0 with $\mathcal{L}_{r_2}$ or $\mathcal{L}_{r_3}$ we have

$$\begin{aligned}\int_\Gamma \left|\frac{dz}{z}\right| &= \int_{\varphi(\Gamma)} \left|\frac{\psi'(\xi)}{\psi(\xi)}\right| |d\xi| \ge \int_{\varphi(\Gamma)} \left|\frac{\psi'(\xi)}{\psi(\xi)}\right| d|\xi| \\ &\ge \min\left\{\int_{r_1}^{r_2} \frac{(1-r)dr}{(1+r)\,r}, \int_{r_3}^{r_1} \frac{(1-r)dr}{(1+r)\,r}\right\} = \log\bigl(k_\pi(r_2)/k_\pi(r_1)\bigr).\end{aligned}$$

Using the inequality $|\log(1+\xi)| \le -\log(1 - |\xi|)$, $\xi \in \mathbb{D}$, we conclude that for any $z \in B(z_0, r_1, r_2)$,

$$\begin{aligned}\bigl|\log\bigl(z/z_0\bigr)\bigr| &= \bigl|\log\bigl(1 + (z - z_0)/z_0\bigr)\bigr| \\ &\le -\log\bigl(1 - |z - z_0|/|z_0|\bigr) < \log\bigl(k_\pi(r_2)/k_\pi(r_1)\bigr),\end{aligned}$$

i.e., all $z \in B(z_0, r_1, r_2)$ satisfy condition (2.5). Therefore inclusion (2.4) holds.

Let $r \in (0, e^{-\tau/N})$. Set $r' := re^{\tau/N}$ and $r'' := re^{-\tau/N}$. Consider an arbitrary function h subject to the following conditions: h is analytic in S, $h(0) = 0$, and $|h(z) - z| < |z|\big(1 - k_\pi(r)/k_\pi(r')\big)$ for all $z \in \overline{S_r} \setminus \{0\}$.

Set $r_1 := |z_0|$, $r_2 := |z_0|e^{\tau/N}$ for some $z_0 \in \overline{S_r} \setminus \{0\}$. Since $k_\pi(x)/k_\pi(xe^{\tau/N})$ increases with $x \in (0, r]$, the Schwarz lemma can be applied to the function $h(z) - z$ to conclude that $h(z_0) \in B(z_0, r_1, r_2)$ for all $z_0 \in \overline{S_r} \setminus \{0\}$. Therefore (2.4) implies the following inclusions

$$h(\overline{S_r}) \subset S_{r'}, \quad h(\mathcal{L}_r) \subset S_{r'} \backslash \overline{S_{r''}}. \tag{2.7}$$

By considering the function $\big(h(z) - z\big)/z$ with $f_{\lambda_0}(w)$ substituted for z it is easy to check that since the function f_{λ_0} is an automorphism of S_r for any $r \in (0, 1]$ (see Remark 1.4), the above argument can be applied to $h(z) := f_\lambda(f_{\lambda_0}^{-1}(z))$ for all $\lambda \in \mathrm{D}(\lambda_0, \varepsilon_N(\tau))$ and $r \in (0, r_*]$. Thus (2.7) implies that for any $\lambda \in \mathrm{D}(\lambda_0, \varepsilon_N(\tau))$,

$$f_\lambda\left(\overline{S_{r_j}}\right) \subset S_{r_{j+1}}, \qquad j = 0, 1, \ldots, N-1, \tag{2.8}$$

$$f_\lambda\left(\overline{S_{r_j}} \backslash S_{r_{-j}}\right) \subset S_{r_{j+1}} \backslash \overline{S_{r_{-(j+1)}}}, \qquad j = 0, 1, \ldots, N-1, \tag{2.9}$$

where $r_j := r_0 e^{j\tau/N}$, $j = 0, \pm 1, \ldots, \pm N$. Applying (2.8) repeatedly, we see that $f_\lambda^N(\overline{S_{r_0}}) \subset S_{r_N}$. Similarly, (2.9) implies that $f_\lambda^N(\mathcal{L}_{r_0}) \subset S_{r_N} \backslash \overline{S_{r_{-N}}}$. The former means that the function $s_N(z, \lambda)$ is well defined and analytic for all $z \in \overline{S_{r_0}}$ and $\lambda \in \mathrm{D}(\lambda_0, \varepsilon_N(\tau))$, while the latter means that inequality (2.3) holds for indicated values of λ. This completes the proof of Lemma 2.4. $\square$

Lemma 2.5. *Under the conditions of Lemma* 2.4, *the following equality holds*

$$A_N(z_0) := \left.\frac{\partial \log s_N(z_0, \lambda)}{\partial \lambda}\right|_{\lambda = \lambda_0} = \sum_{k=0}^{N-1} G\big(\lambda_0^k \varphi(z_0)\big), \quad z_0 \in \mathcal{L}_{r_0}, \tag{2.10}$$

where

$$G(\xi) := \frac{u\big(\psi(\xi)\big)}{\lambda_0 \xi \psi'(\lambda_0 \xi)}.$$

Proof. Consider the following function of $n+1$ independent variables

$$g_n(z; \lambda_1, \ldots, \lambda_n) := \begin{cases} \big(f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}\big)(z), & n \in \mathbb{N}, \\ z, & n = 0. \end{cases}$$

Note that

$$A_N(z_0) = \frac{\varphi'\left(f_{\lambda_0}^N(z_0)\right)}{s_N(z_0, \lambda_0)} \cdot \left.\frac{\partial g_N(z_0; \lambda, \ldots, \lambda)}{\partial \lambda}\right|_{\lambda = \lambda_0} \qquad \text{and}$$

$$\left.\frac{\partial g_N(z_0; \lambda, \ldots, \lambda)}{\partial \lambda}\right|_{\lambda = \lambda_0} = \sum_{k=0}^{N-1} g'_{N,k+1}(z_0; \lambda_0, \ldots, \lambda_0),$$

where $g'_{n,j}$ stands for $(\partial/\partial\lambda_j) g_n$. Using the equality

$$g_N(z; \lambda_1, \ldots, \lambda_n) = g_{N-j}\Big(f_{\lambda_j}\big(g_{j-1}(z; \lambda_1, \ldots, \lambda_{j-1})\big); \lambda_{j+1}, \ldots, \lambda_N\Big),$$

we get

$$g'_{N,k+1}(z_0;\lambda_0,\dots,\lambda_0) = \bigl(f_{\lambda_0}^{N-k-1}\bigr)'\bigl(f_{\lambda_0}^{k+1}(z_0)\bigr)\cdot u\bigl(f_{\lambda_0}^{k}(z_0)\bigr).$$

Schröder equation (1.1) for $f := f_{\lambda_0}$ allows us to express $f_{\lambda_0}^j$ and $\bigl(f_{\lambda_0}^j\bigr)'$ in terms of φ and ψ. Denoting $z_j := f_{\lambda_0}^j(z_0)$, $j \in \mathbb{N}_0$, we obtain

$$\begin{aligned} g'_{N,k+1}(z_0;\lambda_0,\dots,\lambda_0) &= \lambda_0^{N-k-1}\psi'\bigl(\lambda_0^{N-k-1}\,\varphi(z_{k+1})\bigr)\varphi'(z_{k+1})\,u(z_k)\\ &= \lambda_0^{N-k-1}\psi'\bigl(\lambda_0^{N-k-1}\,\varphi(z_{k+1})\bigr)\,\frac{u(z_k)}{\psi'\bigl(\varphi(z_{k+1})\bigr)}\\ &= \lambda_0^{N-k-1}\psi'\bigl(\lambda_0^{N}\varphi(z_0)\bigr)\,\frac{u\bigl(\psi\bigl(\lambda_0^k\varphi(z_0)\bigr)\bigr)}{\psi'\bigl(\lambda_0^{k+1}\varphi(z_0)\bigr)}. \end{aligned}$$

In the same way, we get

$$\frac{\varphi'\bigl(f_{\lambda_0}^N(z_0)\bigr)}{s_N(z_0,\lambda_0)} = \frac{1}{\psi'\bigl(\lambda_0^N\varphi(z_0)\bigr)\lambda_0^N\varphi(z_0)}.$$

Now one can combine the obtained equalities to deduce (2.10). □

Proposition 2.6. *Let $\tau > 0$ and $\Theta \in (0,\pi/2)$. If a function $v(\varsigma)$ is analytic in $\mathbb{D}$ and satisfies the following inequalities*

$$|v(0)|e^{-\tau} < |v(\varsigma)| < |v(0)|e^{\tau}, \quad \varsigma \in \mathbb{D}, \tag{2.11}$$

$$\vartheta := \bigl|\arg\{v'(0)/v(0)\}\bigr| + \Theta < \pi/2,$$

then the modulus of $t := \pi v'(0)/(4\tau v(0))$ does not exceed 1 and the following inequality holds

$$|v(\varsigma)| \ge |v(0)|, \quad \varsigma \in \Xi(\rho_0), \tag{2.12}$$

where $\Xi(\rho)$ stands for the circular sector $\{\varsigma : |\Im\mathrm{m}\,\varsigma| \le |\varsigma|\sin\Theta \le \rho\sin\Theta\}$ and $\rho_0 := \sqrt{\gamma^2+1}-\gamma$, $\gamma := (1-|t|^2)/(2|t|\cos\vartheta)$.

Proof. Replacing $v(\varsigma)$ with $v(\varsigma)/v(0)$, we can suppose that $v(0) = 1$. The multi-valued function

$$\phi(\xi) := h\left(\exp\left(\frac{i\pi\log\xi}{2\tau}\right)\right), \quad h(z) := -i\,\frac{z-1}{z+1},$$

maps the annulus $\{\xi : e^{-\tau} < |\xi| < e^{\tau}\}$ conformally onto $\mathbb{D}$ (in the sense of [24, p. 248]) and satisfies the conditions $\phi(1) = 0$, $\phi'(1) > 0$. Since the composition $f := \phi \circ v$ can be continued analytically along every path in $\mathbb{D}$, it defines an analytic function $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$. By the Schwarz lemma, $|f'(0)| \le 1$. Since $f'(0) = \phi'(1)v'(0) = \pi v'(0)/(4\tau) = t$, the first part of Proposition 2.6 is proved. To prove the remaining part we note that (2.12) is equivalent to the inequality $\Re\mathrm{e}\, f(\varsigma) \ge 0$. Applying the invariant form of the Schwarz lemma to $f(z)/z$, we obtain

$$\left|\frac{f(\varsigma) - f'(0)\varsigma}{\varsigma - \overline{f'(0)}f(\varsigma)}\right| \le |\varsigma|, \quad \varsigma \in \mathbb{D},$$

It follows that $f(\varsigma)$ lies in the closed disk of radius $R := |\varsigma|^2(1-|t|^2)/(1-|t\varsigma|^2)$ centred at $\sigma_0 := t\varsigma(1-|\varsigma|^2)/(1-|t\varsigma|^2)$. Therefore for the inequality $\Re\mathrm{e}\, f(\varsigma) \geq 0$ to be satisfied, it is sufficient that $\Re\mathrm{e}\,\sigma_0 \geq R$. An easy calculation leads to the following condition

$$\cos(\arg t + \arg\varsigma) \geq \frac{|\varsigma|(1-|t|^2)}{|t|(1-|\varsigma|^2)},$$

which is satisfied for all points of the arc

$$l(\rho) := \{\varsigma : |\Im\mathrm{m}\,\varsigma| \leq |\varsigma|\sin\Theta = \rho\sin\Theta\}, \quad \rho \in (0,1),$$

provided

$$\cos\vartheta \geq \frac{\rho(1-|t|^2)}{|t|(1-\rho^2)}. \tag{2.13}$$

The right-hand of (2.13) increases with $\rho \in (0,1)$ and $\rho := \rho_0$ satisfies (2.13). Therefore inequality (2.12) holds for all $\varsigma \in \bigcup_{\rho\in[0,\rho_0]} l(\rho) = \Xi(\rho_0)$. This completes the proof of Proposition 2.6. □

Proof of Lemma 2.2. Consider the function $s_N(z,\lambda)$ introduced in Lemma 2.4. This lemma states that $s_N(z,\lambda)$ is well defined and analytic for all $z \in \overline{S_{r_0}}$ and $\lambda \in \mathrm{D}(\lambda_0, \varepsilon_N(\tau))$ and satisfies inequality (2.3). According to Remark 1.4, $f_{\lambda_0}(\mathcal{L}_r) = \mathcal{L}_r$ for all $r \in [0,1)$. Consequently $|s_N(z,\lambda_0)| = |\varphi(z)|$, $z \in S$. Therefore for any $z_0 \in \mathcal{L}_{r_0}$ the function $v(\varsigma) := 1/s_N\big(z_0, \lambda_0(1-\varepsilon_N(\tau)\varsigma)\big)$ is analytic in $\mathbb{D}$ and satisfies inequality (2.11).

Let us employ now Proposition 2.6. To this end we compute the logarithmic derivative of $v(\varsigma)$ at $\varsigma = 0$. By Lemma 2.5,

$$\frac{v'(0)}{v(0)} = \lambda_0\varepsilon_N(\tau)A_N(z_0) = \lambda_0\varepsilon_N(\tau)\sum_{k=0}^{N-1} G\big(\lambda_0^k\varphi(z_0)\big).$$

Consider the sum $E_N := \sum_{k=0}^{N-1} G(\lambda_0^k\varphi(z_0))/N$. It can be regarded as an approximate value of the integral $E_* := \int_0^1 G\big(r_0 e^{2\pi i(t+t_0)}\big)\,dt$, where $t_0 \in \mathbb{R}$ is an arbitrary number, which does not affect E_*:

$$E_* = \frac{1}{2\pi i}\int\limits_{|\xi|=r_0} \frac{G(\xi)}{\xi}\,d\xi = \operatorname*{Res}_{\xi=0}\frac{G(\xi)}{\xi} = G(0) = \frac{1}{\lambda_0}.$$

Applying Theorem B to the points $x_n := x_n^\beta$, $\beta := (\arg\varphi(z_0))/(2\pi) - t_0$, and the function $\phi(t) := G\big(r_0 e^{2\pi i(t+t_0)}\big)$, we get the following estimate

$$|E_N - E_*| < Q_{\beta,N}\int_0^1 \big|(d/dt)G\big(r_0 e^{2\pi i(t+t_0)}\big)\big|\,dt.$$

Since $t_0 \in \mathbb{R}$ is arbitrary real, we have

$$|E_N - E_*| \leq Q_N\int_0^1 \big|(d/dt)G\big(r_0 e^{2\pi i(t+t_0)}\big)\big|\,dt. \tag{2.14}$$

The function under the sign $\int_0^1 |\cdot|dt$ is

$$\frac{dG\big(r_0 e^{2\pi i(t+t_0)}\big)}{dt} = 2\pi i\, \xi G'(\xi) = 2\pi i\, J(t+t_0), \quad \xi := r_0 e^{2\pi i(t+t_0)}.$$

From (2.14) it follows that

$$\left| \frac{1}{N} \cdot \sum_{k=0}^{N-1} G(\lambda_0^k \varphi(z_0)) - \frac{1}{\lambda_0} \right| \le a_N,$$

and hence,

$$\left| \frac{1}{N} \cdot \frac{v'(0)}{v(0)} - \varepsilon_N(\tau) \right| \le a_N \varepsilon_N(\tau). \tag{2.15}$$

Since by condition $0 \le a_N < 1$, inequality (2.15) implies that

$$\left| \frac{v'(0)}{v(0)} \right| \ge N(1-a_N)\varepsilon_N(\tau), \tag{2.16}$$

$$\left| \arg \frac{v'(0)}{v(0)} \right| \le \arcsin a_N. \tag{2.17}$$

Now if we recall that validity of (2.11) has been already verified and take into account (2.16), (2.17), we see that the conditions of Proposition 2.6 are satisfied. Therefore, by elementary reasoning we see that (2.12) holds for all $\varsigma \in \Xi\big(\Lambda_N\big(\tau, \varepsilon_N(\tau)\big)\big)$. In terms of s_N this means that

$$|s_N(z_0,\lambda)| \le |s_N(z_0,\lambda_0)| = r_0, \quad \lambda \in \Xi_0, \tag{2.18}$$

where

$$\Xi_0 := \big\{\lambda : \big|\lambda - \lambda_0\big| < \varepsilon_*, \big|\arg(1-\lambda/\lambda_0)\big| < \Theta\big\}.$$

Since $z_0 \in \mathcal{L}_{r_0} = \partial S_{r_0}$ is arbitrary in the above arguments, by the maximum modulus theorem, inequality (2.18) implies that $|\varphi(f_\lambda^N(z))| < r_0$ for all $z \in S_{r_0}$ and $\lambda \in \Xi_0$. Therefore for indicated values of λ we have $f_\lambda^N\big(S_{r_0}\big) \subset S_{r_0}$. This completes the proof of Lemma 2.2. □

2.2. Proof of Theorem 1.2

Suppose that the sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfies the conditions of Theorem 1.2. Then every subsequence of f_n also meets these conditions. So we have only to prove that $S := \mathcal{A}^*(0, f_0, U)$ is the kernel of the sequence $A_n := \mathcal{A}^*(0, f_0, U)$, that is:

(i) any compact set $K \subset S$ lies in all but a finite number of A_n's;
(ii) S is the largest domain that contains $z = 0$ and satisfies condition (i).

Now we employ Lemma 2.2 in order to prove (i). To this end we should fix any $r_0 \in (0,1)$ such that $S_{r_0} \supset K$, specify appropriate values of N and τ, and trace the dependence on the choice of n_*. As a result we would prove that

$$\varepsilon_*^0 := \inf_{n_* \in \mathbb{N}} \varepsilon_* > 0. \tag{2.19}$$

Since $\lambda_n \to \lambda_0$ as $n \to +\infty$, (2.19) would imply that $K \subset S_{r_0} \subset \mathcal{A}^*(z_0, f_n, U)$ for all $n \in \mathbb{N}$ large enough.

Set $\tau := (1+r_0)/(2r_0)$. In view of condition (ii) of Theorem 1.2,

$$L := \sup_{n_*\in\mathbb{N}} \left(\int_0^1 |J(t)|\, dt\right) < +\infty.$$

Since by Remark 2.1, $Q_N \to 0$ as $N \to +\infty$, there exists $N \in \mathbb{N}$ such that

$$Q_N < \frac{\sin(\pi/4 - \Theta/2)}{2\pi L}.$$

Fix any such value of N. Then $a_N < \sin(\pi/4 - \Theta/2) < \sin(\pi/2 - \Theta)$. Hence Lemma (2.2) is applicable to the specified values of N and τ.

Let us estimate ε_* from below. In view of condition (ii) of Theorem 1.2,

$$\varepsilon_0 := \inf_{n_*\in\mathbb{N}} \varepsilon_N(\tau) > 0.$$

Denote $b := \pi\varepsilon_N(\tau)N(1-a_N)/(4\tau)$, $b_1 := \min\{1, b\}$.

Since $\vartheta = \Theta + \arcsin a_N < \pi/4 + \Theta/2 < \pi/2$, we have

$$\begin{aligned}
\Lambda_N\big(\tau, \varepsilon_N(\tau)\big) &\geq \frac{\sqrt{1 + 2b_1^2\cos 2\vartheta + b_1^4} - 1 + b_1^2}{2b_1\cos\vartheta} \\
&\geq \frac{\sqrt{1 + 2b_1^2\cos 2\vartheta + (b_1^2\cos 2\vartheta)^2} - 1 + b_1^2}{2b_1\cos\vartheta} \\
&> b_1\cos\vartheta > b_1\cos(\pi/4 + \Theta/2) \\
&\geq \cos(\pi/4 + \Theta/2)\min\left\{1, \frac{\pi\varepsilon_0 N\big(1 - \sin(\pi/4 - \Theta/2)\big)}{4\tau}\right\} =: C_0.
\end{aligned}$$

The constant C_0 is positive and does not depend on n_*. From the inequality $\varepsilon_* > \varepsilon_0 C_0$ it follows that (2.19) takes place. This proves assertion (i).

To prove (ii) let us assume the converse. Then there exists a domain $S' \not\subset S$, $0 \in S'$, satisfying (i). Let $z_0 \in S' \setminus S$ and $\Gamma \subset S'$ be a curve that joins points $z = 0$ and z_0. Consider any domain D such that $\Gamma \subset D$ and $K := \overline{D} \subset S'$. By the assumption, $K \subset A_n$ for all n large enough. Now we claim that

$$D \subset E(f_0, U). \tag{2.20}$$

Consider an arbitrary $\zeta_0 \in D$. Suppose that $\zeta_0 \notin E(f_0, U)$. Then there exists $j_0 \in \mathbb{N}$ such that $f_0^{j_0}$ is well defined (and so analytic) in some domain $D_0 \ni \zeta_0$, $D_0 \subset D$, with $f_0^j(\zeta_0) \in U$, $j < j_0$, but $f_0^{j_0}(\zeta_0) \notin U$. Since the sequence f_n converges to f_0 uniformly on each compact subset of U, the sequence $f_n^{j_0}$ converges to $f_0^{j_0}$ uniformly on each compact subset of D_0. According to Hurwitz's theorem, this means that $f_n^{j_0}(D_0) \not\subset U$ for all $n \in \mathbb{N}$ large enough. Consequently, $D \not\subset E(f_n, U)$ for large n. At the same time, $K = \overline{D} \subset A_n \subset E(f_n, U)$ for all n large enough. This contradiction proves (2.20).

The remaining part of the proof depends on the properties of the domain U. Since $U \subset \mathbb{C}$, we have three possibilities:

(Hyp) The domain U is hyperbolic. Then by Montel's criterion, $\mathcal{F}(f_0, U)$ coincides with the interior of $E(f_0, U)$. Since $D \ni 0$ is connected, we conclude that $z_0 \in \Gamma \subset D \subset S$. With this fact contradicting the assumption, the proof of (ii) for the hyperbolic case is completed.

(Euc) The domain U coincides with $\mathbb{C}$. The functions f_n, $n \in \mathbb{N}_0$, are entire functions.

(Cyl) The domain U is the complex plane punctured at one point.

Let us prove (ii) for case (Euc). Since $\Gamma \cap \partial S \neq \emptyset$ and $\partial S \subset \mathcal{J}(f_0, \mathbb{C})$, we have $D \cap \mathcal{J}(f_0, \mathbb{C}) \neq \emptyset$. The classical result proved for entire functions by I.N. Baker [25] asserts that the Julia set coincides with the closure of the set of all repelling periodic points. Therefore, D contains a periodic point of f_0 different from 0. Owing to Hurwitz's theorem, the same is true for f_n provided n is large enough. This leads to a contradiction, because the immediate basin of attraction $\mathcal{A}^*(0, f_n, U)$ contains no periodic points except for the fixed point $z = 0$. Assertion (ii) is now proved for case (Euc).

It remains to consider case (Cyl). Similarly to case (Euc), we need only to show that $D \setminus \{0\}$ contains a periodic point. By means of linear transformations we can assume that $U = \mathbb{C} \setminus \{1\}$. From (2.20) it follows that functions

$$\phi_n(z) := \frac{f_0^n(z) - z}{f_0^n(z) - 1}, \quad n \in \mathbb{N},$$

does not assume values 1 and ∞ in D. Since $D \cap \mathcal{J}(f_0, U) \neq \emptyset$, the family $\{\varphi_n\}_{n \in \mathbb{N}}$ is not normal in D. Hence, due to Montel's criterion, there exists $z_1 \in D$ and $n_0 \in \mathbb{N}$ such that $\phi_{n_0}(z_1) = 0$ and so $z_1 \in D$ is a periodic point of f_0. This completes the proof of (ii) for case (Cyl).

By now (i) and (ii) are shown to be true. Theorem 1.2 is proved. □

2.3. Proof of Theorem 1.3

Fix any $r_0 \in (0, 1)$. As in the proof of Theorem 1.2 one can make use of Lemma 2.2 to show that there exist $n_1, N \in \mathbb{N}$ such that $f_n^N(S_{r_0}) \subset S_{r_0}$ for all $n > n_1$. By Remark 1.4 the function φ_0 maps S conformally onto a Euclidian disk centred at the origin. It is convenient to rescale the dynamic variable, by replacing f_k, $k \in \mathbb{N}_0$, with $r f_k(z/r)$ for some constant $r > 0$, so that $\varphi_0(S) = \mathbb{D}$ (or equivalently $\varphi_0 = \varphi$). Then the functions $g_n(\zeta) := (1/r_0)\big(\varphi_0 \circ f_n^N \circ \varphi_0^{-1}\big)(r_0\zeta)$, $n > n_1$, are defined and analytic in $\mathbb{D}$. Furthermore, $g_n(0) = 0$ and $g_n(\mathbb{D}) \subset \mathbb{D}$ for any $n > n_1$. Let us observe that for any analytic function f with a geometrically attractive or Siegel fixed point z_0 the Kœnigs function φ associated with the pair (z_0, f) is the same as that of the pair (z_0, f^N). Hence it is easy to see that the function $\phi_n(\zeta) := \varphi_n\big(\varphi_0^{-1}(r_0\zeta)\big)/r_0$ is the Kœnigs function associated with $(0, g_n)$. Since $S = \varphi_0^{-1}(\mathbb{D})$ and $r_0 \in (0, 1)$ is arbitrary, it suffices to prove that $\phi_n(\zeta) \to \zeta$ as $n \to +\infty$ uniformly on each compact subset of $\mathbb{D}$.

According to Remark 1.4, the function f_0 is a conformal automorphism of S. Therefore, with f_n converging to f_0 uniformly on each compact subset of $U \supset S$, there exists $n_2 \geq n_1$ such that for all $n > n_2$ functions f_n^N and consequently g_n are univalent in S_{r_0} and in $\mathbb{D}$, respectively. It follows (see, e.g., [26]) that ϕ_n, $n > n_2$, are also univalent in $\mathbb{D}$. The convergence of f_n to f_0 implies also that g_n converges to g_0, $g_0(\zeta) := \lambda_0^N \zeta$, uniformly on each compact subset of $\mathbb{D}$.

We claim that there exists a sequence $\{r_n \in (0,1)\}_{n\in\mathbb{N}}$ converging to 1 such that for all $n > n_2$ the domain $\phi_n(r_n\mathbb{D})$ is contained in some disk $\{\xi : |\xi| < R_n\}$ that lies in $\phi_n(\mathbb{D})$. Owing to the Carathéodory convergence theorem and normality of the family $\{\phi_n : n \in \mathbb{N}, n > n_2\}$, this statement would imply convergence of the sequence ϕ_n to the identity map and hence the proof of Theorem 1.3 would be completed.

By p/q and p'/q' let us denote some successive convergents of the number $\alpha_n := \big(\arg g_n'(0)\big)/(2\pi) = (\arg \lambda_n^N)/(2\pi)$ (regardless of whether α_n is irrational or not). Put $\Omega_n := \phi_n(\mathbb{D})$, $\kappa_n := -\log|g_n'(0)| = -N\log|\lambda_n|$, $a_n := \kappa_n(q-1)$, and $b_n := \pi(1/q + 2/q')$. Consider a point $\zeta_0 \in \mathbb{D}$ and make use of the following inequality (see, e.g., [24, p. 117, inequal. (18)]) from the theory of univalent function

$$\left|\log \frac{\zeta\phi_n'(\zeta)}{\phi_n(\zeta)}\right| \leq \log\frac{1+|\zeta|}{1-|\zeta|}, \quad \zeta \in \mathbb{D},$$

to obtain

$$\int_\Gamma \left|\frac{dw}{w}\right| \geq -\log\big(4k_\pi(|\zeta_0|)\big), \quad k_\pi(z) := \frac{z}{(1+z)^2}, \; z \in \mathbb{D}, \tag{2.21}$$

where Γ is any rectifiable curve that joins $\xi_0 := \phi_n(\zeta_0)$ with $\partial\Omega_n$ and lies in Ω_n except for one of the endpoints. The equality in (2.21) can occur only if ϕ_n is a rotation of the Koebe function $k_0(z) := z/(1-z)^2$ and Γ is a segment of a radial half-line. It follows that Ω_n contains the annular sector

$$\Sigma := \big\{\xi_0 e^{x+iy} : |x| \leq a_n,\, |x| \leq b_n,\, x, y \in \mathbb{R}\big\}$$

provided $|\zeta_0| \leq r_n := k_\pi^{-1}\big((1/4)\exp(-\sqrt{a_n^2+b_n^2})\big)$. Moreover, Ω_n is invariant under the map $\zeta \mapsto \lambda_n^N\zeta$. Indeed,

$$\lambda_n^N\zeta = \lambda_n^N\phi_n\big(\phi_n^{-1}(\zeta)\big) = \phi_n\big(g_n\big(\phi_n^{-1}(\zeta)\big)\big) \in \Omega_n$$

for all $\zeta \in \Omega_n$. Denote

$$\Sigma_0 := \big\{\xi_0 e^{x+iy} : |x| \leq \pi/q, |x| \leq b_n,\, x, y \in \mathbb{R}\big\}, \quad \lambda_* := e^{-\kappa + 2\pi i p/q}.$$

Since p and q are coprime integers, the union of the annular sectors $\lambda_*^j\Sigma_0$, $j = 0, 1, \ldots, q-1$, contains the circle $\xi_0\mathbb{T}$, $\mathbb{T} := \partial\mathbb{D}$. The inequality from the theory of continued fractions $|\alpha_n - p/q| \leq 1/(qq')$ implies that

$$\lambda_*^j\Sigma_0 \subset \big(\lambda_n^N\big)^j\Sigma, \quad j = 0, 1, \ldots, q-1.$$

Therefore, for any $\xi_0 \in \phi_n(r_n\mathbb{D})$ the domain Ω_n contains the circle $\xi_0\mathbb{T}$. It follows that $\phi_n(r_n\mathbb{D})$ is a subset of some disk $\{\xi : |\xi| < R_n\}$ contained in Ω_n.

It remains to choose the successive convergents p/q and p'/q' of α_n in such a way that $r_n \to 1$ as $n \to +\infty$. To this end we fix some successive convergents p/q and p'/q' of $\alpha_* := (\arg \lambda_0^N)/(2\pi)$ and note that p/q and p'/q' are also successive convergents of α_n provided n is large enough, because $\alpha_n \to \alpha_*$ as $n \to +\infty$. Using the fact that $\kappa_n \to 0$ as $n \to +\infty$ and that the denominators of convergents of the irrational number α_* forms unbounded increasing sequence, we see that it is possible to choose p/q for each n in such a way that $\sqrt{a_n^2 + b_n^2} \to 0$ and, consequently, $r_n \to 1$ as $n \to +\infty$. The proof of Theorem 1.3 is now completed. □

3. Proof of Theorem 1.5

In this section we sketch the proof of Theorem 1.5. First of all we note that the proof of Lemma 2.2 does not use the fact that the dependence of $f_\lambda[n_*]$ (see equation (2.1)) on the parameter λ is linear. So Lemma 2.2 can be applied to any analytic family f_λ satisfying conditions (i)–(iii) on page 168, provided some notations are modified to a new (more general) setting. First of all we have to redefine $u(z) := \partial f_\lambda(z)/\partial\lambda|_{\lambda=\lambda_0}$. Then fix any $r \in (0,1)$ and consider the modulus of continuity for the family $h_\lambda := f_\lambda/f_{\lambda_0}$ calculated at $\lambda = \lambda_0$,

$$\omega_r(\delta) := \sup\Big\{\big|1 - f_\lambda(z)/f_{\lambda_0}(z)\big| : z \in S_r, \lambda \in W \cap \mathrm{D}(\lambda_0, \delta)\Big\}, \quad \delta > 0.$$

This quantity, as a function of δ, is defined, continuous, and increasing on the interval $I^* := (0, \delta^*)$, $\delta^* := \operatorname{dist}(\lambda_0, \partial W)$, with $\lim_{\delta\to+0} \omega_r(\delta) = 0$. Therefore there exists an inverse function $\omega_r^{-1} : (0, \epsilon^*) \to (0, +\infty)$, where $\epsilon^* := \lim_{\delta\to\delta^*-0} \omega_r(\delta)$. If $\epsilon^* \neq +\infty$, then we set $\omega_r^{-1}(\epsilon) := \delta^*$ for all $\epsilon \geq \epsilon^*$. Now we can redefine $\varepsilon_N(\tau)$ as

$$\varepsilon_N(\tau) := \omega_{r_*}^{-1}\big(1 - k_\pi(r_*)/k_\pi(r^*)\big), \quad r_* := r_0 e^{\tau(1-1/N)}, \ r^* := r_0 e^\tau.$$

Finally, define Θ to be equal to the half-angle of Δ. To apply Lemma 2.2 we need the following

Proposition 3.1. *For any $n \in \mathbb{N}$ the following inequality holds*

$$Q_{q_n} < (1/q_n + 1/q_{n+1})/2. \tag{3.1}$$

Proof. Fix $n \in \mathbb{N}$. Due to the inequality $|\alpha_0 - p_n/q_n| < 1/(q_n q_{n+1})$ there exists $\gamma \in (0, 1/q_{n+1})$ such that

$$|\alpha_0 - p_n/q_n| < \gamma/q_n. \tag{3.2}$$

Let $\beta_0 := \big(1/q - (-1)^n\gamma\big)/2$. Taking into account that p_n and q_n are coprime integers one can deduce by means of the inequalities $\gamma < 1/q_{n+1} < 1/q_n$, $(-1)^n(\alpha_0 - p_n/q_n) > 0$, and (3.2) that

$$Q_{\beta_0, q_n} < (1/q_n + 1/q_{n+1})/2. \tag{3.3}$$

This proves the proposition. □

Now let us show how Theorem 1.5 can be proved. Fix $r_0 \in (0,1)$. Define $\varepsilon(r_0)$ in the following way.

According to Proposition 3.1, $0 < a_N < \sin\big(\pi/4 - \theta/2\big)$ for

$$N := \ell\left(2\pi \int_0^1 \big|J(t)\big|\,dt \Big/ \sin\big(\pi/4 - \theta/2\big)\right), \quad \tau := \log\frac{1+2r_0}{3r_0}.$$

Hence, Lemma 2.2 can be used with the specified values of N and τ. Therefore, we can set $\varepsilon(r_0) := \varepsilon_*$, so that statement (i) in Theorem 1.5 becomes true. Let us show that statement (ii) of this theorem is also true, assuming that r_0 is sufficiently close to 1.

Since $\overline{S} \subset U$ there exists $C_1 > 0$ such that

$$\left|1 - \frac{f_\lambda(\psi(\xi))}{f_{\lambda_0}(\psi(\xi))}\right| < C_1|\lambda - \lambda_0|$$

for all $\xi \in \mathbb{D}$ and $\lambda \in \mathrm{D}(\lambda_0, \varepsilon^0)$, where $\varepsilon^0 > 0$ is chosen so that $\overline{\mathrm{D}(\lambda_0, \varepsilon^0)} \subset W$. It follows that $\omega_r^{-1}(s) \ge \min\big\{\varepsilon^0, s/C_1\big\}$, for all $s > 0$, $r \in (0,1)$. Elementary calculations show that

$$1 - \frac{k_\pi(r_*)}{k_\pi(r^*)} \ge 1 - \exp\left(-\frac{\tau(1-r^*)}{N(1+r^*)}\right) \ge C_2\frac{(1-r_0)^2}{N}$$

for some constant $C_2 > 0$. Combining these two inequalities we obtain $\varepsilon_N(\tau) \ge C_3(1-r_0)^2/N$, where $C_3 := C_2/C_1$. Now we estimate $\Lambda_N(\tau, \varepsilon_N(\tau))$ in the same way as in the proof of Theorem 1.2 to conclude that $\varepsilon_* \ge C(1-r_0)^3/N$ for some constant $C > 0$. To complete the proof we use the following inequalities (see, e.g., [24, p. 52]):

$$\left|\frac{\xi\psi''(\xi)}{\psi'(\xi)} - \frac{2r^2}{1-r^2}\right| \le \frac{4r}{1-r^2}, \quad 0 \le r = |\xi| < 1,$$

$$\left|\frac{\psi'(\xi)}{\psi'(0)}\right| \ge \frac{1-r}{(1+r)^3}, \quad 0 \le r = |\xi| < 1,$$

which imply that $N \le \ell\big((1-r_0)^{-\gamma}\big)$ for all $r_0 < 1$ sufficiently close to 1. □

4. Essentiality of conditions in Theorem 1.2

In this section we show that conditions (i) and (ii) in Theorem 1.2 are essential. As for condition (i) this can be regarded as a consequence of lower semi-continuity of the Julia set.

Example 1. Consider the family $f_\lambda(z) := \lambda z + z^2$ in the whole complex plane ($U := \mathbb{C}$). The map $\lambda \mapsto \mathcal{J}(f_\lambda, \mathbb{C})$ is lower semi-continuous [5], i.e.,

$$\mathcal{J}(f_{\lambda_*}, \mathbb{C}) \subset \bigcap_{\varepsilon>0} \bigcup_{\delta>0} \bigcap_{|\lambda-\lambda_*|<\delta} O_\varepsilon\big(\mathcal{J}(f_\lambda, \mathbb{C})\big) \text{ for any } \lambda_* \in \mathbb{C},$$

where $O_\varepsilon(\cdot)$ stands for the ε-neighbourhood of a set. Let $\lambda_0 := e^{2\pi i\alpha_0}$, $\alpha_0 \in \mathbb{R}\backslash\mathbb{Q}$, and $\alpha_n \in \mathbb{Q}$ converge to α_0 as $n \to +\infty$. The point $z_0 := 0$ is a parabolic fixed point of $f_{\lambda_n^0}$, $\lambda_n^0 := \exp(2\pi i\alpha_n)$, and so $0 \in \mathcal{J}(f_{\lambda_n^0}, \mathbb{C})$. Due to lower semi-continuity of $\lambda \mapsto \mathcal{J}(f_\lambda, \mathbb{C})$ at the points λ_n^0, there exists a sequence $\{\mu_n \in (0,1)\}_{n\in\mathbb{N}}$ such that $D(0,1/n)\cap\mathcal{J}(f_{\lambda_n},\mathbb{C})\neq\emptyset$, $\lambda_n:=\mu_n\lambda_n^0$, $n\in\mathbb{N}$. It follows that $\mathcal{A}^*(0,f_{\lambda_n},\mathbb{C})\to\{0\}$ as to the kernel. Assume that f_{λ_0}, $\lambda_0:=\exp(2\pi i\alpha_0)$, has a Siegel point at $z_0=0$. This is the case if α_0 is a Brjuno number ([14, Th. 6], see also [15]). The sequence $f_n:=f_{\lambda_n}$ satisfies all conditions of Theorem 1.2 except for condition (i), but the conclusion of Theorem 1.2 fails to be true. Therefore condition (i) is an essential one.

It is known [27, p. 44] that condition (ii) can be omitted in Theorem 1.2 provided that the multiplier of the Siegel fixed point $\lambda_0:=f_0'(z_0)$ equals to $\exp(2\pi i\alpha_0)$ for some Brjuno number α_0. However, if no such assumptions concerning α_0 are made, condition (ii) cannot be omitted. This fact is demonstrated by the following

Example 2. Let α_0 be an irrational real number. By q_n denote the denominator of the nth convergent of α_0. Consider the sequence of polynomials

$$f_n(z):=\frac{\lambda_0\left(z+z^{q_n+1}\right)}{1+1/2^{q_n}},\quad \lambda_0:=e^{2\pi i\alpha_0},$$

converging to $f_0(z)=\lambda_0 z$ uniformly on each compact subset of $\mathbb{D}$.

We claim that the sequence of domains $\mathcal{A}^*(0,f_n,\mathbb{D})$ does not converge to $\mathcal{A}^*(0,f_0,\mathbb{D})=\mathbb{D}$ as to the kernel, provided the growth of q_n is sufficiently rapid. Assume the converse. Then for all $n\in\mathbb{N}$ large enough, say for $n>n_0$, the inclusion $D_{48}\subset\mathcal{A}^*(0,f_n,\mathbb{D})$ holds, where $D_j:=j/(j+1)\mathbb{D}$, $j\in\mathbb{N}$. It follows that $f_n^m(D_{48})\subset\mathbb{D}$ for all $n>n_0$, $n\in\mathbb{N}$, and $m\in\mathbb{N}$. Hence the family $\Phi:=\{f_n^m\}_{n>n_0,\,n,m\in\mathbb{N}_0}$ is normal in the disk D_{48}. In particular, there exist constants $C_1>1$, $C_2>0$ such that

$$\left|(f_n^m)'(z)\right|<C_1,\quad z\in D_8,\quad n>n_0,\ n,m\in\mathbb{N}_0, \tag{4.1}$$

$$\left|(f_n^m)''(z)\right|<C_2,\quad z\in D_8,\quad n>n_0,\ n,m\in\mathbb{N}_0. \tag{4.2}$$

Furthermore, by the Schwarz lemma,

$$f_n^m(D_4)\subset D_5,\ f_n^m(D_6)\subset D_7\quad n>n_0,\ n,m\in\mathbb{N}_0, \tag{4.3}$$

Consider functions $g_n:=f_n^{q_n}$, $\tilde{g}_n:=\tilde{f}_n^{q_n}$,

$$\tilde{f}_n(z):=\frac{\exp\left(2\pi i p_n/q_n\right)}{1+1/2^{q_n}}\left(z+z^{q_n+1}\right),\quad z\in\mathbb{D},\quad n>n_0,\quad n\in\mathbb{N},$$

where p_n stands for the numerator of the nth convergent of α_0. Apply the following inequality

$$\left|\tilde{f}_n(z)-f_n(z)\right|=\left|f_n(z)\right|\cdot\left|\lambda_0-\exp\left(2\pi i p_n/q_n\right)\right| \le 4\pi\left|\alpha-\frac{p_n}{q_n}\right|\le\frac{4\pi}{q_nq_{n+1}},\quad z\in\mathbb{D}, \tag{4.4}$$

to prove that

$$\left|\tilde{g}_n(z)-g_n(z)\right|<\frac{4\pi C_1}{q_{n+1}},\quad z\in D_4, \tag{4.5}$$

for all $n\in\mathbb{N}$ large enough.

Since $q_n \to +\infty$ as $n \to +\infty$, there exists $n_1 \in \mathbb{N}$, $n_1 \geq n_0$, such that

$$\frac{4\pi}{q_n q_{n+1}} < \frac{1}{72} \quad \text{and} \quad \frac{4\pi C_1}{q_{n+1}} < \frac{1}{42}, \qquad n > n_1, \quad n \in \mathbb{N}.$$

We shall show that for all $n > n_1$, $n \in \mathbb{N}$, and $k = 1, 2, \ldots, q_n - 1$ the following implication holds

$$\Big(P(j),\ j = 1, 2, \ldots, k\Big) \Longrightarrow P(k+1), \tag{4.6}$$

where

$$\begin{aligned} P(j): \Big[\ &\big|\tilde{f}_n^{j-1}(z)\big| < 1, \quad z \in D_4, \quad \text{and} \\ &\big|\tilde{f}_n^{j}(z) - f_n^{j}(z)\big| < \frac{4j\pi C_1}{q_n q_{n+1}}, \quad z \in D_4.\ \Big] \end{aligned} \tag{4.7}$$

Now let $n > n_1$ and $P(j)$ take place for all $j = 1, 2, \ldots, k$. Relations (4.3), (4.4), and (4.7) imply the following inclusions

$$\tilde{f}_n(D_6) \subset D_8, \qquad \tilde{f}_n^{j}(D_4) \subset D_6, \quad j = 1, 2, \ldots, k. \tag{4.8}$$

For $j := k$ the latter guarantees that $|\tilde{f}_n^{k}(z)| < 1$, $z \in D_4$. Fix any $z \in D_4$ and denote $w_j := \tilde{f}_n^{j}(z)$, $\tilde{\xi}_j := \tilde{f}_n(w_j)$, $\xi_j := f_n(w_j)$. According to (4.3) and (4.8), we have $w_j \in D_6$, $\tilde{\xi}_j, \xi_j \in D_8$, $j = 1, 2, \ldots, k$. Taking this into account, from (4.1) and (4.4), we get the following inequality

$$\begin{aligned} \Big|\tilde{f}_n^{k+1}(z) - f_n^{k+1}(z)\Big| &\leq \sum_{j=0}^{k} \Big|\big(f_n^{k-j} \circ \tilde{f}_n^{j+1}\big)(z) - \big(f_n^{k-j+1} \circ \tilde{f}_n^{j}\big)(z)\Big| \\ &= \sum_{j=0}^{k} \Big|\big(f_n^{k-j} \circ \tilde{f}_n\big)(w_j) - \big(f_n^{k-j} \circ f_n\big)(w_j)\Big| \\ &= \sum_{j=0}^{k} \Big|f_n^{k-j}(\tilde{\xi}_j) - f_n^{k-j}(\xi_j)\Big| < \sum_{j=0}^{k} C_1|\tilde{\xi}_j - \xi_j| \leq \frac{4(k+1)\pi C_1}{q_n q_{n+1}}. \end{aligned}$$

Therefore, (4.7) holds also for $j := k + 1$. This proves implication (4.6).

For $j := 1$ inequality (4.7) follows from (4.4). Hence $P(1)$ is valid. Owing to (4.6), $P(1)$ implies $P(q_n)$. Therefore, inequality (4.5) holds for all $n > n_1$.

The functions $\tilde{g}_n$ have the fixed point $\tilde{z}_* := 1/2$. Now we apply (4.5) to show that if

$$q_{n+1} \geq 2^{q_n}, \qquad n \in \mathbb{N}, \tag{4.9}$$

then for any sufficiently large $n \in \mathbb{N}$ the function g_n has also a fixed point $z_* \in D_3 \setminus \{0\}$. Straightforward calculation gives

$$\tilde{g}_n'(\tilde{z}_*) = l_n := \left(\frac{1 + (q_n + 1)/2^{q_n}}{1 + 1/2^{q_n}}\right)^{q_n} > 1.$$

From (4.2), (4.5), and the Cauchy integral formula it follows that

$$|\tilde{g}_n''(z)| < C_3 := C_2 + 51200\pi C_1/q_{n+1}, \quad z \in D_3, \quad n > n_1.$$

Now we assume that $n \in \mathbb{N}$ is large enough and apply Rouché's theorem to the functions $\tilde{g}_n(z) - z$ and $g_n(z) - z$ in the disk $B_n := \{z : |z - 1/2| < \rho_n\}$, where $\rho_n := (l_n - 1)/(2C_3)$. Since $B_n \subset D_3$, we have

$$\Re\mathrm{e}\, \frac{d}{dz}\Big(\tilde{g}_n(z) - z\Big) > \frac{l_n - 1}{2}, \qquad z \in B_n.$$

It follows that

$$|\tilde{g}_n(z) - z| \geq \frac{(l_n - 1)\rho_n}{2}, \qquad z \in \partial B_n. \tag{4.10}$$

Inequalities (4.5), (4.9), and (4.10) imply that $|\tilde{g}_n(z) - z| > |\tilde{g}_n(z) - g_n(z)|$ for all $z \in \partial B_n$. Consequently, $g_n(z) - z$ vanishes at some point $z_* \in B_n$. At the same time, the immediate basin $\mathcal{A}^*(0, f_n, \mathbb{D})$ contains no periodic points of f_n except for the fixed point at $z_0 = 0$. Therefore, $D_3 \not\subset \mathcal{A}^*(0, f_n, \mathbb{D})$ for large n. This fact implies that the sequence $\mathcal{A}^*(0, f_n, \mathbb{D})$ does not converge to $\mathbb{D}$ as to the kernel.

It is easy to see that the prescribed sequence f_n satisfies all conditions of Theorem 1.2 with $U := \mathbb{D}$ except for condition (ii), but the conclusion fails to hold. This shows that (ii) is also an essential condition in Theorem 1.2.

References

[1] Bergweiler W., An introduction to complex dynamics, *Textos de Matematica Serie B*, **6**, Universidade de Coimbra, 1995.

[2] Eremenko A., Lyubich M., The dynamics of analytic transformations, *Leningr. Math. J.*, **1** (1990), No. 3, 563–634 (in English); Russian original: *Algebra i analiz*, **1** (1989), No. 3, 1–70.

[3] Bergweiler W., Iteration of meromorphic functions, *Bull. Amer. Math. Soc., NS*, **29** (1993), No. 2, 151–188.

[4] Baker I.N., Domínguez P., Herring M.E., Dynamics of functions meromorphic outside a small set, *Ergod. Th. & Dynam. Sys.*, **21** (2001), No. 3, 647–672.

[5] Douady A., Does a Julia set depend continuously on the polynomial? Proc. Symp. in Appl. Math., **49** (1994) ed. R. Devaney, 91–138.

[6] Kisaka M., Local uniform convergence and convergence of Julia sets, *Nonlinearity*, **8** (1995), No. 2, 237–281.

[7] Kriete H., Repellors and the stability of Julia sets, *Mathematica Gottingensis*, 04/95 (1995), http://www.uni-math.gwdg.de/preprint/mg.1995.04.ps.gz.

[8] Krauskopf B., Kriete H., A note on non-converging Julia sets, *Nonlinearity*, **9** (1996), 601–603.

[9] Kriete H., Continuity of filled-in Julia sets and the closing lemma, *Nonlinearity*, **9** (1996), No. 6, 1599–1608.

[10] Krauskopf B., Kriete H., Hausdorff Convergence of Julia Sets, *Bull. Belg. Math. Soc.*, **6** (1999), No. 1, 69–76.

[11] Wu Sh., Continuity of Julia sets, *Sci. China, Ser. A*, **42** (1999), No. 3, 281–285.

[12] Kriete H., Approximation of indifferent cycles, *Mathematica Gottingensis*, 03/96 (1996), http://www.uni-math.gwdg.de/preprint/mg.1996.03.ps.gz.

[13] Siegel C., Iteration of analytic function, *Ann. of Math.*, **43** (1942), No. 2, 607–612.

[14] Brjuno A., Analytic forms of differential equations, *Trans. Mosc. Math. Soc.*, **25** (1971), 131–288.

[15] Yoccoz J.C., Petits diviseurs en dimension 1, *Astérisque*, **231** (1995) (in French).

[16] Milnor J., *Dynamics in one complex variable. Introductory lectures*, 2nd edition, Vieweg, 2000.

[17] Bargmann D., Conjugations on rotation domains as limit functions of the geometric means of the iterates, *Ann. Acad. Sci. Fenn. Math.*, **23** (1998), No. 2, 507–524.

[18] Buff X., Petersen C., On the size of linearization domains, *Preprint* (2007) http://www.picard.ups-tlse.fr/~buff/Preprints/Preprints.html.

[19] Bukhshtab A. A., *Theory of numbers*, "Prosvyascheniye", Moscow, 1966 (in Russian).

[20] Coppel W. A., *Number Theory. Part A*, Springer, 2006.

[21] Hedenmalm H., Shimorin S., Weighted Bergman spaces and the integral means spectrum of conformal mappings, *Duke Mathematical Journal* **127** (2005), 341–393.

[22] Gumenuk P.A., Siegel disks and basins of attraction for families of analytic functions, *Izvestiya Saratovskogo Universiteta, NS, Ser. Matem. Mekh. Inform.*, **5** (2005), No. 1, 12–26 (in Russian).

[23] Sobol' I.M., *Multidimensional quadrature formulas and Haar functions*, "Nauka", Moscow, 1969 (in Russian).

[24] Goluzin G.M., *Geometric theory of functions of a complex variable*, 2^{nd} ed., "Nauka", Moscow, 1966 (in Russian); German transl.: Deutscher Verlag, Berlin, 1957; English transl.: Amer. Math. Soc., 1969.

[25] Baker I.N., Repulsive fixpoints of entire functions, *Math. Z.*, **104** (1968), 252–256.

[26] Goryainov V., Kœnigs function and fractional iterates of probability generating functions, *Sb. Math.,* **193** (2002), No. 7, 1009–1025; translation from *Mat. Sb.* **193** (2002), No. 7, 69–86.

[27] Buff X., Disques de Siegel & Ensembles de Julia d'aire strictement positive, *Diplôme D'Habilitation à diriger des Recherches,* Univ. Paul Sabatier, Toulouse (in French), http://www.picard.ups-tlse.fr/~buff.

Pavel Gumenyuk
Department of Mathematics, University of Bergen
Johannes Brunsgate 12,
5008 Bergen, Norway

and

Department of Mechanics and Mathematics
Saratov State University,
Astrakhanskaya 83
410012 Saratov, Russia
e-mail: Pavel.Gumenyuk@math.uib.no
Gumenuk@sgu.ru