OUTAGE PROBABILITY OF SELECTION COMBINING IN AN EXPONENTIALLY CORRELATED LOGNORMAL SHADOWING ENVIRONMENT

Fadel F. Digham and Mohamed-Slim Alouini

Department of Electrical and Computer Engineering
University of Minnesota
Minneapolis, MN 55455, USA
E-mails: <fdigham,alouini@ece.umn.edu>

ABSTRACT
Outage probability of selection combining (SC) in a correlated lognormal shadowing environment is studied. The uplink scenario is addressed and the shadowed signals at the base stations are assumed to be exponentially correlated. The result presented for such a correlation model is limited to three base stations. The effect of correlation on the outage probability is studied and the numerical results show that the outage probability is sensitive to the variations of the correlation coefficient in its entire range \(0 \leq p \leq 1\). The difference in the system performance from the outage probability perspective at the two limiting cases of zero and full correlation is up to two order of magnitude. Finally, the outage probability of SC diversity systems with two and three antennas are compared for different scenarios of practical interest.

1. INTRODUCTION
In radio mobile applications, the propagation environment is characterized by three nearly independent factors: path loss, shadowing and multipath fading [1]. The first is a deterministic effect which manifests itself as a decay of the average power at the receiver with the distance. The second yields a slow variation of the signal due to large obstacles such as buildings and mountains which might exist between a mobile station (MS) and a base station (BS) in a mobile network. The third represents a faster scale of signal variations than those in case due to shadowing.

Many diversity combining schemes are then used at the receiving part to mitigate the two latter effects; namely shadowing and fading. For instance, macro-diversity schemes in which the signal is captured or transmitted from different BSs are used to combat the effect of shadowing. These schemes are typically implemented in the form of selection combining (SC) in which the diversity path with the maximum received signal to noise ratio (SNR) is selected. One of the most important criterion to evaluate the performance of such a scheme is the outage probability which is defined as the probability that the received signals on all diversity paths fall below a predefined threshold. In this paper, we are interested in evaluating the outage probability of SC diversity scheme in a correlated lognormal shadowing environment. Previous work in this topic assumed independent or uniformly correlated shadowing samples at the different radio ports [2]. In this paper, we rather consider an exponential correlation model which has been verified by Gudmundson [3].

This paper is organized as follows. In section 2, the system model is described. The outage probability is calculated in section 3. Numerical examples are given in section 4. Finally, concluding remarks are presented in section 5.

2. SYSTEM MODEL
We focus on an exponentially correlated shadowing environment which is of great interest in certain applications such as communication along a highway. The system under study consists of a linear array of BSs on a highway in which a MS is moving. The up-link (from
MS to BS) scenario is addressed. The mobile is assumed to be connected to the three closest BSs. These BSs are reliably connected (through wireline connection) to a base station controller (BSC) at which SC is performed. The shadowed signal power (local mean) $\Omega$ follows a lognormal distribution when expressed in normal units or equivalently follows a Gaussian probability density function (PDF) when expressed in dB units [1]. Hence,

$$f_{\Omega(dB)}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad (1)$$

where $\mu$ is the area mean in dB and $\sigma$ is the shadow standard deviation in dB. Moreover, the samples of the shadowing process are considered to be exponentially correlated [3]. For our setup, the exponential correlation model is exact at the two ends of the highway (i.e., when the mobile is getting into or leaving the highway) as shown in part (a) of figure 1 while it will represent a lower bound on the system performance when the mobile is in between the two ends as in parts (b) and (c) in figure 1.

3. OUTAGE PROBABILITY

The system is considered to be in outage if the received local mean $\Omega_i$ ($i = 1, 2, 3$) at each operating BS (we consider here the three closest ones to the MS) falls below a certain threshold $\lambda_i$. Therefore, the outage probability $P_{out}$ can be written as

$$P_{out} = p(\Omega_1 < \lambda_1, \Omega_2 < \lambda_2, \Omega_3 < \lambda_3) \quad (2)$$

$$= \int_{-\infty}^{\lambda_1} \int_{-\infty}^{\lambda_2} \int_{-\infty}^{\lambda_3} f(\Omega_1, \Omega_2, \Omega_3) \, d\Omega_1 d\Omega_2 d\Omega_3, \quad (3)$$

where $f(\Omega_1, \Omega_2, \Omega_3)$ is the joint PDF of the three correlated Gaussian variates with correlation coefficients $\rho_{12}, \rho_{13}$ and $\rho_{23}$. To the best of our knowledge, evaluating the expression in (2) in a closed form for arbitrary correlation coefficients is still an open problem even for the case of three variates [4] which is of interest to us. To tackle this problem for the case in hand (exponential correlation), we follow the trick given in [5, Eq.33] and used also in [2] to evaluate the outage probability in the uniformly correlated case. Hence, we assume that $\rho_{ij}$ can be written in the form:

$$\rho_{ij} = \alpha_i \alpha_j, \quad i \neq j, \quad (4)$$

where $0 \leq \alpha_i \leq 1$. Then, we can generally decorrelate $N$ zero-mean unit-variance normal variates ($\Omega_i, i = 1, 2, \ldots, N$) by generating $N + 1$ independent zero-mean unit-variance normal variates $X_1, X_2, \ldots, X_N$ and $Y$ by the transformation (with $N = 3$ in our case) [5, Eq. 33]

$$\Omega_i = \sqrt{1 - \alpha_i^2}X_i + \alpha_i Y, \quad i = 1, 2, 3. \quad (5)$$

To compensate for the mean $\mu_i$ and standard deviation $\sigma_i$ in (5), the new $\Omega_i$ can be written as

$$\Omega_i = \sigma_i \left( \sqrt{1 - \alpha_i^2}X_i + \alpha_i Y \right) + \mu_i, \quad i = 1, 2, 3. \quad (6)$$

Substituting (6) into the basic definition of $P_{out}$ in (2) yields [5]

$$P_{out} = \int_{-\infty}^{\infty} \prod_{i=1}^{3} P(X_i < \zeta_i | Y = y)f(y)dy$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{3} F(\zeta_i)f(y)dy, \quad (7)$$

where

$$\zeta_i = \lambda_i - (\mu_i + \sigma_i \alpha_i y) / \sigma_i \sqrt{1 - \alpha_i^2}, \quad (8)$$

and $f(\cdot)$ and $F(\cdot)$ are the standard normal PDF and cumulative distribution function (CDF), respectively. The expression in (7) can be rewritten as

$$P_{out} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \prod_{i=1}^{3} [1 - Q(\zeta_i)] e^{-\frac{y^2}{2}} dy, \quad (9)$$

where $Q(\cdot)$ is the well-tabulated function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{t^2}{2}} dt.$$

The expression in (9) can be used for constant correlation model (by setting $\alpha_i = \sqrt{p}$) as it was done in [2]. Here, we are interested in the values of $\alpha_i$ to accomplish an exponential correlation model. In the case of three variates, the correlation matrix $C$ for such a model is given by

$$C = \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix}. \quad (10)$$

Due to the nature of the exponential model, we can assume the values of $\alpha_i$ to be

$$\alpha_i = \begin{cases} \rho^{ai+b} & i : \text{odd}, \\ \rho^{ai+d} & i : \text{even}. \end{cases} \quad (11)$$

Knowing that $\rho_{ij} = \alpha_i \alpha_j$ and using (10) and (11), we can solve for $a, b, c$ and $d$ and hence for $\alpha_i$ which can be obtained as

$$\alpha_i = \begin{cases} \rho & i = 1, 3, \\ 1 & i = 2. \end{cases} \quad (12)$$
Using these values of $\alpha_i$, one can now easily evaluate the $P_{\text{out}}$ in (9). Note that it might seem from (8) that we are not able to substitute for $\alpha_i = 1$. Fortunately, we can use $\alpha_i = 1 - \epsilon$ where $\epsilon$ is a very small positive number. As a check, in case of full correlation ($\rho = 1$), our expression of $P_{\text{out}}$ with $\alpha_i = 0.999999$ for all $i$ matches numerically the expression resulting from the full correlation ($\Omega_1 \equiv \Omega_2 \equiv \Omega_3 \equiv \Omega$) which is given by

$$P_{\text{out}1} = P(\Omega < \lambda) = 1 - Q \left( \frac{\lambda - \mu}{\sigma} \right),$$

(13)

where the subscript 1 refers to the full correlation case ($\rho = 1$).

In addition, when evaluating (9) with the zero correlation case, the resulting expression exactly matches the one obtained from the reduced expression

$$P_{\text{out}0} = P(\Omega < \lambda) = 1 - Q \left( \frac{\lambda - \mu}{\sigma} \right),$$

(14)

which holds for the independent diversity path case and in which the subscript 0 refers to zero correlation ($\rho = 0$) case.

Unfortunately, following the same approach (assuming $\rho_{ij} = \alpha_i \alpha_j$) does not provide a solution for the case of four exponentially correlated radio ports. This is simply because in such a case, the following system of equations

$$\begin{align*}
\alpha_1 \alpha_2 &= \rho, \\
\alpha_2 \alpha_3 &= \rho, \\
\alpha_3 \alpha_4 &= \rho, \\
\alpha_1 \alpha_3 &= \rho^2, \\
\alpha_2 \alpha_4 &= \rho^2, \\
\text{and} \quad \alpha_1 \alpha_4 &= \rho^3
\end{align*}$$

should be satisfied. However, there is no solution to satisfy this set of equations. Therefore, the approach presented here can not help evaluate $P_{\text{out}}$ for more than three exponentially correlated shadowing samples.

4. NUMERICAL EXAMPLES

For numerical purposes we choose $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. It is worth mentioning that the expression of $P_{\text{out}}$ in (9) is numerically stable and converges for small integrating limits (e.g., from -30 to 30).

Figure 2 illustrates the effect of the correlation coefficient $\rho$ on the calculated $P_{\text{out}}$ at different values of threshold levels $\lambda$. It is clear that $P_{\text{out}}$ increases as $\lambda$ increases for fixed $\rho$ as expected. Also, $P_{\text{out}}$ increases as $\rho$ increases for fixed $\lambda$, as expected too. Moreover, it can be inferred that at a given $\lambda$, $P_{\text{out}}$ is sensitive to the change of the correlation coefficient at both low and large values of $\rho$. It can also be inferred that the variations in $P_{\text{out}}$ with respect to the correlation coefficient $\rho$ at low threshold values are within one order of magnitude and these variations start to shrink as $\lambda$ increases. In other words, the slope of $P_{\text{out}}$ with respect to $\lambda$ increases as $\rho$ decreases.

Figure 3 illustrates the system performance at higher standard deviation value ($\sigma = 9$ dB). From figures 2 and 3, it can be inferred that the system with $\sigma = 9$ dB has a degradation ranging from 1.5 to 2 dB from the outage threshold perspective compared to the system with 6 dB standard deviation. Moreover, it can be noticed that $P_{\text{out}}$ curves tend to have lower slopes with respect to $\lambda$ for higher $\sigma$ values.

The effect of $\sigma$ variations on the calculated $P_{\text{out}}$ is shown in figure 4 for different $\rho$ values and $\lambda = 10$ dB. It is shown that there is about one order of magnitude difference between the two correlation limits in the range of $\sigma$ values of interest.

It is of interest to quantify the diversity gain obtained by going from two BSs to three BSs. As such, we compare in this section our results with that in the case of two correlated radio ports ($\alpha_1 = \alpha_2 = \sqrt{\rho}$) for different scenarios. The probability of outage is compared for two and three macroscopic antenna diversity in figures 5 and 6 for the following two scenarios:

- we consider the case when the mobile is located in the middle between two antennas ($\mu_1 = \mu_2 = 15$ dB and $\mu_3 = 9$ dB) as shown in part (b) of figure 1. In such a case, it can be inferred from figure 5 that the two systems are equivalent at high $\rho$ values while a three-antenna system exhibits about 0.5 dB gain over a two-antenna system from the outage threshold perspective.
- we also consider the case where the mobile station is close to one antenna or radio port ($\mu_1 = 20$ dB and $\mu_2 = \mu_3 = 12$ dB) as illustrated in part (c) of figure 1. In this case, it can be inferred from figure 6 that the two systems are again equivalent at high correlation values while a three-antenna system introduces about 1 dB gain over a two-antenna system from the outage threshold perspective.

5. CONCLUSION

Outage probability of SC diversity was calculated for an exponentially correlated lognormal shadowing environment. More specifically, the case of three exponentially correlated antennas was addressed. Similar to [2], our expression contains one infinite integral.
which converges rapidly for small values of integration limits. A first set of numerical results showed that the outage probability is sensitive to the small variations in the correlation coefficient in its entire range \((0 \leq \rho \leq 1)\). They also showed that the difference in the calculated outage probability at the two limits of zero and full correlation varies from one order to half an order of magnitude at low and high values of outage threshold \(\lambda\), respectively. It has also been shown that an increase of \(\sigma\) with 3 dB led to a system degradation up to 2 dB from the outage threshold perspective. In another set of numerical results we compared the performance of three and two-antenna diversity systems from the outage probability perspective and for different scenarios describing the position of the mobile with respect to the diversity antennas. It has been shown that there is no need for three-antenna systems if the underlying environment exhibits a high degree of correlation. On the other hand, at low values of the correlation coefficient, a three-antenna system exhibits a gain ranging roughly from 0.5 to 1 dB from the outage threshold perspective.

6. REFERENCES


Figure 4: Effect of the correlation coefficient on the outage probability at different standard deviation values ($\mu_1 = \mu_2 = 15 \, \text{dB}$, $\mu_3 = 9 \, \text{dB}$, and $\lambda = 10 \, \text{dB}$).

Figure 5: Outage probability for two and three antennas at different threshold levels ($\mu_1 = \mu_2 = 15 \, \text{dB}$ $\mu_3 = 9 \, \text{dB}$, and $\sigma = 6 \, \text{dB}$).

Figure 6: Outage probability for two and three antennas at different threshold levels ($\mu_1 = 20 \, \text{dB}$, $\mu_2 = \mu_3 = 12 \, \text{dB}$, and $\sigma = 6 \, \text{dB}$).