WEIGHTED DIRECTIONAL DISTANCE FILTERS

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ABSTRACT

A class of weighted directional distance filters for color image filtering is introduced. This nonlinear filtering tool for vector-valued signals is derived from the directional distance filter using two independent weight vectors for vector distance and vector angle domains. Thus, in the weighted directional distance approach, the contribution of each input sample to a sum of distances and a sum of angles, each separately, necessary to determine a filter output can be controlled by two nonnegative real weights. The filter output is the sample that results in a very small sum of weighted vector distances and simultaneously in a very small sum of weighted vector angles, both determined to other samples in the input set. Note that weighted directional distance filters represent a generalized vector filtering class for vector median filters, weighted vector median filters, basic vector directional filters, weighted vector directional filters and directional distance filters.

The behavior of the proposed weighted directional distance filters is analyzed and studied for a wide range of the impulse noise corruption.

1. INTRODUCTION

In real color images, the independent processing of R,G,B color channels may introduce the changes of color chromaticity, i.e. color artifacts, since the reordering of vectors’ components results in new samples [8]. In the case of vector processing of color images, the inherent correlation between color channels is respected and the input samples are processed as a sequence of vectors. Thus, the filter output is the sample from the input set.

When impulses, bit errors and outliers are introduced to an image, the effective denoising methodologies are based on order-statistics. In the case of color images, many vector filtering algorithms involve the minimization of error criteria [4],[8]. A class of vector median filters utilizes the distances between input vectors, whereas a class of vector directional filters operates on the vector directions (angles). Finally, directional distance filters combine vector distances and vector angles.

In this paper, a new class of weighted directional distance filters provides the generalized vector filtering tool for the impulse noise suppression in color images. Note that the proposed method is based on two independent weight vectors for distance and angle domains.

2. RELEVANT VECTOR FILTERS

Let \( y(x) : \mathbb{Z} \rightarrow \mathbb{Z}^n \) represent a multichannel image, with an image dimension \( l \) and a number of channels \( m \). If \( m \geq 2 \), it is the case of \( m \)-channel image processing. In the case of standard color images \( l = 2 \) and \( m = 3 \). Let also \( W = \{ x \in \mathbb{Z}^l ; i = 1,2,\ldots,N \} \) represent a filter window of a finite, generally odd size \( N \).

Since the input set \( x_1, x_2,\ldots, x_N \) represents the set of vectors, many operations and methodologies related to scalar samples cannot be directly applied to the set of vectors. The typical example is the sample ordering holding an important position in the field of impulse noise filtering. By this operation, atypical image samples are moved to the borders of the ordered set and the median value represents the noise-free sample with the highest probability from samples of the input set. Since the ordering of vector-valued samples has no natural basis [10], the direct extension of the scalar sorting algorithms to the set of vector samples is impossible. For that reason, vector samples are ordered according to the ordering criterion such as the distance function.

2.1. Vector Median Filters

Vector median filter (VMF) firstly introduced in [1] still represents a very popular filtering tool for vector data. The vector median of the set \( x_1, x_2,\ldots, x_N \) is defined as the sample \( x_{\text{med}} \in \{ x_1, x_2,\ldots, x_N \} \) such that

\[
\sum_{j=1}^{N} \| x_{\text{med}} - x_j \| \leq \sum_{j=1}^{N} \| x_j - x_j \| \quad \text{for} \quad j = 1,2,\ldots,N
\]

where \( L \) characterizes the used norm (e.g. Euclidean). In order to unify the vector filter definitions, let us consider the second VMF definition.

Let \( x_1, x_2,\ldots, x_N \) be the input set and \( L_1, L_2,\ldots, L_N \) the set of sums of vector distances such that

\[
L_i = \sum_{j=1}^{N} \| x_i - x_j \| \quad \text{for} \quad i = 1,2,\ldots,N
\]

Note that each \( L_i \), i.e. the sum of vector distances, is associated with the input sample \( x_i \), \( i = 1,2,\ldots,N \). Since \( L_1, L_2,\ldots, L_N \) are scalar values, their ordered set can be written simply as

\[
L_{(1)} \leq L_{(2)} \leq \ldots \leq L_{(N)}
\]

If the same ordering is implied to the input set \( x_1, x_2,\ldots, x_N \), then VMF output is given by the sample...
from the input set minimizing the sum of vector distances with other vectors. Shortly, if the ordered input set is described as
\[ \mathbf{x}^{(1)} \leq \mathbf{x}^{(2)} \leq \ldots \leq \mathbf{x}^{(N)} \] (4)
then the sample \( \mathbf{x}^{(i)} \) represents the VMF output.

### 2.2. Vector Directional Filters

Vector directional filters (VDFs) [7] operate on the direction of image vectors and the VDF output is determined according to these directions in the vector space. By above operation, image vectors with atypical directions in the vector space are eliminated and VDFs result in optimal estimates in the sense vectors’ directions, so that VDFs preserve the color chromaticity well.

Let each input sample \( \mathbf{x}_i \), for \( i = 1, 2, \ldots, N \), be associated with a sum of vector angles
\[ \alpha_i = \sum_{j=1}^{N} A(\mathbf{x}_i, \mathbf{x}_j) \] (5)
where
\[ A(\mathbf{x}_i, \mathbf{x}_j) = \cos^{-1} \left( \frac{\mathbf{x}_i \cdot \mathbf{x}_j}{|\mathbf{x}_i| \cdot |\mathbf{x}_j|} \right) \] (6)
represents the angle between two \( m \)-dimensional vectors \( \mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{im}) \) and \( \mathbf{x}_j = (x_{j1}, x_{j2}, \ldots, x_{jm}) \).

If \( \alpha_1, \alpha_2, \ldots, \alpha_N \), i.e. the sums of vector angles, serve as an ordering criterion, i.e.
\[ \alpha_{(1)} \leq \alpha_{(2)} \leq \ldots \leq \alpha_{(N)} \] (7)
and the same ordering is applied to the input set \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N \), this operation results in (4). The sample \( \mathbf{x}^{(i)} \), i.e. the sample that minimizes the sum of angles with other vectors, represents the output of basic vector directional filter (BVDF). Since the BVDF passes to the filter output the sample associated with minimal angle distance \( \alpha_{(i)} \), it preserves the color chromaticity better than the VMF.

### 2.3. Directional Distance Filters

If the minimization formula is expressed through a minimization of products
\[ \Omega_i = L_i \alpha_i \text{ for } i = 1, 2, \ldots, N \] (8)
\[ \Omega_i = \left( \sum_{j=1}^{N} ||\mathbf{x}_i - \mathbf{x}_j|| \right) \left( \sum_{j=1}^{N} A(\mathbf{x}_i, \mathbf{x}_j) \right) \text{ for } i = 1, 2, \ldots, N \] (9)
and the filter output is given by the sample \( \mathbf{x}^{(i)} \) associated with \( \Omega_{(i)} \), i.e. the minimum value from products \( \Omega_1, \Omega_2, \ldots, \Omega_N \) such their ordered set is simply written as
\[ \Omega_{(1)} \leq \Omega_{(2)} \leq \ldots \leq \Omega_{(N)} \] (10)
then the sample \( \mathbf{x}^{(i)} \) determines the output of directional distance filter (DDF) [2].

Although the minimization of products \( L_i \alpha_i \), for \( i = 1, 2, \ldots, N \), does not necessarily imply a minimum for either of \( L_i \) and \( \alpha_i \), it results in very small values for both of them [2]. For that reason, the product minimization will select as the filter output the vector-valued sample that results in a very small sum of vector distances (2) and a very small sum of vector angles (5), simultaneously.

The DDF combines properties of both VMF and BVDF and the used ordering criterion utilizes the sum of vector distances and the sum of vector angles. Thus, the DDF includes the VMF and the BVDF as special cases.

Let us assume the DDF with the power parameter \( p \) introduced to a filter structure so that the power \( 1 - p \) is associated with the sum of vector distances and the power \( p \) (from interval \( (0,1) \)) with the sum of vector angles. Then, the eq. (8) can be simply rewritten [2] as
\[ \Omega_i = L_i^{1-p} \alpha_i^p \text{ for } i = 1, 2, \ldots, N \] (11)
If \( p = 0 \), the DDF operates as the VMF, whereas for \( p = 1 \), the DDF is equivalent to the BVDF. For \( p = 0.5 \), the definition (11) is identical with (8), since both the sum of vector distances and the sum of vector angles are equally important.

### 3. PROPOSED METHOD

The introducing of the weight vector to a filter structure can significantly improve the filter performance [3],[4],[11]. The drawback of the methodology based on the association of input samples with corresponding weights lies in the filter optimization necessary to determine the appropriate weight vector for given filtering purposes. The success of the best weights setting depends on various factors such as the mechanism generating the original signal, the nature of the corruption, the training data availability, optimization approach, etc. On the other hand, the introduced weighted filtering function allows to achieve a higher design flexibility, the filter adaptation on changed signal statistics (especially in the case of time-varying signals) and the theoretical acquisition in the form of unique methodology and generalization.

#### 3.1. Weighted Directional Distance Filters

Let \( w_1, w_2, \ldots, w_N \) represent a set of nonnegative real weights related to the vector distance domain and \( u_1, u_2, \ldots, u_N \) a set of nonnegative real weights related to the vector angle domain, so that each weight \( w_j \) and \( u_j \), for \( j = 1, 2, \ldots, N \), be associated with the input sample \( \mathbf{x}_j \).

Similarly to definitions of weighted VMFs [3],[11], it is possible to express the sum of weighted vector distances as
\[ J_i = \sum_{j=1}^{N} w_j ||\mathbf{x}_i - \mathbf{x}_j|| \text{ for } i = 1, 2, \ldots, N \] (12)
where \( L \) is a selected norm.
When the set of nonnegative real weights \( u_i, u_j, \ldots, u_N \) is implied to the sum of vector angles, then it is possible to define the weighted angle distances \( \beta \) [4],[5] associated with the input sample \( x_i \), i.e.

\[
\beta_i = \sum_{j=1}^{N} u_j A(x_i, x_j) \quad \text{for } i = 1, 2, \ldots, N
\]

and constitute a recently developed class of weighted vector directional filters [5].

Now, let us consider the combined weighted distance \( \Psi_i \) associated with the sample \( x_i \) as a product of the sums \( J_i \) and \( \beta_i \) so that

\[
\Psi_i = J_i \beta_i \quad \text{for } i = 1, 2, \ldots, N
\]

Besides the possibility to control separately the influence of samples in sums of weighted distances and weighted vector angles, there is allowed the additional filter tuning provided by the power parameter \( p \). Thus, the combined weighted distance \( \Psi_i \) (14) can be rewritten as

\[
\Psi_i = (J_i)^p (\beta_i)^p \quad \text{for } i = 1, 2, \ldots, N
\]

If the combined weighted distances \( \Psi_1, \Psi_2, \ldots, \Psi_N \) serve as an ordering criterion, i.e.

\[
\Psi_1 \leq \Psi_2 \leq \ldots \leq \Psi_N
\]

and the same ordering is implied to the input set \( x_1, x_2, \ldots, x_N \), which results in ordered input sequence defined by (4), the sample \( x^0 \) represents an output of the weighted directional distance filter (WDDF).

Although the weighted VMF minimizes the sum of weighted vector distances and the weighted VDF minimizes the sum of weighted vector angles, the proposed WDDF results in a very small sum of weighted vector distances and simultaneously in a very small sum of weighted vector angles.

### 3.2. Generalization of vector filters

It is clear that the proposed WDDFs can perform a wide range of smoothing operations. Now, it will be shown that a class of WDDFs includes a number of well-known vector filters such as the VMF, the BVDF, the DDF and many more as special cases and represents their generalisation. These filters can be expressed (Table 1) through the setting of the power parameter and the filter weights.

#### Table 1. Special cases of the proposed WDDFs

<table>
<thead>
<tr>
<th>Method</th>
<th>( p )</th>
<th>Filter weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>identity</td>
<td>-</td>
<td>( w_{i(N+1)/2} = N ) ( u_{i(N+1)/2} = N ) others 1s</td>
</tr>
<tr>
<td>VMF</td>
<td>0</td>
<td>( w_i = 1 ) ( i = 1, 2, \ldots, N )</td>
</tr>
<tr>
<td>BVDF</td>
<td>1</td>
<td>( u_i = 1 ) ( i = 1, 2, \ldots, N )</td>
</tr>
<tr>
<td>DDF</td>
<td>0.5</td>
<td>( w_i = 1 ) ( u_i = 1 ) ( i = 1, 2, \ldots, N )</td>
</tr>
<tr>
<td>weighted VMF</td>
<td>0</td>
<td>nonnegative real ( w_i ) ( i = 1, 2, \ldots, N )</td>
</tr>
<tr>
<td>weighted VDF</td>
<td>1</td>
<td>nonnegative real ( u_i ) ( i = 1, 2, \ldots, N )</td>
</tr>
</tbody>
</table>

When the weight vectors are set so that the central weights \( w_{i(N+1)/2} = N \) and \( u_{i(N+1)/2} = N \), whereas other weights are 1s, the proposed method is equivalent to an identity filter and no smoothing will be provided. If the power parameter \( p \) is equal to 0 and the weight vector for the vector distance domain consists of 1s only, the WDDF is equivalent to the VMF. In the case of \( p = 1 \) and the unit weight vector related to the vector angle domain, the WDDF is equivalent to the BVDF. For all weights set to 1s and power parameter \( p = 0.5 \), the proposed method performs the DDF filtering operation. Finally, for the case of non-negative real weights, the proposed WDDF is equivalent to the weighted VMF ( \( p = 0 \) ) and the weighted VDF ( \( p = 1 \)).

#### Table 2. Results related to original Peppers

<table>
<thead>
<tr>
<th>Method</th>
<th>MAE</th>
<th>MSE</th>
<th>CD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identity</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Marginal median</td>
<td>2.946</td>
<td>35.5</td>
<td>6.384</td>
</tr>
<tr>
<td>VMF</td>
<td>2.885</td>
<td>36.7</td>
<td>6.061</td>
</tr>
<tr>
<td>BVDF</td>
<td>3.458</td>
<td>50.2</td>
<td>5.943</td>
</tr>
<tr>
<td>DDF</td>
<td>2.907</td>
<td>37.5</td>
<td>5.846</td>
</tr>
<tr>
<td>Weighted VMF ( w = a )</td>
<td>1.151</td>
<td>14.0</td>
<td>2.360</td>
</tr>
<tr>
<td>Weighted VMF ( w = b )</td>
<td>1.317</td>
<td>16.0</td>
<td>2.713</td>
</tr>
<tr>
<td>Weighted BVDF ( u = a )</td>
<td>1.434</td>
<td>18.8</td>
<td>2.642</td>
</tr>
<tr>
<td>Weighted BVDF ( u = b )</td>
<td>1.686</td>
<td>22.2</td>
<td>2.978</td>
</tr>
<tr>
<td>Weighted DDF ( w = a ), ( u = a )</td>
<td>1.057</td>
<td>13.0</td>
<td>2.354</td>
</tr>
<tr>
<td>Weighted DDF ( w = b ), ( u = b )</td>
<td>1.250</td>
<td>15.3</td>
<td>2.710</td>
</tr>
<tr>
<td>Weighted DDF ( w = a ), ( u = b )</td>
<td>1.141</td>
<td>13.9</td>
<td>2.538</td>
</tr>
<tr>
<td>Weighted DDF ( w = b ), ( u = a )</td>
<td>1.133</td>
<td>13.8</td>
<td>2.469</td>
</tr>
</tbody>
</table>

#### Table 3. Results related to original Lena

<table>
<thead>
<tr>
<th>Method</th>
<th>MAE</th>
<th>MSE</th>
<th>CD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identity</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Marginal median</td>
<td>3.123</td>
<td>43.5</td>
<td>4.745</td>
</tr>
<tr>
<td>VMF</td>
<td>3.190</td>
<td>45.4</td>
<td>4.534</td>
</tr>
<tr>
<td>BVDF</td>
<td>3.605</td>
<td>52.7</td>
<td>4.605</td>
</tr>
<tr>
<td>DDF</td>
<td>3.288</td>
<td>46.8</td>
<td>4.536</td>
</tr>
<tr>
<td>Weighted VMF ( w = a )</td>
<td>1.463</td>
<td>21.3</td>
<td>2.709</td>
</tr>
<tr>
<td>Weighted VMF ( w = b )</td>
<td>1.550</td>
<td>20.5</td>
<td>2.197</td>
</tr>
<tr>
<td>Weighted BVDF ( u = a )</td>
<td>1.784</td>
<td>24.3</td>
<td>2.304</td>
</tr>
<tr>
<td>Weighted BVDF ( u = b )</td>
<td>1.891</td>
<td>23.8</td>
<td>2.207</td>
</tr>
<tr>
<td>Weighted DDF ( w = a ), ( u = a )</td>
<td>1.503</td>
<td>21.5</td>
<td>2.129</td>
</tr>
<tr>
<td>Weighted DDF ( w = b ), ( u = b )</td>
<td>1.588</td>
<td>20.6</td>
<td>2.255</td>
</tr>
<tr>
<td>Weighted DDF ( w = a ), ( u = b )</td>
<td>1.527</td>
<td>20.5</td>
<td>2.173</td>
</tr>
<tr>
<td>Weighted DDF ( w = b ), ( u = a )</td>
<td>1.524</td>
<td>20.4</td>
<td>2.156</td>
</tr>
</tbody>
</table>
4. EXPERIMENTAL RESULTS

As the test images are used the well known images Peppers (Figure 1a) and Lena. The noisy version (10% impulse noise) of image Peppers is shown in Figure 1b. Mathematically, impulse noise [6],[8] is defined by

\[
\begin{cases} \nu_{ij} = x_{ij} & \text{with probability } p_v \\ o_{ij} = x_{ij} & \text{with probability } 1 - p_v \end{cases}
\]

where \(i,j\) characterise the sample position, \(o_{ij}\) is the sample from the original image, \(x_{ij}\) represents the sample from the noisy image, \(p_v\) is a corruption probability and \(\nu = (\nu_x,\nu_y,\nu_z)\) is a random noise vector. Since the components of \(\nu\) are generated independently, the gray impulse, i.e. the equivalence \(\nu_x = \nu_y = \nu_z\) of all components of \(\nu\), can occur in the special case, only.

The performance of the methods is evaluated by three objective criteria [6],[9] such as the mean absolute error (MAE), the mean square error (MSE) and color difference (CD) were used. In general, the MAE and the MSE for gray-scale images are given by

\[
MAE = \frac{1}{KL} \sum_{i=1}^{K} \sum_{j=1}^{L} |p_{ij} - x_{ij}|
\]

\[
MSE = \frac{1}{KL} \sum_{i=1}^{K} \sum_{j=1}^{L} (o_{ij} - x_{ij})^2
\]

where \(o_{ij}\) is the sample from original image, \(x_{ij}\) is the sample from filtered (noisy) image, \(i,j\) are indices of the sample position and \(K,L\) characterise an image size. In the case of color images, MAE and MSE criteria are understood as a mean over color channels. In general, the MAE is a mirror of the signal-detail preservation, whereas the MSE expresses the noise attenuation capability. The third criterion CD [9] expresses the color chromaticity preservation. The CD is defined by

\[
\Delta E_c = \sqrt{(\Delta L)^2 + (\Delta u)^2 + (\Delta v)^2}
\]

where \(\Delta L, \Delta u\) and \(\Delta v\) represent the difference between original and noisy images in \(L,u\) and \(v\) color channels. In the case of CD, the threshold value was established around 2.9 that characterizes the senselessness of human eyes to the color distortion.

Achieved results are summarized in Figure 1, Figure 2 and Tables 2-9. In order to emphasis the difference of the filter performance, besides filter outputs, Figures 1-2 show the estimation error of selected filters. Although used filters suppress the noise effectively, their outputs are simultaneously characterized by the significant estimation error related to worsen estimation properties on image edges and details. In the dependence on the chosen weight vectors, the output of the proposed WDDFs can be characterized by the excellently preserved image details and edges with the simultaneous noise suppression. The excellent estimation property of WDDF with \(w = [1,1,2,2,5,2,2,1,1]\) and \(w = [1,1,1,3,1,1,1,1,1]\) is demonstrated by a very small estimation error shown in Figure 2f and evaluated in Tables 2-9.
Note that the following weight vectors were used:
\[a = \{1, 1, 2, 2, 5, 2, 2, 1, 1\}\]
\[b = \{1, 1, 1, 1, 3, 1, 1, 1, 1\}\]
5. CONCLUSION

The new nonlinear vector filtering class for the impulse noise suppression in color images was provided. The novelty of the proposed weighted directional distance filters with a high design flexibility lies in the introduction of the weight vectors to a directional distance filter structure and the control of the sample influence to sums for vector distance and vector angle domains. Thus, it is possible to improve significantly the performance of widely used vector filters. Note that these filters can be expressed as special cases of the proposed method.

Since it is difficult to determine the most appropriate weight vectors for given filtering purposes, the future tasks will be related to the extension of some optimization methods used in the weighted median filter design to their vector forms.

6. REFERENCES