A PARTIAL RESOLUTION OF THE PUNCTUAL HILBERT SCHEME OF A NONSINGULAR SURFACE

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1. Introduction

The punctual Hilbert scheme of a nonsingular surface is a variety whose closed points correspond to subschemes of finite length n, say, supported at a fixed point on the surface. It is singular in general. A less singular model has been suggested by A. S. Tikhomirov [8], namely a certain component of the variety parameterizing flags $\xi_1 \subset \xi_2 \subset \cdots \subset \xi_n$ of subschemes, where each ξ_i has length i and is supported at the chosen point. It is not obvious, however, how to determine whether a given flag belongs to this particular component. In this paper we show that a necessary, and at least for $n \leq 7$ sufficient, condition is that the associated filtration of ideals $I_1 \supset I_2 \supset \cdots \supset I_n$ has the multiplicative property $I_i I_j \subseteq I_{i+j}$. The variety parameterizing such flags can be algorithmically computed. In particular we find that the suggested model for the punctual Hilbert scheme is singular for $n = 5$. This corrects an assertion of S. A. Tikhomirov's paper [9], where nonsingularity is erroneously claimed for $n = 5$. In [8], A. S. Tikhomirov showed that the model is nonsingular for $n \leq 4$, a result we also obtain here.

In sections 2–4 we construct a scheme parameterizing flags of subschemes in a more general setting. In sections 5–6 we specialize to the case of a nonsingular surface.

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2. Punctual Hilbert schemes of flags

Let k be an algebraically closed field. By a scheme we shall mean a locally Noetherian scheme over k. Product of schemes means product over k throughout. If Y_1 and Y_2 are closed subschemes of a third scheme X, the expression $Y_1 \cap Y_2$ denotes their scheme theoretic intersection and $Y_1 \subseteq Y_2$ means scheme theoretic inclusion. By a map of schemes we always mean a morphism in the category of schemes.

Let $(A, \mathfrak{m})$ be a local Artinian k-algebra of finite type. Then $X = \operatorname{Spec} A$ is a projective scheme, hence the Hilbert scheme $\operatorname{Hilb}^n(X)$ parameterizing subschemes $\xi \subset X$ of length n exists [5].

Introduce the following notation: For a map of schemes $f\colon Y' \to Y$, let

$$f_X\colon Y' \times X \to Y \times X$$

denote the product of f with the identity map on X. Furthermore, for any scheme Y, let

$$i_Y\colon Y \to Y \times X$$

denote the closed immersion obtained by identifying

$$Y \cong Y \times \operatorname{Spec}(A/\mathfrak{m}) \subset Y \times X.$$

To make formulas slightly more readable, we write i_*^Y in place of $(i_Y)_*$ for push forward along i_Y.

We want to construct a scheme $\operatorname{Flag}^n(X)$ parameterizing complete flags of subschemes

$$\xi_1 \subset \cdots \subset \xi_n \subset X$$

such that each ξ_i has length i.

DEFINITION 2.1. The *Hilbert functor of complete flags* in X of length n is the contravariant functor

$$\underline{\operatorname{Flag}}^n(X)\colon \mathbf{Sch}_k \to \mathbf{Sets}$$

from the category of locally Noetherian schemes over k to the category of sets that associates to a scheme T the set of n-tuples of families

$$T \times \operatorname{Spec}(A/\mathfrak{m}) = W_1 \subset \cdots \subset W_n \subset T \times X,$$

with W_i being defined by the ideal sheaf $\mathcal{J}_i \subset \mathcal{O}_{T\times X}$, such that

(I) each W_i is flat and finite of degree i over T

(II) $i_T^*(\mathcal{J}_i/\mathcal{J}_{i+1})$ is an invertible sheaf on T for $i = 1, 2, \ldots, n-1$.

REMARK 2.2. For k-valued points, condition (II) is automatic, thus a scheme representing $\underline{\operatorname{Flag}}^n(X)$ does parameterize complete flags of subschemes in X. In fact, a k-valued point consists of subschemes $\xi_i \subset X$ of length i, defined by ideals

$$I_n \subset \cdots \subset I_1 = \mathfrak{m} \subset A.$$

The sheaf $i_T^*(\mathcal{J}_i/\mathcal{J}_{i+1})$ is now nothing but the k-vector space $I_i/(I_{i+1} + \mathfrak{m}I_i)$. Consider the obvious inclusions

$$I_{i+1} \subseteq \mathfrak{m}I_i + I_{i+1} \subseteq I_i.$$

By Nakayama's lemma, the rightmost inclusion must be strict. By the assumption on lengths, the leftmost inclusion must then be an equality, that is, $\mathfrak{m}I_i \subseteq I_{i+1}$. Thus

$$I_i/(I_{i+1} + \mathfrak{m}I_i) = I_i/I_{i+1}$$

which is one-dimensional.

Similarly one can show that condition (II) is automatic for any reduced locally Noetherian base scheme T, but we shall not need this fact.

In the next section we shall prove the following result.

THEOREM 2.3. *There exists a scheme* $\operatorname{Flag}^n(X)$ *representing* $\underline{\operatorname{Flag}}^n(X)$.

3. Construction of $\operatorname{Flag}^n(X)$

We construct $\operatorname{Flag}^n(X)$ by induction on n. For $n = 1$ we clearly have $\operatorname{Flag}^1(X) = \operatorname{Spec} k$, with universal family

$$Z_1 = \operatorname{Spec} k \times \operatorname{Spec} k \subset \operatorname{Spec} k \times X.$$

The main idea is the following: A closed point in $\operatorname{Flag}^n(X)$ corresponds to a filtration of ideals $I_1 \supset \cdots \supset I_n$. Consider a closed point in $\mathsf{P}(I_n/\mathfrak{m}I_n)$, that is a vector space quotient

$$I_n/\mathfrak{m}I_n \to k \to 0.$$

Such a quotient is also a homomorphism of A-modules, hence the kernel of the composite

$$I_n \to I_n/\mathfrak{m}I_n \to k$$

is an ideal I_{n+1}. The extended filtration $I_1 \supset \cdots \supset I_n \supset I_{n+1}$ defines a closed point in $\operatorname{Flag}^{n+1}(X)$, and conversely any point arises in this way. The rest of this section is a straightforward globalization of this "fibrewise" construction.

Suppose now, for some fixed n, there exists a scheme $F = \operatorname{Flag}^n(X)$ representing $\underline{\operatorname{Flag}}^n(X)$, and let

$$Z_1 \subset \cdots \subset Z_n \subset F \times X$$

denote the universal flag, with Z_i defined by the ideal sheaf $\mathcal{I}_i \subset \mathcal{O}_{F\times X}$. Define the coherent $\mathcal{O}_F$-module

$$\mathcal{E}_n = i_F^* \mathcal{I}_n$$

and let

$$\pi\colon \mathsf{P}(\mathcal{E}_n) \to F$$

denote the structure map. We want to show that $\mathsf{P}(\mathcal{E}_n)$ represents $\underline{\mathrm{Flag}}^{n+1}(X)$ by exhibiting a universal flag

$$\widetilde{Z}_1 \subset \cdots \subset \widetilde{Z}_{n+1} \subset \mathsf{P}(\mathcal{E}_n) \times X.$$

For $i = 1, \dots, n$, simply let

$$\widetilde{Z}_i = \pi_X^{-1}(Z_i) \subset \mathsf{P}(\mathcal{E}_n) \times X$$

which, since Z_i is flat over F, is defined by the ideal sheaf

$$\widetilde{\mathcal{I}}_i = \pi_X^* \mathcal{I}_i.$$

Furthermore, we define

$$\widetilde{Z}_{n+1} \subset \mathsf{P}(\mathcal{E}_n) \times X$$

by the ideal sheaf $\widetilde{\mathcal{I}}_{n+1}$, constructed as follows: Let

$$\phi_1\colon \widetilde{\mathcal{I}}_n \to i_*^{\mathsf{P}(\mathcal{E}_n)} i_{\mathsf{P}(\mathcal{E}_n)}^* \widetilde{\mathcal{I}}_n = i_*^{\mathsf{P}(\mathcal{E}_n)} \pi^* \mathcal{E}_n \tag{1}$$

be the canonical surjection and let

$$\phi_2\colon i_*^{\mathsf{P}(\mathcal{E}_n)} \pi^* \mathcal{E}_n \to i_*^{\mathsf{P}(\mathcal{E}_n)} \mathcal{O}(1) \tag{2}$$

be the map obtained by applying $i_*^{\mathsf{P}(\mathcal{E}_n)}$ to the universal quotient

$$\pi^* \mathcal{E}_n \to \mathcal{O}(1) \to 0 \tag{3}$$

on $\mathsf{P}(\mathcal{E}_n)$. Then define $\widetilde{\mathcal{I}}_{n+1}$ to be the kernel of $\phi_2 \circ \phi_1$. The horizontal row in the following diagram is then exact:

$$\begin{array}{ccccccccc} & & & & & & i_*^{\mathsf{P}(\mathcal{E}_n)} \pi^* \mathcal{E}_n & & \\ & & & & & \overset{\phi_1}{\nearrow} & \downarrow{\scriptstyle \phi_2} & & \\ 0 & \longrightarrow & \widetilde{\mathcal{I}}_{n+1} & \longrightarrow & \widetilde{\mathcal{I}}_n & \longrightarrow & i_*^{\mathsf{P}(\mathcal{E}_n)} \mathcal{O}(1) & \longrightarrow & 0 \end{array} \tag{4}$$

By the short exact sequence in (4) we see that $i_{\mathsf{P}(\mathcal{E}_n)}^*(\widetilde{\mathcal{I}}_n/\widetilde{\mathcal{I}}_{n+1})$ is invertible, hence condition (II) in definition 2.1 is fulfilled. The same exact sequence may be rewritten

$$0 \to i_*^{\mathsf{P}(\mathcal{E}_n)} \mathcal{O}(1) \to \mathcal{O}_{\widetilde{Z}_{n+1}} \to \mathcal{O}_{\widetilde{Z}_n} \to 0$$

from which we see that $\widetilde{Z}_{n+1}$ is flat and finite of degree $n+1$ over $\mathsf{P}(\mathcal{E}_n)$, hence condition (I) is satisfied as well.

The following theorem ends the induction step and thus proves theorem 2.3:

Theorem 3.1. *The flag $\widetilde{Z}_1 \subset \cdots \subset \widetilde{Z}_{n+1}$ constructed above has the following universal property: For any scheme T and any T-valued point*

$$T \times \operatorname{Spec}(A/\mathfrak{m}) = W_1 \subset \cdots \subset W_{n+1} \subset T \times X$$

of $\underline{\mathrm{Flag}}^{n+1}(X)$, there exists a unique map

$$f\colon T \to \mathsf{P}(\mathcal{E}_n)$$

such that $W_i = f^{-1}(\widetilde{Z}_i)$ for each i. Hence $\mathsf{P}(\mathcal{E}_n)$ represents $\underline{\mathrm{Flag}}^{n+1}(X)$.

Proof. Let $\mathcal{J}_i \subset \mathcal{O}_{T\times X}$ be the sheaf of ideals defining W_i. By the induction hypothesis we have assumed that F represents $\operatorname{Flag}^n(X)$, so the families $W_1, \dots, W_n$ determine a unique map $g\colon T \to F$ such that $W_i = g_X^{-1}(Z_i)$ for $i = 1, \dots, n$. Since Z_i is flat over F, the inverse image $g_X^{-1}(Z_i)$ is defined by $g_X^*\mathcal{I}_i$, hence $\mathcal{J}_i = g_X^*\mathcal{I}_i$. We want to show that g extends uniquely to a map f in the diagram

$$\begin{array}{ccc} & & \mathsf{P}(\mathcal{E}_n) \\ & \overset{f}{\nearrow} & \downarrow \pi \\ T & \xrightarrow{\;g\;} & F \end{array} \tag{5}$$

such that $f_X^{-1}(\widetilde{Z}_{n+1}) = W_{n+1}$, or equivalently $f_X^*(\widetilde{\mathcal{I}}_{n+1}) = \mathcal{J}_{n+1}$. Extending g to a map f in the diagram (5) is equivalent to giving a quotient

$$g^*\mathcal{E}_n \to \mathcal{L} \to 0 \tag{6}$$

where $\mathcal{L}$ is an invertible sheaf on T. In fact, f is then the unique map such that (6) is obtained by applying f^* to the universal quotient (3).

Uniqueness: Assume there exists an f in diagram (5) such that $f_X^*(\widetilde{\mathcal{I}}_{n+1}) = \mathcal{J}_{n+1}$. We want to show that this determines the quotient (6) uniquely. This can be seen by applying $f^* i^*_{\mathsf{P}(\mathcal{E}_n)}$ to diagram (4). Firstly, applying $i^*_{\mathsf{P}(\mathcal{E}_n)}$ to the map ϕ_1 in (1) we obtain the identity map on

$$i^*_{\mathsf{P}(\mathcal{E}_n)}\widetilde{\mathcal{I}}_n = i^*_{\mathsf{P}(\mathcal{E}_n)}\pi_X^*\mathcal{I}_n = \pi^* i_F^*\mathcal{I}_n = \pi^*\mathcal{E}_n. \tag{7}$$

Furthermore, applying $i^*_{\mathsf{P}(\mathcal{E}_n)}$ to ϕ_2 in (2) we recover the universal quotient (3). Thus, the result of applying $i^*_{\mathsf{P}(\mathcal{E}_n)}$ to diagram (4) is the following diagram:

$$\begin{array}{ccccccc} & & & & \pi^*\mathcal{E}_n & & \\ & & & // & \downarrow & & \\ i^*_{\mathsf{P}(\mathcal{E}_n)}\widetilde{\mathcal{I}}_{n+1} & \longrightarrow & i^*_{\mathsf{P}(\mathcal{E}_n)}\widetilde{\mathcal{I}}_n & \longrightarrow & \mathcal{O}(1) & \longrightarrow & 0 \end{array}$$

Now applying f^* and using the identity $i_T^* f_X^* = f^* i^*_{\mathsf{P}(\mathcal{E}_n)}$, we obtain

$$\begin{array}{ccccccc} & & & & g^*\mathcal{E}_n & & \\ & & & // & \downarrow & & \\ i_T^*\mathcal{J}_{n+1} & \longrightarrow & i_T^*\mathcal{J}_n & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \end{array}$$

where $\mathcal{L} = f^*\mathcal{O}(1)$. Hence f corresponds to the quotient

$$i_T^*\mathcal{J}_n \to i_T^*(\mathcal{J}_n/\mathcal{J}_{n+1}) \to 0 \tag{8}$$

and is thus uniquely determined by the families W_i.

Existence: Simply define $\mathcal{L} = i_T^*(\mathcal{J}_n/\mathcal{J}_{n+1})$ and let f be the unique map corresponding to the quotient (8). This makes sense, since $\mathcal{L}$ is invertible by assumption. It remains only to check that we have $f_X^*\widetilde{\mathcal{I}}_{n+1} = \mathcal{J}_{n+1}$. For this, apply f_X^* to the short exact sequence in (4) to obtain

$$f_X^*\widetilde{\mathcal{I}}_{n+1} \to \mathcal{J}_n \to i_*^T\mathcal{L} \to 0. \tag{9}$$

Now observe that the canonical map $\mathcal{J}_n/\mathcal{J}_{n+1} \to i_*^T\mathcal{L}$ is an isomorphism, under which the rightmost map in (9) may be identified with the canonical map $\mathcal{J}_n \to \mathcal{J}_n/\mathcal{J}_{n+1}$. Thus the kernel is $f_X^*\widetilde{\mathcal{I}}_{n+1} = \mathcal{J}_{n+1}$, that is, $f_X^{-1}(\widetilde{Z}_{n+1}) = W_{n+1}$.

Proposition 3.2. *The scheme* $\operatorname{Flag}^n(X)$ *is connected.*

PROOF. If $f\colon X \to Y$ is a closed continuous surjective map of topological spaces, it is elementary that X is connected if both Y and the fibers of f are. We apply this to the structure map

$$\mathsf{P}(\mathcal{E}_n) \to \mathrm{Flag}^n(X).$$

This map is proper and the fibers are projective spaces. Hence $\mathrm{Flag}^{n+1}(X) = \mathsf{P}(\mathcal{E}_n)$ is connected if $\mathrm{Flag}^n(X)$ is. The conclusion follows by induction on n.

4. Punctual Hilbert schemes of multiplicative flags

DEFINITION 4.1. A k-valued point in $\mathrm{Flag}^n(X)$, corresponding to a filtration of ideals

$$I_n \subset \cdots \subset I_1 = \mathfrak{m} \subset A$$

is *multiplicative* if we have $I_i I_j \subseteq I_{i+j}$ for all $i + j \leq n$.

We next construct a subscheme of $\mathrm{Flag}^n(X)$, parameterizing only multiplicative flags in X.

DEFINITION 4.2. The *Hilbert functor of multiplicative complete flags* in X of length n is the contravariant functor

$$\underline{\mathrm{Mult}}^n(X)\colon \mathbf{Sch}_k \to \mathbf{Sets}$$

from the category of locally Noetherian schemes over k to the category of sets that associates to a scheme T the set of n-tuples of families

$$T \times \mathrm{Spec}(A/\mathfrak{m}) = W_1 \subset \cdots \subset W_n \subset T \times X,$$

with W_i being defined by the ideal sheaf $\mathcal{J}_i \subset \mathcal{O}_{T\times X}$, such that

(I) each W_i is flat and finite of degree i over T
(II) $i_T^*(\mathcal{J}_i/\mathcal{J}_{i+1})$ is an invertible sheaf on T for all i
(III) $\mathcal{J}_i\mathcal{J}_j \subseteq \mathcal{J}_{i+j}$ for all $i + j \leq n$.

We want to show that the condition $\mathcal{J}_i\mathcal{J}_j \subseteq \mathcal{J}_{i+j}$ is closed, in the strong sense that $\underline{\mathrm{Mult}}^n(X)$ is a closed subfunctor of $\underline{\mathrm{Flag}}^n(X)$. This is a consequence of the following lemma:

LEMMA 4.3. *Let $\pi\colon Y \to S$ be a morphism of locally Noetherian schemes and let $W, Z \subseteq Y$ be closed subschemes such that Z is flat and finite over S. Then there exists a unique S-scheme*

$$i\colon S' \to S$$

such that

(I) *$Z \times_S S' \subseteq W \times_S S'$*
(II) *if $T \to S$ is any S-scheme satisfying $Z \times_S T \subseteq W \times_S T$ then there exists a unique morphism $g\colon T \to S'$ over S.*

Furthermore, i is a closed immersion.

PROOF. Suppose the lemma holds whenever S is affine. Then we may apply the lemma to each S_α in an affine open cover $\{S_\alpha\}$ of S. Thus there exists closed immersions $i_\alpha\colon S'_\alpha \to S_\alpha$, uniquely determined by properties (I) and (II) when replacing S, W and Z with S_α, $W \cap S_\alpha$ and $Z \cap S_\alpha$. Again applying the lemma to an affine open cover of each intersection $S_\alpha \cap S_\beta$, we see that the immersions $\{i_\alpha\}$ agree on the overlaps. Hence they may be glued to form the required closed immersion $i\colon S' \to S$. Thus we may assume S is affine.

Since Z is finite over S, Z is affine as well. Then we may choose a free presentation

$$\mathcal{O}_Z^n \xrightarrow{\phi} \mathcal{O}_Z \to \mathcal{O}_{Z\cap W} \to 0 \tag{10}$$

where $Z \cap W$ denotes the scheme theoretic intersection. Let $f\colon T \to S$ be any morphism, and let $\widetilde{Z} = Z \times_S T$ and $\widetilde{W} = W \times_S T$. We claim the condition $\widetilde{Z} \subseteq \widetilde{W}$ is equivalent to requiring $f^*\pi_*\phi = 0$: Form the fibre square

$$\begin{array}{ccc} Y \times_S T & \xrightarrow{\widetilde{f}} & Y \\ {\scriptstyle\widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ T & \xrightarrow{f} & S \end{array}.$$

Then applying $\widetilde{f}^*$ to (10) gives a free presentation of the structure sheaf of $\widetilde{Z} \cap \widetilde{W}$:

$$\mathcal{O}^n_{\widetilde{Z}} \xrightarrow{\widetilde{f}^*\phi} \mathcal{O}_{\widetilde{Z}} \to \mathcal{O}_{\widetilde{Z}\cap\widetilde{W}} \to 0$$

Thus the condition $\widetilde{Z} \subseteq \widetilde{W}$, or equivalently $\widetilde{Z} \cap \widetilde{W} = \widetilde{Z}$, is the same thing as requiring $\widetilde{f}^*\phi = 0$. Now the restriction of $\widetilde{\pi}$ to $\widetilde{Z}$ is finite, hence affine, so $\widetilde{f}^*\phi = 0$ if and only if $\widetilde{\pi}_*\widetilde{f}^*\phi = 0$. Furthermore, as Z is flat over S, $\widetilde{\pi}_*\widetilde{f}^*\phi = f^*\pi_*\phi$. Hence $\widetilde{Z} \subseteq \widetilde{W}$ if and only if $f^*\pi_*\phi = 0$ as claimed.

Since Z is flat and finite over S,

$$\pi_*\mathcal{O}^n_Z \xrightarrow{\pi_*\phi} \pi_*\mathcal{O}_Z \tag{11}$$

is a map of locally free sheaves of finite rank on S. Thus $\pi_*\phi$ can be locally represented by a matrix of regular functions, hence its vanishing locus has a canonical structure of a closed subscheme $i\colon S' \to S$. Then $i^*\pi_*\phi = 0$, so i has property (I). Furthermore, if a morphism $f\colon T \to S$ satisfies $f^*\pi_*\phi = 0$, then the image in $\mathcal{O}_T$ of the ideal sheaf defining $S' \subset S$ is zero, which says that f factors through i. So i has property (II).

THEOREM 4.4. *$\underline{\mathrm{Mult}}^n(X)$ is a closed subfunctor of $\underline{\mathrm{Flag}}^n(X)$.*

PROOF. Let S denote a scheme and h_S its functor of points. Consider a cartesian diagram

$$\begin{array}{ccc} h & \longrightarrow & \underline{\mathrm{Mult}}^n(X) \\ \downarrow & & \downarrow \\ h_S & \longrightarrow & \underline{\mathrm{Flag}}^n(X) \end{array}$$

where h is the fibre product functor. We claim there exists a closed subscheme $S' \subseteq S$ and an isomorphism $h \cong h_{S'}$ such that the map $h \to h_S$ is compatible with the inclusion map $h_{S'} \to h_S$.

The image of a morphism $T \to S$ under the given map $h_S \to \underline{\mathrm{Flag}}^n(X)$ is a flag

$$W_1 \subset \cdots \subset W_n \subset X \times T. \tag{12}$$

Let $\mathcal{J}_i \subset \mathcal{O}_{X\times T}$ denote the ideal sheaf corresponding to W_i. By definition, h is the subfunctor of h_S whose T-valued points are the morphisms $T \to S$ such that the corresponding flag (12) has the multiplicative property

$$\mathcal{J}_i\mathcal{J}_j \subseteq \mathcal{J}_{i+j} \text{ for all } i + j \leq n. \tag{13}$$

Thus our claim is that there is a closed subscheme $S' \subseteq S$ such that $T \to S$ factors through S' if and only if property (13) holds. This can be seen as follows:

The image of the identity map id_S under the given map $h_S \to \underline{\mathrm{Flag}}^n(X)$ is a flag

$$Z_1 \subset \cdots \subset Z_n \subset X \times S \tag{14}$$

over S, with Z_i corresponding to some ideal sheaf $\mathcal{I}_i \subset \mathcal{O}_{X\times S}$. For any morphism $T \to S$, the corresponding flag (12) is just the pullback of the flag (14) along $T \to S$. Thus the existence of $S' \subseteq S$ is a consequence of lemma 4.3, applied to $Y = X \times S$, $W = V(\mathcal{I}_i\mathcal{I}_j)$ and $Z = Z_{i+j}$, for each i and j.

Corollary 4.5. *There exists a closed subscheme* $\mathrm{Mult}^n(X) \subseteq \mathrm{Flag}^n(X)$ *representing* $\underline{\mathrm{Mult}}^n(X)$.

Remark 4.6. The scheme $\mathrm{Mult}^n(X)$ can be constructed more explicitly in the same fashion that we constructed $\mathrm{Flag}^n(X)$: Consider the universal flag

$$Z_1 \subset \cdots \subset Z_n \subset \mathrm{Flag}^n(X) \times X,$$

with Z_i defined by the ideal sheaf $\mathcal{I}_i$. Denote by

$$W_1 \subset \cdots \subset W_n \subset \mathrm{Mult}^n(X) \times X$$

their restriction to $\mathrm{Mult}^n(X)$, with W_i defined by the ideal sheaf $\mathcal{J}_i$. In section 3 we constructed $\mathrm{Flag}^{n+1}(X)$ as $\mathsf{P}(\mathcal{E}_n)$, where $\mathcal{E}_n = i_F^*\mathcal{I}_n$. Thus $\mathrm{Mult}^{n+1}(X)$ is the maximal subscheme of $\mathsf{P}(\mathcal{E}_n)$ such that the restriction of the universal flag has the multiplicative property. This is precisely the universal property of

$$\pi\colon \mathsf{P}(\mathcal{F}_n) \to \mathrm{Mult}^n(X)$$

where

$$\mathcal{F}_n = \mathcal{J}_n / \textstyle\sum_{i=0}^{n-1} \mathcal{J}_{i+1}\mathcal{J}_{n-i},$$

considered as a coherent sheaf on $\mathrm{Mult}^n(X) \cong W_1 \subset \mathrm{Mult}^n(X) \times X$. Thus we have an isomorphism $\mathrm{Mult}^{n+1}(X) \cong \mathsf{P}(\mathcal{F}_n)$ over $\mathrm{Mult}^n(X)$. The universal multiplicative flag

$$\widetilde{W}_1 \subset \cdots \subset \widetilde{W}_{n+1} \subset \mathrm{Mult}^{n+1}(X) \times X$$

is defined by ideals $\widetilde{\mathcal{J}}_1 \supset \cdots \supset \widetilde{\mathcal{J}}_{n+1}$ where $\widetilde{\mathcal{J}}_i = \pi_X^*\mathcal{J}_i$ for $i \leq n$, whereas $\widetilde{\mathcal{J}}_{n+1}$ is the kernel of the canonical map

$$\widetilde{\mathcal{J}}_n \to i_*^{\mathsf{P}(\mathcal{F}_n)}\mathcal{O}(1)$$

where $\mathcal{O}(1)$ now denotes the tautological invertible sheaf on $\mathsf{P}(\mathcal{F}_n)$.

Proposition 4.7. *The scheme* $\mathrm{Mult}^n(X)$ *is connected.*

Proof. Using the construction of $\mathrm{Mult}^n(X)$ in remark 4.6, the proof of 3.2 can be repeated.

5. Punctual Hilbert schemes of points on a nonsingular surface

For the rest of this text we consider the following situation: Assume k has characteristic zero. Fix an algebraic surface S over k and a nonsingular point $p \in S$. Let $\mathcal{O}_{S,p}$ denote the local ring at p and let $\mathfrak{m}_p \subset \mathcal{O}_{S,p}$ denote its maximal ideal. Any subscheme $\xi \subset S$ of length n and supported at p is contained in the $(n-1)$'st infinitesimal neighbourhood $X = \operatorname{Spec} \mathcal{O}_{S,p}/\mathfrak{m}_p^n$. Thus the scheme $\mathrm{Hilb}^n(X)$ parameterizes length n subschemes of S supported at p. We let

$$H(n) = \mathrm{Hilb}^n(X)_{\mathrm{red}}$$

denote the underlying reduced subscheme. We suppress S and p from the notation, as the definition of $H(n)$ only depends on the $(n-1)$'st infinitesimal neighbourhood of p, whose isomorphism class is independent of the choices of S and p.

It is well known that $H(n)$ is irreducible and has dimension $n-1$ (proved by Briançon [1] over the complex numbers, see e.g. Ellingsrud and Lehn [2] for a proof in a more general setting). However, it is singular in general. For instance, $H(3)$ is isomorphic to the projective cone over the twisted cubic in P^3. In the rest of this paper we present work towards finding a natural resolution of singularities of $H(n)$.

Following Le Barz [7], we make the following definition:

Definition 5.1. A subscheme $\xi \subset S$, supported at p, is *curvilinear* if there exists a curve C which contains ξ and is nonsingular at p.

It is well known ([1], [6]) that the subset of $H(n)$ consisting of curvilinear subschemes is open, dense and nonsingular. The following result is also well known:

Lemma 5.2. *Let $\xi \subset S$ be a subscheme supported at a point p. If ξ is curvilinear, there is a unique flag*

$$\xi_1 \subset \cdots \xi_{n-1} \subset \xi$$

with ξ_i of length i. In fact, ξ_i is the intersection of ξ with the $(i-1)$'st infinitesimal neighbourhood of p in S.

Proof. Suppose C is a nonsingular curve through p containing ξ, locally defined by the ideal $J \subset \mathcal{O}_{X,p}$. Let $\xi_i \subset \xi$ be a subscheme of length i and let $I \subset I_i \subset \mathcal{O}_{X,p}$ be the ideals defining ξ and ξ_i. Then we have $\mathfrak{m}_p^i \subseteq I_i$, hence

$$J + \mathfrak{m}_p^i \subseteq I + \mathfrak{m}_p^i \subseteq I_i.$$

But the left hand side is the ideal defining the $(i-1)$'st infinitesimal neighbourhood of p in C, which has colength i since C is nonsingular. Since the right hand side ideal I_i has colength i also, the inclusions are actually equalities. In particular $I_i = I + \mathfrak{m}_p^i$, which shows that ξ_i is uniquely determined as the intersection of ξ with the $(i-1)$'st infinitesimal neighbourhood of p in S.

Define

$$HF(n) = \mathrm{Flag}^n(X)_{\mathrm{red}}$$

which is a reduced scheme whose closed points correspond to flags of subschemes in S supported at p. The canonical map

$$\mathrm{Flag}^n(X) \to \mathrm{Hilb}^n(X)$$

induces a map

$$\rho_n \colon HF(n) \to H(n).$$

Proposition 5.3. *There is a unique component $HF'(n) \subseteq HF(n)$ which is mapped birationally onto $H(n)$ by ρ_n.*

Proof. Let $U \subseteq H(n)$ be the open subset corresponding to curvilinear subschemes. By lemma 5.2, the fibre $\rho_n^{-1}(\xi)$ is a single point for every (closed) point $\xi \in U$. Hence ρ_n is bijective over U. Since ρ_n is proper and U is nonsingular, Zariski's main theorem [4, prop. 4.4.1] shows that ρ_n is an isomorphism over U. Thus the closure $HF'(n)$ of $\rho_n^{-1}(U)$ in $HF(n)$ is the unique component mapping birationally onto $H(n)$.

Denote by

$$\rho'_n \colon HF'(n) \to H(n)$$

the restricted map. We call this a partial resolution of $H(n)$. This construction has been studied by Tikhomirov in [8], where he proves that ρ'_n is a resolution of singularities for $n \leq 4$. The problem addressed in the next section is how to determine whether a given flag belongs to the component $HF'(n)$. This leads us to a different proof of Tikhomirov's result (theorem 6.1) and also the new result that $HF'(5)$ is singular (theorem 6.2).

Define

$$HMF(n) = \mathrm{Mult}^n(X)_{\mathrm{red}}$$

which is a reduced scheme whose closed points correspond to multiplicative flags of subschemes in S supported at p. Since $\mathrm{Mult}^n(X)$ is a closed subscheme of $\mathrm{Flag}^n(X)$, we find that $HMF(n)$ is a closed subscheme of $HF(n)$. The motivation for studying $HMF(n)$ is the following observation:

Proposition 5.4. *Any (closed) point in $HF'(n)$ is multiplicative, hence $HF'(n)$ is contained in $HMF(n)$.*

PROOF. Denote by $U \subseteq H(n)$ the open set consisting of curvilinear points. Let $V \subseteq HF'(n)$ denote the inverse image of U by the map $\rho'_n \colon HF'(n) \to H(n)$. By definition, $HF'(n)$ is the closure of V in $HF(n)$.

First consider a (closed) point in V, that is, a flag

$$\xi_1 \subset \cdots \subset \xi_n$$

with ξ_n curvilinear. Then, if ξ_i corresponds to the ideal $I_i \subset \mathcal{O}_{X,p}$ we have

$$I_i = \mathfrak{m}_p^i + I_n \quad \text{for all } i$$

by lemma 5.2. Then it is obvious that $I_i I_j \subseteq I_{i+j}$.

Thus $V \subset HMF(n)$. Since $HMF(n)$ is closed in $HF(n)$ and $HF'(n)$ is the closure of V, we have $HF'(n) \subset HMF(n)$.

QUESTION 5.5. Is the converse to proposition 5.4 true, i.e. do we have an equality $HF'(n) = HMF(n)$? As $HF'(n)$ is a component of $HF(n)$, this is equivalent to asking whether $HMF(n)$ is irreducible.

The calculations in section 6 show that the answer to the question is positive for $n \leq 7$. For higher n we do not know. We remark that $HMF(n)$ is at least connected, by proposition 4.7.

6. Examples

To describe $HMF(n)$, we follow the construction of $\operatorname{Mult}^n(X)$ in remark 4.6. More explicitly, let $U = \operatorname{Spec} A$ be an affine open subset of $\operatorname{Mult}^n(X)$. We want to describe an affine open cover for the inverse image of U in $\operatorname{Mult}^{n+1}(X)$, denoted $\operatorname{Mult}^{n+1}(X)\big|_U$. With notation as in remark 4.6, the family W_i is defined over U by the ideal $J_i = \Gamma(U \times X, \mathcal{J}_i)$ in the affine coordinate ring of $U \times X$. Then

$$\operatorname{Mult}^{n+1}(X)\big|_U = \mathsf{P}(M)$$

where

$$M = \Gamma(U, \mathcal{F}_n) = J_n / \textstyle\sum_{v=0}^{n-1} J_{v+1} J_{n-v}$$

considered as an A-module. To give concrete equations for $\mathsf{P}(M)$, choose a free presentation

$$A^r \xrightarrow{(g_{ij})} A^s \xrightarrow{(f_j)} M \to 0.$$

Then $\mathsf{P}(M) = \operatorname{Proj} R$ where

$$R = A[t_1, \dots, t_s] / (\textstyle\sum_j g_{1j} t_j, \dots, \sum_j g_{rj} t_j). \tag{15}$$

Thus $\mathsf{P}(M)$ is covered by the affine open subsets $V_i = \operatorname{Spec} R_i$ where R_i is the degree 0 part of the localization R_{t_i}. The universal quotient is the homomorphism

$$M \otimes R_i \to R_i \to 0$$

sending $f_j \otimes 1$ to $T_j = t_j / t_i$ (in particular $f_i \otimes 1 \mapsto 1$). Hence, on V_i the universal flag is defined by ideals

$$\widetilde{J}_1 \supset \cdots \supset \widetilde{J}_{n+1}$$

where $\widetilde{J}_v = J_v R_i$ for $v \leq n$, and

$$\widetilde{J}_{n+1} = (T_j f_i - f_j)_{j \neq i} + (\textstyle\sum_{v=0}^{n-1} J_{v+1} J_{n-v}) R_i.$$

As long as the rings R_i are nilpotent-free, this gives an algorithm for computing an open cover of $HMF(n)$. Otherwise we should divide by the nilradical to get the underlying reduced scheme. It turns out that in all our examples, i.e. whenever $n \leq 7$, $\operatorname{Mult}^n(X)$ is already reduced, hence $HMF(n) = \operatorname{Mult}^n(X)$. We do not know whether this is true for arbitrary n.

Clearly, $\operatorname{Mult}^2(X) = HMF(2) \cong H(2) \cong \mathsf{P}^1$. The next result describes $HMF(3)$ and $HMF(4)$. We are going to use the following (well known and easy to derive) classification of punctual subschemes of length 2 and 3 on a nonsingular surface: For a suitable choice of local parameters, any subscheme of length two may be defined by an ideal of the form

$$(x, y^2) \subset \mathcal{O}_{S,p}.$$

Thus any such subscheme is curvilinear. For subschemes of length three, there are two types: Firstly there are the curvilinear ones, which for a suitable choice of local parameters may be defined by an ideal of the form

$$(x, y^3) \subset \mathcal{O}_{S,p}.$$

Secondly there is just one non curvilinear subscheme of length three, namely the first infinitesimal neighbourhood of p, defined by

$$\mathfrak{m}_p^2 = (x^2, xy, y^2) \subset \mathcal{O}_{S,p}.$$

THEOREM 6.1. *For $n = 2$ and 3 the sheaf $\mathcal{F}_n$ is locally free of rank 2, hence $HMF(n+1)$ is a P^1-bundle over $HMF(n)$. In particular, $HMF(3)$ and $HMF(4)$ are nonsingular.*

PROOF. Any point in $HMF(2)$ is curvilinear, hence $\mathcal{F}_2$ has rank two everywhere. Thus it is locally free.

A punctual subscheme of length 3 is either the first order infinitesimal neighbourhood of p or it is curvilinear. Consider a point in $HMF(3)$, that is a filtration of ideals

$$I_3 \subset I_2 \subset I_1 = \mathfrak{m}_p.$$

If I_3 is curvilinear, then

$$I_3/(I_1 I_3 + I_2^2) = I_3/I_1 I_3$$

is two dimensional as before. If not, then $I_3 = (x^2, xy, y^2)$. For a suitable choice of local parameters we may assume $I_2 = (x, y^2)$. Then

$$I_1 I_3 + I_2^2 = (x^2, y^3, xy^2)$$

and hence

$$I_3/(I_1 I_3 + I_2^2) = \langle xy, y^2 \rangle$$

is two dimensional. Thus $\mathcal{F}_3$ has rank two everywhere.

The surface $HMF(3)$ can be determined completely. In fact it is isomorphic to the minimal ruled surface F_3. For this, let $R = k[a_0, a_1]$, then $HMF(2) = H(2) = \operatorname{Proj} R$ with universal family defined by the ideal

$$J = (a_1 y - a_0 x, x^2, xy, y^2) \subset R \otimes_k \mathcal{O}_{X,p}. \tag{16}$$

Then the sheaf $\mathcal{F}_2$ corresponds to the graded R-module N with generators

$$f = a_1 y - a_0 x \qquad g = x^2$$
$$h = xy \qquad k = y^2$$

where f has degree 1 and the rest have degree 0. The relations are

$$a_1 h = a_0 g \qquad a_1 k = a_0 h.$$

From this we conclude that N is isomorphic to $R(-1) \oplus R(2)$ in positive degrees, where f generates the summand corresponding to $R(-1)$, and g, h and k generate the summand corresponding to $R(2)$. Thus

$$\mathcal{F}_2 = \mathcal{O}_{\mathsf{P}^1}(-1) \oplus \mathcal{O}_{\mathsf{P}^1}(2)$$

and the associated projective bundle is F_3.

Finally, we remark that $HF(4)$ is reducible, so $HMF(4) = HF'(4)$ is not the only component. In fact, above the rational curve in $HF(3) = HMF(3)$ consisting of filtrations of the form

$$\mathfrak{m}_p^2 = I_3 \subset I_2 \subset I_1 = \mathfrak{m}_p$$

where I_2 varies freely in a P^1, every fibre in $HF(4)$ is a P^2. Thus the inverse image of this curve has dimension 3, which therefore cannot be contained in the irreducible three dimensional variety $HMF(4)$. To give an explicit example, the ideals

$$(x^2, xy, y^3) \subset (x^2, xy, y^2) \subset (x^2, y) \subset (x, y)$$

define a point in $HF(4)$ which is not multiplicative.

For $n = 5$ we obtain the following, which corrects [9, Theorem 1].

THEOREM 6.2. *$HMF(5)$ is singular along a curve, but irreducible.*

PROOF. We compute the restriction of $HMF(5)$ to a particular open affine chart $U_4 \subset HMF(4)$. By the same method one can compute an open cover explicitly.

With notation as in equation (16), let $U_2 \subset HMF(2)$ be the open affine subset defined by $a_0 \neq 0$. Then

$$U_2 = \operatorname{Spec} k[a]$$

where $a = a_1/a_0$, and the universal flag is defined by the ideals

$$J_1 = (x, y) \qquad J_2 = (ay - x, y^2). \tag{17}$$

Carrying through the recipe given above, we find

$$HMF(3)\big|_{U_2} = \operatorname{Proj} k[a][b_0, b_1]$$

where the generators b_i correspond to t_i in equation (15). We define the open affine $U_3 \subset HMF(3)$ by $b_0 \neq 0$, then the universal flag on U_3 is defined by ideals $J_1 \supset J_2 \supset J_3$, where J_1 and J_2 are the ideals in (17) and

$$J_3 = (b(ay - x) - y^2, (ay - x)x, (ay - x)y)$$

where $b = b_1/b_0$. (We should really write $J_1 k[a, b]$ and $J_2 k[a, b]$ in place of J_1 and J_2, but this shouldn't cause any confusion.) Since

$$a\left((ay - x)y\right) - (ay - x)x = (ay - x)^2 \in J_2^2$$

we find that U_3 trivializes $\mathcal{F}_3$ and

$$HMF(4)\big|_{U_3} = \operatorname{Proj} k[a, b][c_0, c_1].$$

where again the new coordinates c_i correspond to t_i in equation (15). Define $U_4 \subset HMF(4)$ by $c_0 \neq 0$, then the universal flag is defined over U_4 by

$$J_4 = (c(b(ay - x) - y^2) - (ay - x)y, b(ay - x)y - y^3, (ay - x)^2)$$

where $c = c_1/c_0$, together with J_1, J_2, J_3 as above.

Now we are in position to describe the restriction of $HMF(5)$ to U_4. The module

$$M = J_4/(J_1 J_4 + J_2 J_3)$$

is generated by

$$\begin{aligned} f &= c(b(ay - x) - y^2) - (ay - x)y \\ g &= b(ay - x)y - y^3 \\ h &= (ay - x)^2 \end{aligned}$$

and the element $bh - cf$ is contained in $J_2 J_3$, thus

$$HMF(5)\big|_{U_4} = \operatorname{Proj} k[a, b, c][F, G, H]/(bH - cF).$$

In fact, since this is irreducible, reduced and of dimension four, the found relation $bh - cf$ is the only one.

Thus $HMF(5)\big|_{U_4}$ is irreducible and singular along a curve. Repeating the calculations while moving U_4 around proves the statement.

By the same procedure one may test the irreducibility of $HMF(n)$, and hence question 5.5, for higher n. The explicit calculations get rather involved, but with the aid of the computer program Singular [3], using a primary decomposition algorithm, it has been verified that $HMF(n)$ is irreducible for $n \leq 7$, and also that $\mathrm{Mult}^n(X)$ is already reduced. At 8 points we stopped due to lack of computer power.

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