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Hilbert schemes of linear group orbits in the affine plane

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"The proof of this theorem is not easy."
David Hilbert, 1890

Preface

I have had the opportunity to be working with a topic that combines classical material, both easy to grasp and quite the opposite, with modern ongoing research. The objects studied are simple singularities, or Klein singularities, Du Val singularities, rational doublepoints ... The richness of names gives an indication of the wide range of possible angles of view. Moreover, the celebrated ADE classification reveals links to as different subjects as Lie algebras, finite subgroups of SL(2), quivers, Von Neumann algebras plus some physics and other stuff I know nothing about. And of course the platonic solids. McKay's observation from 1980, the so called McKay correspondence, deals with the link to finite subgroups of SL(2). In 1996 Ito and Nakamura found a viewpoint connected with the Hilbert scheme of points in the plane, and their work is the starting point of my thesis.

The McKay correspondence works as a guiding problem throughout this text. There are two rather different parts: The first, shorter part is chapter 2, where a slight variation over Ito/Nakamura's construction is formulated. Some results are shown, whereas other conjectures and questions are left open. The second part, which is chapters 3–6, consists of explicit calculations, verifying the conjectures from chapter 2 in special cases. This part uses a strategy from [ES88], utilizing torus actions and associated cell decompositions.

Make the following conventions: All schemes considered are defined over an algebraically closed field k of characteristic zero. For an affine scheme X , write $k[X]$ for the coordinate ring, i.e. the ring of global sections of the structure sheaf $\mathcal{O}_X$. In general the notation follows (of course) Hartshorne [Har77].

I would like to thank everyone at the mathematics department at the University of Oslo, and in particular the algebraic geometry and topology groups—both students and employees. Most of all I am grateful to my supervisor, professor Geir Ellingsrud, both for all the time he has spent teaching, explaining and discussing with me, and for showing me the way into a rich and challenging subject.

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Contents

1 Background **5**

1.1 Simple singularities 5

1.2 McKay's observation 5

1.3 Ito/Nakamura's construction 6

1.4 Generalizations 7

2 McKay correspondence **8**

2.1 Statement 8

2.2 Equivariant free resolutions 9

2.3 Partial proof of the theorem 12

2.3.1 Irreducibility 12

2.3.2 Global socle 13

2.3.3 Adjacency 14

3 Preliminaries on torus actions **16**

3.1 Torus actions and cell decompositions 16

3.2 Morphisms to $\mathrm{Hilb}^n(\mathbf{A}^2)$ 19

4 Cyclic groups **22**

4.1 Cell decompositions 23

4.2 Affine charts 25

4.3 Remarks 28

5 Binary dihedral groups **30**

5.1 A cell decomposition 32

5.2 Stratification of G-$\mathrm{Hilb}(\mathbf{A}^2)$ 36

5.3 Remarks 43

6 Abelian subgroups of GL(2) **47**

6.1 Fixpoints 47

6.2 Affine charts 48

Chapter 1

Background

1.1 Simple singularities

The objects studied in the present paper are simple singularities and their resolutions. There are many characterizations of such singularities. In this text the following will be taken as the definition.

Definition 1.1. A point p on a surface X is a *simple singularity* if it is analytically isomorphic to the singularity at the origin in the quotient $\mathbf{A}^2/G$ for some finite group $G \subset \mathrm{SL}(2)$ acting the natural way on $\mathbf{A}^2$.

It is well known that such a singularity admits a unique minimal resolution $\pi : \widetilde{X} \to X$, i.e. such that every other resolution factors through π. Furthermore, the components of the exceptional fibre $\pi^{-1}(0)$ are smooth rational curves intersecting transversally.

The conjugacy classes of finite subgroups of SL(2) consist of two countable families, the cyclic and binary dihedral groups, and the three binary platonic groups, the binary tetrahedron, octahedron and icosahedron group. The corresponding quotient singularities are said to be of type A_n, D_n, E_6, E_7 and E_8. The naming comes from the following construction: Given the resolution $\pi : \widetilde{X} \to X$ one may draw a graph, with one vertex for each component of the exceptional fibre, and with an edge connecting two vertices whenever the corresponding curves in $\widetilde{X}$ intersect. The resulting graph, called the *dual graph* of π, is one of the Dynkin diagrams A_n, D_n, E_6, E_7 and E_8.

Of interest are also the (non-simple) singularities $\mathbf{A}^2/G$ for finite subgroups $G \subset \mathrm{GL}(2)$. The abelian case is treated in chapter 6.

1.2 McKay's observation

Definition 1.2. Given a finite subgroup $G \subset \mathrm{SL}(2)$, the *canonical representation Q* is the representation of degree 2 given by the inclusion.

With G as above, let $\{V_i\}$ be the set of irreducible representations and let $V_i \otimes Q = \bigoplus a_{ij} V_j$ be the isotypical decomposition. The integers a_{ij} thus defined turn out to have the properties that $a_{ij} = a_{ji}$ and each a_{ij} is either 0 or 1 (this is not true for subgroups of GL(2) in general). Thus one may construct a graph as follows.

Definition 1.3. The *representation graph* of a finite subgroup $G \subset \mathrm{SL}(2)$ consists of a vertex for each irreducible representation, and an edge connecting the representations V_i and V_j whenever V_j occurs in $V_i \otimes Q$.

McKay's observation [McK80] is that the representation graph is an "extended" Dynkin diagram: If one removes the vertex corresponding to the trivial representation then the result equals the dual graph of the minimal resolution of $\mathbf{A}^2/G$. Thus there is a bijective correspondence between the irreducible components of the exceptional fibre and the irreducible representations of the group, preserving the structure given by the edges in the two graphs. This is called the *McKay correspondence*. The correspondence can be verified by checking each case, although such a procedure isn't very illuminating. The haunt for a natural way of realizing the McKay correspondence has triggered a lot of research. Of importance is the construction by Gonzalez-Springberg and Verdier [GSV83] where the correspondence is interpreted in terms of K-theory. Roughly speaking, it says that K-theory on the resolution of $\mathbf{A}^2/G$ equals G-equivariant K-theory on $\mathbf{A}^2$. This viewpoint opens for possible generalizations in several directions.

1.3 Ito/Nakamura's construction

The following construction is described by Ito and Nakamura [IN99].

Let n be the order of the group $G \subset \mathrm{SL}(2)$. There is a natural inclusion $\mathbf{A}^2/G \hookrightarrow \mathrm{Sym}^n(\mathbf{A}^2)$ into the symmetric product of n copies of the plane, sending an orbit to its n points. The Hilbert-Chow morphism

$$\mathrm{Hilb}^n(\mathbf{A}^2) \to \mathrm{Sym}^n(\mathbf{A}^2) \tag{1.1}$$

sending a zero-dimensional subscheme to its support, counted with multiplicities, is known to be a resolution of singularities [Fog68]. The idea is to use this morphism to construct a resolution of the simple singularity. The inverse image of $\mathbf{A}^2/G$ is the fixpoint locus $\mathrm{Hilb}^n(\mathbf{A}^2)^G$ under the induced action of G on $\mathrm{Hilb}^n(\mathbf{A}^2)$. Now this subscheme may not be connected, but there is a distinguished component.

Definition 1.4. The *Hilbert scheme of G-orbits*, G-$\mathrm{Hilb}(\mathbf{A}^2)$, is the unique component of $\mathrm{Hilb}^n(\mathbf{A}^2)^G$ containing the points corresponding to free G-orbits in $\mathbf{A}^2$.

Proposition 1.5 ([IN99]). *The restriction of the Hilbert-Chow morphism to G-$\mathrm{Hilb}(\mathbf{A}^2)$ is a minimal resolution of $\mathbf{A}^2/G$ (considered as a subscheme of $\mathrm{Sym}^n(\mathbf{A}^2)$).*

A few remarks on the nature of G-$\mathrm{Hilb}(\mathbf{A}^2)$ is in order: At the level of sets, view (the closed points of) the Hilbert scheme of points as a set of ideals.

$$\mathrm{Hilb}^n(\mathbf{A}^2) = \{I \subset k[\mathbf{A}^2] \mid \dim_k k[\mathbf{A}^2]/I = n\} \tag{1.2}$$

Proposition 1.6 ([Nak99]). *The closed points of G-$\mathrm{Hilb}^n(\mathbf{A}^2)$ are the ideals I in $\mathrm{Hilb}^n(\mathbf{A}^2)^G$ such that $k[\mathbf{A}^2]/I \cong k[G]$ as $k[G]$-modules.*

Thus, if one defines a "scheme theoretic orbit" to be a finite G-invariant subscheme $Z \subset \mathbf{A}^2$ with coordinate ring isomorphic to $k[G]$, the scheme G-Hilb$(\mathbf{A}^2)$ may be viewed as a parameter space of such orbits.

Ito/Nakamura's construction of the McKay correspondence is now as follows. Consider the minimal resolution given by the Hilbert-Chow morphism

$$f : G\text{-Hilb}(\mathbf{A}^2) \to \mathbf{A}^2/G \tag{1.3}$$

with reduced exceptional fibre $E = f^{-1}(0)_{\text{red}}$ decomposing into irreducible components E_i. Denote the maximal ideal corresponding to the origin in $\mathbf{A}^2$ by $\mathfrak{m}$ and the ideal generated by non-constant invariants by $\mathfrak{n}$. Let $I \in E_i$ be an ideal corresponding to a smooth point of E, then associate to E_i the representation $I/(\mathfrak{m}I + \mathfrak{n})$, which turns out to be irreducible and independent of the choice of I. This association realizes the isomorphism of graphs described by McKay. In Ito/Nakamura's work this is verified by case-by-case calculations.

1.4 Generalizations

The construction of G-Hilb$(\mathbf{A}^2)$ for $G \subset \mathrm{SL}(2)$ can be generalized to G-Hilb(X) for other groups G acting on other schemes X. In general proposition 1.6 may not be true, so there are two possible definitions of G-Hilb$(\mathbf{A}^2)$ — either using definition 1.4 or taking the characterization in proposition 1.6 as the definition. In the cases mentioned below, they are equivalent (for this, see [BKR99]). In any case, there are a number of questions to be raised: Is G-Hilb(X) smooth, is it a resolution of singularities and can the McKay correspondence be generalized? Having the construction of Gonzalez-Springberg and Verdier in mind, such generalizations are commonly formulated in terms of K-theory or derived categories.

The case G-Hilb$(\mathbf{A}^3)$ for $G \subset \mathrm{SL}(3)$ is the most well-studied generalization. The map induced by the Hilbert-Chow morphism is indeed a resolution, which may be surprising, since $\mathrm{Hilb}^n(\mathbf{A}^3)$ is far from smooth. Furthermore, a McKay correspondence in terms of equivalence of derived categories is constructed by Bridgeland and others in [BKR99]. For abelian groups one may construct G-Hilb$(\mathbf{A}^3)$ as a toric resolution. However, such a resolution is no longer unique, so one may ask which one is the "distinguished one", a question that is answered explicitly by Craw and Reid [CR99].

Another case of interest is G-Hilb$(\mathbf{A}^2)$ for any finite, small (i.e. without reflections) $G \subset \mathrm{GL}(2)$. G-Hilb$(\mathbf{A}^2)$ is then a minimal resolution. A generalization of the McKay correspondence to this setting does exist and gives a bijection between the components of the exceptional fibre and a *subset* of the irreducible representations, called "special" representations. A description in terms of derived categories is given by Ishii [Ish00].

The present paper is mainly devoted to the study of G-Hilb$(\mathbf{A}^2)$ for $G \subset \mathrm{SL}(2)$. The techniques used may also be applied to the general case $G \subset \mathrm{GL}(2)$, but are restricted to dimension 2 because of a heavy dependence upon the Hilbert-Burch theorem. A short treatment in the case of abelian subgroups of $\mathrm{GL}(2)$ is given in chapter 6.

Chapter 2

Construction of the McKay correspondence

In this chapter a construction of the McKay correspondence is given. It is slightly different from the one given by Ito/Nakamura, but shown to be equivalent to theirs. The results given here are partly conjectural. The idea is to set up a framework in which the correspondence can be given a good formulation, and prove as much as possible in this general setting. The conjectures are verified in special cases by explicit calculations (independent of the ones by Ito/Nakamura) in the next chapters.

In the following, the ideals in the exceptional fibre E of the resolution $G\text{-Hilb}(\mathbf{A}^2) \to \mathbf{A}^2/G$ are studied. Note that E consists of the ideals supported at the origin, so one has

$$E = \{I \in G\text{-Hilb}(\mathbf{A}^2) \mid I \subset \mathfrak{m}\}. \tag{2.1}$$

It is convenient to work with the local ring $k[\![x,y]\!]$ in place of $k[\mathbf{A}^2] = k[x,y]$. To be able to transfer the results back to $k[\mathbf{A}^2]$ one should make some observations. First, a G-invariant ideal $I \subset k[x,y]$ gives rise to the G-invariant ideal $J = Ik[\![x,y]\!]$, and if $k[x,y]/I$ is finite dimensional as a k-vector space, then it is isomorphic to $k[\![x,y]\!]/J$ as a $k[G]$-module (the basis for the first vector space maps to a basis for the latter under the inclusion $k[x,y] \subset k[\![x,y]\!]$). Furthermore, the two $k[G]$-modules $\mathrm{Soc}(k[x,y]/I)$ and $I/\mathfrak{m}I$ shall be of interest, and these are isomorphic to $\mathrm{Soc}(k[\![x,y]\!]/J)$ and $J/\mathfrak{m}J$ respectively. Here $\mathfrak{m}$ denotes the ideal generated by x and y in both rings and the socle is its annihilator. So let $R = k[\![x,y]\!]$ for the rest of this chapter and make the convention that "an ideal I in E" means the ideal generated by I in $k[\![x,y]\!]$. Then everything said about the $k[G]$-modules $\mathrm{Soc}(R/I)$ and $I/\mathfrak{m}I$ also applies when replacing R by $k[x,y]$.

2.1 Statement

Theorem 2.1 (McKay correspondence). *Let $G \subset \mathrm{SL}(2)$ be a finite subgroup and let*

$$f : G\text{-Hilb}(\mathbf{A}^2) \to \mathbf{A}^2/G \tag{2.2}$$

be the resolution given by the Hilbert-Chow morphism with reduced exceptional fibre $E = f^{-1}(0)_{\mathrm{red}}$ *decomposing into irreducible components* E_i. *Let* $I \in E_i$ *be a smooth point of* E. *Then the following holds.*

(i) $\operatorname{Soc}(R/I)$ *is irreducible.*

(ii) Up to isomorphism, the representation $\operatorname{Soc}(R/I)$ *is independent of the choice of* $I \in E_i$ *along smooth points of* E.

(iii) This association gives a bijection between the set of irreducible components $\{E_i\}$ *and the set of irreducible nontrivial representations* $\{V_i\}$.

(iv) Two components E_i *and* E_j *intersect if and only if the corresponding representations* V_i *and* V_j *are adjacent in the representation graph, i.e.* V_i *occurs in* $Q \otimes V_j$.

The representation $\operatorname{Soc}(R/I)$ is shown to be the same as Ito/Nakamura's $I/(\mathfrak{m}I + \mathfrak{n})$ in proposition 2.5. The theorem is thus a reformulation of their work. In the following, an independent partial proof is given: Modulo two conjectures given, it is shown that the association $E_i \mapsto V_i$ is well defined, and *if it is a bijection*, then it is in fact a realization of the isomorphism of graphs observed by McKay. In the next chapters it is verified that this construction indeed is a bijection in the A_n and D_n cases.

2.2 Equivariant free resolutions

In this section the existence of an isomorphism of representations $\operatorname{Soc}(R/I) \cong I/(\mathfrak{m}I + \mathfrak{n})$ is proved. This result follows from studying G-equivariant free resolutions of I. Apart from showing that theorem 2.1 is equivalent to the work by Ito/Nakamura, the isomorphism is significant for the attempts at proving the theorem, given later in this chapter.

Since R is local there exists a unique minimal free resolution of I. In fact such a resolution can be made G-equivariant.

Proposition 2.2. *Let* G *be a finite group acting on a local ring* A, *keeping the maximal ideal* $\mathfrak{m}$ *invariant. Let* M *be a finitely generated* $A[G]$*-module. Then there exists a minimal free resolution of* M *as an* A*-module such that the free* A*-modules and the maps between them also carry the structure of* $A[G]$*-modules and -homomorphisms.*

Proof. This follows from the standard construction of a minimal resolution, taking care to make all maps equivariant. By Nakayama's lemma, there exists a minimal generating set $f_1, \ldots, f_r$ of M as an A-module, corresponding to a basis $\overline{f}_1, \ldots, \overline{f}_r$ of $M/\mathfrak{m}M$ as a vector space over $K = A/\mathfrak{m}$. Let $F_1 = A \otimes_K M/\mathfrak{m}M$, which is free as A-module. Let $\varphi : F_1 \to M$ be any A-module-homomorphism such that there is a commutative diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{\varphi} & M \\ & \searrow & \downarrow \\ & & M/\mathfrak{m}M \end{array} \tag{2.3}$$

(for instance defining $\varphi(a \otimes \overline{f}_i) = af_i$) where the diagonal map is the composite $F_1 \to F_1 \otimes_A K \cong M/\mathfrak{m}M$. For any $g \in G$, φ may be replaced by the map $g \cdot \varphi$ taking $a \in F_1$ to $g\varphi(g^{-1}a)$ without affecting the commutativity of the diagram, because the other two maps are equivariant. So one may set

$$\varphi_1 = \frac{1}{|G|} \sum_{g \in G} g \cdot \varphi \tag{2.4}$$

which is a G-equivariant map $F_1 \to M$, still giving a commutative diagram. Now replace M by the kernel of this map and construct F_2 and φ_2 the same way. Continuing like this gives the resolution. □

The Hilbert-Burch theorem applied to $I \in E$ states that the free resolution is of the form

$$0 \to F_2 \to F_1 \to I \to 0 \tag{2.5}$$

where $F_2 = R^m$ and $F_1 = R^{m+1}$ as R-modules, and where the (matrix of the) second nontrivial map consists of the $m \times m$ minors of the first, with proper signs inserted. In fact, the precise $R[G]$-module structure of the free R-modules can be determined.

Proposition 2.3. *Let I be a point in the exceptional fibre and let*

$$0 \to F_2 \to F_1 \to I \to 0 \tag{2.6}$$

be a minimal, G-equivariant, free resolution of I. Then

$$F_1 \otimes_R k \cong \operatorname{Soc} R/I \tag{2.7}$$

$$F_2 \otimes_R k \cong I/\mathfrak{m}I \tag{2.8}$$

as $k[G]$-modules.

Proof. Apply $- \otimes_R k$ to the given resolution to obtain the exact sequence

$$0 \to \operatorname{Tor}_1(I,k) \to R^m \otimes k \xrightarrow{0} R^{m+1} \otimes k \to I \otimes k \to 0. \tag{2.9}$$

Because of the assumed minimality, the middle map is zero as indicated. Since $I \otimes k \cong I/\mathfrak{m}I$ in a G-equivariant way, the last statement is proved. To calculate $\operatorname{Tor}_1(I,k)$, use the short exact sequence

$$0 \to I \to R \to R/I \to 0 \tag{2.10}$$

which gives

$$0 \to \operatorname{Tor}_2(R/I,k) \to \operatorname{Tor}_1(I,k) \to 0 \tag{2.11}$$

in a G-equivariant way. The free resolution of k

$$0 \to R \xrightarrow{\binom{-y}{x}} R^2 \xrightarrow{(x\ y)} R \tag{2.12}$$

shows that $\operatorname{Tor}_2(R/I,k)$ is the second homology module of

$$0 \to R/I \to (R/I)^2 \to R/I. \tag{2.13}$$

So $\operatorname{Tor}_2(R/I,k) = \ker(R/I \to (R/I)^2) = \operatorname{Soc}(R/I)$. □

Let $R(G)$ denote the representation ring of G and let $K^G(R)$ denote the Grothendieck group of finitely generated $R[G]$-modules, that is, the free abelian group generated by all finitely generated $R[G]$-modules, divided by the relations $M - M' - M''$ for all extensions $0 \to M' \to M \to M'' \to 0$. There is an isomorphism of groups $R(G) \cong K^G(R)$ described by Gonzalez-Springberg and Verdier [GSV83], sending a representation V to the $R[G]$-module $V \otimes_k R$. The inverse map sends an $R[G]$-module M to the element $\sum(-1)^i \operatorname{Tor}_i^R(M,k)$ of $R(G)$. For M projective this is just $M \otimes_R k$.

In view of this isomorphism, proposition 2.3 implies that $F_1 \cong R \otimes_k \operatorname{Soc}(R/I)$ and $F_2 \cong R \otimes_k I/\mathfrak{m}I$ as $R[G]$-modules.

Under the isomorphism $R(G) \cong K^G(R)$ just described, the free resolution in proposition 2.3 carries over to a relation between the representations $\operatorname{Soc}(R/I)$ and $I/\mathfrak{m}I$. For this a lemma is helpful.

Lemma 2.4. *Let M be a finitely generated $R[G]$-module supported in $\mathfrak{m}$. Then*

$$M = \sum_{i \geq 0} \mathfrak{m}^i M / \mathfrak{m}^{i+1} M \tag{2.14}$$

in $K^G(R)$. In particular, the class of M is determined by its $k[G]$-module structure.

Proof. The short exact sequences

$$0 \to \mathfrak{m}^{i+1} M \to \mathfrak{m}^i M \to \mathfrak{m}^i M/\mathfrak{m}^{i+1} M \to 0 \tag{2.15}$$

show that $\mathfrak{m}^i M = \mathfrak{m}^i M/\mathfrak{m}^{i+1} M + \mathfrak{m}^{i+1} M$ in $K^G(R)$. Since M is supported in $\mathfrak{m}$ there exists an integer N such that $\mathfrak{m}^N M = 0$, thus $M = \sum_{i=0}^{N-1} \mathfrak{m}^i M/\mathfrak{m}^{i+1} M$.

The last statement follows when noting that $M \cong \bigoplus_i \mathfrak{m}^i M/\mathfrak{m}^{i+1} M$ as $k[G]$-modules, which follows from the same short exact sequence. ☐

Proposition 2.5. *There is an isomorphism $I/\mathfrak{m}I \cong \operatorname{Soc}(R/I) \oplus k$ of $k[G]$-modules.*

Proof. The minimal free resolution

$$0 \to R \otimes \operatorname{Soc}(R/I) \to R \otimes I/\mathfrak{m}I \to R \to R/I \to 0 \tag{2.16}$$

gives the relation

$$R \otimes I/\mathfrak{m}I = R \otimes \operatorname{Soc}(R/I) + R - R/I \tag{2.17}$$

in $K^G(R)$. Furthermore, by lemma 2.4, $R/I = k[G]$ in K-theory since they are isomorphic as $k[G]$-modules. In $R(G)$ this gives the relation

$$I/\mathfrak{m}I = \operatorname{Soc}(R/I) + k - k[G] \tag{2.18}$$

Thus the lemma follows if $k[G] = 0$ in $R(G)$. For this, consider the G-equivariant Koszul complex

$$0 \to R \xrightarrow{\binom{-y}{x}} R \otimes_k Q \xrightarrow{(x\ y)} R \to R/\mathfrak{m} \to 0. \tag{2.19}$$

Then one may apply $-\otimes_k k[G]$ to obtain the exact sequence

$$0 \to R[G] \to R \otimes_k (k[G] \otimes_k Q) \to R[G] \to k[G] \to 0. \tag{2.20}$$

Now $k[G] \otimes_k Q \cong k[G]^{\oplus 2}$, in fact $k[G] \otimes_k V \cong k[G]^{\oplus \deg V}$ for any representation V, since the character χ of $k[G]$ is given by $\chi(g) = 0$ for any $g \neq 1$ and $\chi(1) = \deg V$. Thus, in $K^G(R)$ there is a relation

$$R[G] - R[G]^{\oplus 2} + R[G] - k[G] = 0 \tag{2.21}$$

showing that $k[G] = 0$. □

Remark 2.6. The representation $\mathrm{Soc}(R/I)$ never contains the trivial representation. For the regular representation R/I contains exactly one copy of the trivial representation, namely $\langle 1 \rangle$. But 1 generates R/I as an R-module, so $1 \in \mathrm{Soc}(R/I)$ would imply $R/I = \langle 1 \rangle$.

In Ito/Nakamura's work, the trivial summand in $I/\mathfrak{m}I$ is killed off by dividing out all invariants: By proposition 2.5 and the subsequent remark, $\mathrm{Soc}(R/I) \cong I/(\mathfrak{m}I + \mathfrak{n})$, so the correspondence formulated in theorem 2.1 is the same as the one formulated by Ito/Nakamura.

2.3 Partial proof of the theorem

2.3.1 Irreducibility

First consider point (i) in theorem 2.1, the irreducibility of the representation $\mathrm{Soc}(R/I)$.

Proposition 2.7. *Let I be an ideal in E. Then the representation $\mathrm{Soc}\, R/I$ is either irreducible or a sum of two distinct irreducibles.*

Proof. The tangent space at I in $\mathrm{Hilb}^n(\mathbf{A}^2)$ is $\mathrm{Hom}_R(I, R/I)$, hence the tangent space at I in G-$\mathrm{Hilb}(\mathbf{A}^2)$ is the G-invariant part, or $\mathrm{Hom}_{R[G]}(I, R/I)$. Clearly,

$$\mathrm{Hom}_{k[G]}(I/\mathfrak{m}I, \mathrm{Soc}(R/I)) \cong \mathrm{Hom}_{R[G]}(I, \mathrm{Soc}(R/I)) \subset \mathrm{Hom}_{R[G]}(I, R/I) \tag{2.22}$$

where the isomorphism is composition with $I \to I/\mathfrak{m}I$ on the left and the inclusion is composition with $\mathrm{Soc}(R/I) \hookrightarrow R/I$ on the right. The leftmost map is an isomorphism since the kernel of an R-module-homomorphism

$$I \to \mathrm{Soc}(R/I) \tag{2.23}$$

necessarily contains $\mathfrak{m}I$, and furthermore x and y act trivially on both $I/\mathfrak{m}I$ and $\mathrm{Soc}(R/I)$. The result follows from a dimension count: Suppose $\mathrm{Soc}(R/I) = \bigoplus a_i V_i$ is the isotypical decomposition. Then, by remark 2.6, the trivial representation V_0 doesn't occur, and $I/\mathfrak{m}I \ominus V_0 = \bigoplus a_i V_i$. By Schur's lemma

$$\dim \mathrm{Hom}_{k[G]}(I/\mathfrak{m}I, \mathrm{Soc}(R/I)) = \sum a_i^2. \tag{2.24}$$

On the other hand, the dimension of the tangent space $\mathrm{Hom}_{R[G]}(I, R/I)$ is two since G-$\mathrm{Hilb}(\mathbf{A}^2)$ is a smooth surface, so by (2.22) one has $\sum a_i^2 \leq 2$. Thus, a_i is nonzero for at most two indices i, for which $a_i = 1$. □

It seems reasonable to suggest the following sharpening of the previous proposition.

Conjecture 2.8. *Let I be an ideal in E. Then the tangent space to E at I is*

$$T_I(E) = \mathrm{Hom}_{k[G]}(I/\mathfrak{m}I, \mathrm{Soc}(R/I)). \tag{2.25}$$

Corollary 2.9. *Assume conjecture 2.8 is true. Then, if I is a smooth point in E, $\mathrm{Soc}(R/I)$ is irreducible. Otherwise $\mathrm{Soc}(R/I)$ is the sum of two distinct irreducibles.*

In particular, point (i) of theorem 2.1 follows from the conjecture.

In [IN99] it is proved by case-by-case computations that $I/(\mathfrak{m}I + \mathfrak{n})$ is irreducible at smooth points of E, and the sum of two irreducibles at singular points. Thus, in view of proposition 2.5, the corollary is true. This provides some evidence for the conjecture.

2.3.2 Global socle

Let E_i be an exceptional component in G-Hilb$(\mathbf{A}^2)$, and let $\pi : Z \to E_i$ be the restriction of the universal family to E_i, such that the fibre over $p \in E_i$ is the subscheme of $\mathbf{A}^2$ corresponding to the point p itself. Let I be the ideal corresponding to p. Then the fibre over p has coordinate ring R/I, and there is a homomorphism of representations

$$R/I \to R/I \otimes_k Q \tag{2.26}$$

sending $a \in R/I$ to $ax \otimes y - ay \otimes x$. Note that for this map to be equivariant, it is essential that G is a subgroup of $\mathrm{SL}(2)$. The representation $\mathrm{Soc}(R/I)$ is the kernel of this map. The idea is to give a "global" version of this construction, in terms of sheaves on E_i, which fibrewise approximates this situation.

So consider the G-equivariant morphism of $\mathcal{O}_{E_i}$-modules

$$\pi_*\mathcal{O}_Z \to (\pi_*\mathcal{O}_Z) \otimes_k Q \tag{2.27}$$

defined locally as above, sending $a \in \mathcal{O}_Z(\pi^{-1}(U))$ to $ax\otimes y - ay\otimes x$ for any affine open $U \subset E_i$. Let $\mathcal{E}$ denote the kernel of this map. This is the candidate for a "global socle". Restricting to the fibre over p one has $R/I = (\pi_*\mathcal{O}_Z) \otimes_{\mathcal{O}_{E_i}} k(p)$, where $k(p)$ denotes the residue field at p. The complex obtained by applying $- \otimes k(p)$

$$\mathcal{E} \otimes k(p) \to \pi_*\mathcal{O}_Z \otimes k(p) \to ((\pi_*\mathcal{O}_Z) \otimes_k Q) \otimes k(p) \tag{2.28}$$

may not be exact. However, the kernel of the rightmost map is by definition $\mathrm{Soc}(R/I)$, so there is a canonical homomorphism of representations $\mathcal{E} \otimes k(p) \to \mathrm{Soc}(R/I)$. This may not be an isomorphism, but due to the fact that E_i is a smooth curve, it is in fact a monomorphism.

Lemma 2.10. *Let $p \in E_i$ correspond to the ideal $I \subset R$. Then the canonical homomorphism $\mathcal{E} \otimes_k k(p) \to \mathrm{Soc}(R/I)$ is injective.*

Proof. Consider the exact sequence of $\mathcal{O}_{E_i}$-modules

$$0 \to \mathcal{E} \to \pi_*\mathcal{O}_Z \to (\pi_*\mathcal{O}_Z) \otimes_k Q \to \mathcal{F} \to 0 \tag{2.29}$$

where $\mathcal{F}$ is the cokernel. The claim is that the inclusion $\mathcal{E} \to \pi_*\mathcal{O}_Z$ is locally split. If this is the case, then tensoring with $k(p)$ preserves the splitting, giving the inclusion $\mathcal{E} \otimes k(p) \to \pi_*\mathcal{O}_Z \otimes k(p)$ with image in $\mathrm{Soc}(R/I)$ as above.

The local splitting occurs as follows. Since π is finite and flat $\pi_*\mathcal{O}_Z$ is locally free. Thus the map $(\pi_*\mathcal{O}_Z) \otimes_k Q \to \mathcal{F}$ is locally the beginning of a free resolution of $\mathcal{F}$. In fact the kernel $\mathcal{G}$ of this map must also be locally free, since the projective dimension of any $\mathcal{O}_{E_i}$-module is locally at most $\dim E_i = 1$, E_i being smooth. So there is a short exact sequence

$$0 \to \mathcal{E} \to \pi_*\mathcal{O}_Z \to \mathcal{G} \to 0 \tag{2.30}$$

where the middle and rightmost modules are locally free. Thus the sequence is locally split. □

Now for the proof of (ii) in theorem 2.1. Assuming (i) is true, the following result is sufficient.

Proposition 2.11. *Let $p_j \in E_i$ $(j = 1, 2)$ be two smooth points of E corresponding to ideals I_j such that the representations $\mathrm{Soc}(R/I_j)$ are irreducible. Then $\mathrm{Soc}(R/I_1) \cong \mathrm{Soc}(R/I_2)$.*

Proof. Let $e_l \in k[G]$ be idempotent elements such that $k[G] = \bigoplus e_l k[G]$ is the isotypical decomposition. Then there is a canonical decomposition of the "socle sheaf" $\mathcal{E} = \bigoplus e_l \mathcal{E}$.

Now the map $\mathcal{E} \otimes k(p_j) \to \mathrm{Soc}(R/I_j)$, which is injective by the previous lemma, must be an isomorphism when $\mathrm{Soc}(R/I_j)$ is irreducible. Thus $\mathcal{E} \otimes k(p_j)$ is also irreducible, equal to one of its direct summands $e_{l_j}(\mathcal{E} \otimes k(p)) \cong (e_{l_j}\mathcal{E}) \otimes k(p)$. But then $\mathcal{E} = e_{l_1}\mathcal{E} = e_{l_2}\mathcal{E}$, so there are in fact isomorphisms

$$\mathrm{Soc}(R/I_1) \cong \mathcal{E} \otimes k(p_1) \cong \mathcal{E} \otimes k(p_2) \cong \mathrm{Soc}(R/I_2) \tag{2.31}$$

proving the proposition. □

2.3.3 Adjacency

Now assume that (iii) in the theorem is true, such that the association $E_i \mapsto V_i$ constructed is a bijection. To shed some light on point (iv), the adjacency condition, one may again utilize the global socle construction.

Proposition 2.12. *Let $p \in E_i \cap E_j$ for $i \neq j$ and assume E_i and E_j correspond to two distinct irreducible representations V_i and V_j. Then $\mathrm{Soc}(R/I) \cong V_i \oplus V_j$.*

Proof. Let $\mathcal{E}$ and $\mathcal{E}'$ be the "socle sheaves" on E_i resp. E_j. Then, by lemma 2.10, there are two injections

$$V_i \cong \mathcal{E} \otimes k(p) \hookrightarrow \mathrm{Soc}(R/I) \hookleftarrow \mathcal{E}' \otimes k(q) \cong V_j. \tag{2.32}$$

By assumption $V_i \ncong V_j$, so $\mathrm{Soc}(R/I)$ contains $V_i \oplus V_j$. By proposition 2.7, $\mathrm{Soc}(R/I)$ is the sum of at most two irreducible representations, hence the isomorphism is proved. □

Hence if two components E_i and E_j intersect, one has the reducible representation $\mathrm{Soc}(R/I)$ at the intersection point. A possible strategy for explaining point (iv) of theorem 2.1 is then to study which representations that can coexist in a socle.

For this consider the G-equivariant product map

$$\mathfrak{m}/\mathfrak{m}^2 \otimes \mathrm{Soc}\, R/I \to I/\mathfrak{m}I. \tag{2.33}$$

Note that this is indeed well defined. The representation $\mathfrak{m}/\mathfrak{m}^2$ is precisely the canonical representation Q. Letting $V = \mathrm{Soc}\, R/I$ and composing on the right with the projection $I/\mathfrak{m}I \to I/(\mathfrak{m}I + \mathfrak{n}) \cong V$, one obtains a homomorphism of representations

$$Q \otimes V \to V \tag{2.34}$$

Now, if V is irreducible, this map is zero by Schur's lemma, since the representation graph of $G \subset \mathrm{SL}(2)$ contains no loops.

Conjecture 2.13. *If $V = \mathrm{Soc}(R/I)$ is the sum of two irreducible representations, the map*

$$Q \otimes V \to V \tag{2.35}$$

defined above is a projection.

If the association $E_i \mapsto V_i$ defined by the socle construction is a bijection between irreducible exceptional components and nontrivial irreducible representations, point (iv) of theorem 2.1 follows from this conjecture. In other words, the bijection between the nodes of the two graphs considered gives an isomorphism of graphs. For this, assume E_i and E_j are two components that intersect, so that $\mathrm{Soc}(R/I) = V_i \oplus V_j$ for $I \in E_i \cap E_j$. The conjecture claims the existence of a projection map $Q \otimes (V_i \oplus V_j) \to V_i \oplus V_j$. Then Schur's lemma implies that V_i occurs in $Q \otimes V_j$, i.e. V_i and V_j are adjacent in the representation graph. Thus there is an edge in the representation graph for each edge in the dual graph of the resolution. Conversely there can't be any other edges present: The exceptional fibre is connected, hence the dual graph is connected. The presence of an extra edge would then imply the existence of a loop in the representation graph, but no such loop exists.

Chapter 3

Preliminaries on torus actions

3.1 Torus actions and cell decompositions

In this section, some techniques from [ES87, ES88] are reviewed, together with some other general results, for later application to G-Hilb($\mathbf{A}^2$). There are no new results in this section.

Let $k[T_0, T_1, T_2]$ be the homogeneous coordinate ring of $\mathbf{P}^2$ and define $\mathbf{A}^2 \subset \mathbf{P}^2$ by $T_0 \neq 0$. Thus $k[\mathbf{A}^2] = k[x, y]$ with $x = T_1/T_0$ and $y = T_2/T_0$. Then $\mathrm{Hilb}^n(\mathbf{A}^2)$ denotes the subscheme of $\mathrm{Hilb}^n(\mathbf{P}^2)$ consisting of finite schemes of length n supported in the chosen $\mathbf{A}^2 \subset \mathbf{P}^2$.

Let $\Gamma \subset \mathrm{SL}(3)$ be the two-dimensional algebraic torus consisting of all diagonal matrices. Then there is a canonical action of Γ on $\mathbf{P}^2$ given by $T_i \mapsto a_i T_i$ for each element $a = \mathrm{diag}(a_0, a_1, a_2)$ in Γ. Then $\mathbf{A}^2$ is invariant under Γ and the action restricts to $x \mapsto \frac{a_1}{a_0} x$ and $y \mapsto \frac{a_2}{a_0} y$. Let λ and μ denote the corresponding characters, such that $\lambda(a) = \frac{a_1}{a_0}$ and $\mu(a) = \frac{a_2}{a_0}$. Note that the induced action on $\mathrm{Hilb}^n(\mathbf{P}^2)$ keeps $\mathrm{Hilb}^n(\mathbf{A}^2)$ invariant.

One may now consider one-parameter subgroups $\mathbf{G}_m \to \Gamma$, inducing an action of $\mathbf{G}_m$ on the schemes $\mathbf{P}^2$, $\mathbf{A}^2$, $\mathrm{Hilb}^n(\mathbf{P}^2)$ and $\mathrm{Hilb}^n(\mathbf{A}^2)$. The reason for studying this is the following result.

Definition 3.1. A *cell decomposition* of a scheme X is a chain of closed subschemes $X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that each $X_i \setminus X_{i-1}$ (with $X_{-1} = \varnothing$) is a disjoint union of locally closed subschemes U_{ij}, each isomorphic to some affine space $\mathbf{A}^{n_{ij}}$. The spaces U_{ij} are called the cells of the decomposition.

Theorem 3.2 ([BB73][BB76]). *Let X be a smooth projective variety with a $\mathbf{G}_m$-action such that the fixpoint locus is a finite set $\{p_i\}$. Then X has a cell decomposition with cells*

$$U_i = \{p \in X \mid \lim_{t \to 0} t \cdot p = p_i\}. \tag{3.1}$$

Furthermore, denote by $T_{p_i}(X)^+$ the subspace of the tangent space $T_{p_i}(X)$ where the weights of the induced $\mathbf{G}_m$-action are positive. Then

$$T_{p_i}(U_i) = T_{p_i}(X)^+. \tag{3.2}$$

This result is applied to $\mathrm{Hilb}^n(\mathbf{P}^2)$ in [ES88]. In this paper it will be applied to some other subschemes of $\mathrm{Hilb}^n(\mathbf{P}^2)$. The finiteness of the fixpoint set requires some discussion.

Lemma 3.3. *Let $I \subset k[x,y]$ be invariant under the Γ-action above. Then I is generated by monomials.*

Proof. Let $f \in I$ be the sum of terms $c_{ij}x^iy^j$. Then

$$\gamma \cdot f = \sum c_{ij}\lambda(\gamma)^i\mu(\gamma)^j x^i y^j \in I \tag{3.3}$$

for all $\gamma \in \Gamma$. Since k is infinite it follows that each term $c_{ij}x^iy^j$ is in I. □

Obviously this shows that there are finitely many Γ-fixpoints in $\mathrm{Hilb}^n(\mathbf{A}^2)$, since there are finitely many monomial ideals of length n. For $\mathrm{Hilb}^n(\mathbf{P}^2)$, one may write any finite scheme $Z \subset \mathbf{P}^2$ of length n as the union of finite schemes of length $\leq n$ supported in some affine $\mathbf{A}^2 \subset \mathbf{P}^2$ and apply the same observation.

For most one-parameter subgroups $\psi : \mathbf{G}_m \to \Gamma$, the fixpoint set under the $\mathbf{G}_m$-action is the same as that under the Γ-action: For the next lemmas, let Γ be any n-dimensional torus (after this it will stay two-dimensional forever!) with n linearly independent characters λ_i. Then ψ is determined by the integers $(a_1, \ldots, a_n) \in \mathbf{Z}^n$ such that $\lambda_i \circ \psi(t) = t^{a_i}$. Thus one may write $\psi \in \mathbf{Z}^n$.

Lemma 3.4. *Let X be a scheme, proper over k, with an action of a torus Γ. Then the fixpoint locus X^Γ is nonempty.*

Proof. By induction on $\dim\Gamma$. First assume $\dim\Gamma = 1$, that is, $\Gamma = \mathbf{G}_m$. For any $p \in X$, $\lim_{t\to 0} t \cdot p$ is a fixpoint, and the limit exists because X is proper.

Induction step: If $\dim\Gamma > 1$ write $\Gamma = \Gamma_1 \times \Gamma_2$ where Γ_i are tori of dimension $< \dim\Gamma$. By induction X^{Γ_1} is nonempty, and since it is Γ_2-invariant and closed the induction hypothesis may be applied again to see that $(X^{\Gamma_1})^{\Gamma_2}$ is nonempty. But this is X^Γ. □

Proposition 3.5. *Let X be a projective scheme with an action of an n-dimensional torus Γ such that the fixpoint locus is finite. If a one-parameter subgroup $\psi : \mathbf{G}_m \to \Gamma$ is chosen outside a finite number of fixed hyperplanes in $\mathbf{Z}^n$, then $X^\Gamma = X^{\mathbf{G}_m}$.*

Proof. If W is a component of $X^{\mathbf{G}_m}$, then W is closed and Γ-invariant by the commutativity of the two actions. Thus, by lemma 3.4 there is a Γ-fixpoint $p \in W$. The induced action of Γ on $T_p(X)$ decomposes into irreducibles with characters $\chi_1, \ldots \chi_r$, and since all fixpoints of Γ are isolated, $\chi_i \neq 0$ for all i. The action of $\mathbf{G}_m$ on $T_p(X)$ has characters $\chi_i \circ \psi$, so p is also an isolated fixpoint for $\mathbf{G}_m$ (that is, $\{p\} = W$) if ψ is chosen outside the hyperplanes defined by $\chi_i \circ \psi = 0$. Repeating this for each point in X^Γ gives the finite number of hyperplanes. □

Remark 3.6. Given a torus action on a scheme X, a subgroup $\mathbf{G}_m \to \Gamma$ will be called *generic* if it is chosen outside the hyperplanes dictated by the previous lemma.

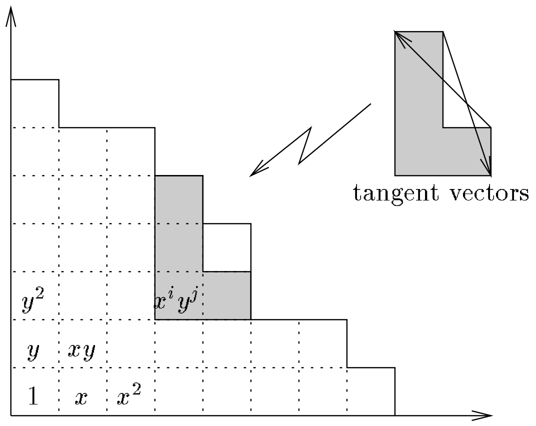

Figure 3.1: Monomial ideal

So any generic one-parameter subgroup $\psi : \mathbf{G}_m \to \Gamma$ induces a cell decomposition of $\mathrm{Hilb}^n(\mathbf{P}^2)$. Moreover, if the subgroup is chosen with some care then $\mathrm{Hilb}^n(\mathbf{A}^2)$ is a union of cells in the decomposition of $\mathrm{Hilb}^n(\mathbf{P}^2)$: If the induced $\mathbf{G}_m$-action on $\mathbf{A}^2$ has *negative weights* on coordinates, that is, $t \cdot x = t^a x$ and $t \cdot y = t^b y$ for all $t \in \mathbf{G}_m$ where a and b are negative integers (for short, just say that $\mathbf{G}_m$ acts with negative weights on $k[\mathbf{A}^2]$), then $\lim_{t\to 0} t \cdot p$ is the origin for any $p \in \mathbf{A}^2$. Furthermore, the complement of $\mathbf{A}^2$ in $\mathbf{P}^2$ is Γ-invariant, so if $p \in \mathbf{P}^2$ is in the line at infinity, then $t \cdot p$ also approaches a point at the infinity. For the Hilbert scheme this means that if $I \in \mathrm{Hilb}^n(\mathbf{A}^2)$, such that $V(I)$ is supported in $\mathbf{A}^2$, then $t \cdot V(I)$ approaches a scheme supported at the origin, thus corresponding to a point in $\mathrm{Hilb}^n(\mathbf{A}^2)$. Conversely, for any I in the complement of $\mathrm{Hilb}^n(\mathbf{A}^2)$, $\lim_{t\to 0} t \cdot V(I)$ has supporting points at the infinity, giving a point also in the complement of $\mathrm{Hilb}^n(\mathbf{A}^2)$. This shows that $\mathrm{Hilb}^n(\mathbf{A}^2)$ is a union of cells. Indeed the same argument shows that if $X \subset \mathrm{Hilb}^n(\mathbf{P}^2)$ is closed and invariant, then $X \cap \mathrm{Hilb}^n(\mathbf{A}^2)$ is a union of cells in the cell decomposition of X, again provided the chosen one-parameter subgroup gives negative $\mathbf{G}_m$-weights on $k[\mathbf{A}^2]$.

The fixpoints in $\mathrm{Hilb}^n(\mathbf{A}^2)$, i.e. the monomial ideals of colength n, will be of importance in the next chapters. Monomial ideals I will be depicted as in figure 3.1: The monomials outside the staircase are in I whereas those inside the staircase give a basis for $k[x,y]/I$.

For any Γ-fixpoint $I \in \mathrm{Hilb}^n(\mathbf{A}^2)$ there is an induced action of Γ on the tangent space at I. This action can be computed. For this, let $\mathcal{S}$ be the points inside the staircase in figure 3.1:

$$\mathcal{S} = \{(i,j) \mid x^i y^j \notin I\} \tag{3.4}$$

Each point x^iy^j determines a *hook*, which is the shaded area in the figure. Let a_j and b_i denote the endpoints of the hook:

$$a_j = \min\{a \mid x^{a_j} y^j \in I\} \tag{3.5}$$

$$b_i = \min\{b \mid x^i y^{b_i} \in I\} \tag{3.6}$$

In this notation, there is the following result.

Proposition 3.7 ([ES87]). *Let λ and μ be two linearly independent characters of Γ. Let T be the tangent space at the Γ-fixpoint I in $\mathrm{Hilb}^n(\mathbf{A}^2)$, with the induced action of Γ. Then, in the representation ring of Γ*

$$T = \sum_{(i,j)\in\mathcal{S}} \left(\lambda^{i-a_j}\mu^{b_i-j-1} + \lambda^{a_j-i-1}\mu^{j-b_i}\right). \tag{3.7}$$

Remark 3.8. The exponents involved can be thought of as the vectors $(i-a_j, b_i-j-1)$ and $(a_j-i-1, j-b_i)$ as indicated in figure 3.1. This makes the formula easy to use in practice.

Also, recall the following well-known result.

Proposition 3.9. *Let X be a smooth scheme over k, with an action of a finite group G. Then the following holds.*

(i) X^G *is smooth*

(ii) $T_p(X^G) = T_p(X)^G$ *for all* $p \in X^G$

3.2 Morphisms to $\mathrm{Hilb}^n(\mathbf{A}^2)$

Let $I \in \mathrm{Hilb}^n(\mathbf{A}^2)$ be a Γ-fixpoint, i.e. a monomial ideal of colength n. In the decomposition of $\mathrm{Hilb}^n(\mathbf{A}^2)$ induced by the one-parameter subgroup $\mathbf{G}_m \to \Gamma$, let U be the cell containing I. In order to find an isomorphism $\mathbf{A}^r \cong U$ explicitly, one may utilize the following construction of morphisms $\mathbf{A}^r \to \mathrm{Hilb}^n(\mathbf{A}^2)$.

One may obviously find a generating set of the form

$$I = (y^{l_1}, x^{k_1}y^{l_2}, \ldots, x^{k_{m-1}}y^{l_m}, x^{k_m}) \tag{3.8}$$

such that $l_i \geq l_{i+1}$ and $k_i \leq k_{i+1}$ (for example taking all the monomials along the "staircase" in figure 3.1). Such a generating set gives rise to a free resolution, possibly non-minimal,

$$0 \to R^m \xrightarrow{A} R^{m+1} \to R \to R/I \to 0 \tag{3.9}$$

where $R = k[x, y]$,

$$A = \begin{pmatrix} x^{i_1} & & & 0 \\ y^{j_1} & x^{i_2} & & \\ & \ddots & \ddots & \\ & & y^{j_{m-1}} & x^{i_m} \\ 0 & & & y^{j_m} \end{pmatrix} \tag{3.10}$$

and the exponents are defined by

$$i_p = k_p - k_{p-1}, \quad j_p = l_p - l_{p+1} \tag{3.11}$$

with $k_0 = l_{m+1} = 0$.

The free resolution above can be modified such that it becomes Γ-equivariant. For this, define $R(\alpha, \beta) = k[x, y]$ with $\gamma \in \Gamma$ acting by $\gamma \cdot x^i y^j = \lambda(\gamma)^{i+\alpha}\mu(\gamma)^{j+\beta}$. Also write $R = R(0, 0)$. Then the free resolution

$$0 \to \bigoplus_{p=1}^{m} R(k_p, l_p) \xrightarrow{A} \bigoplus_{q=1}^{m+1} R(k_{q-1}, l_q) \to R \to R/I \to 0 \tag{3.12}$$

is Γ-equivariant. The idea is, very roughly, to *insert parameters* $u_1, \ldots, u_r$ into A, obtaining a new matrix $\widetilde{A}$, and define a map $\mathbf{A}^r \to \mathrm{Hilb}^n(\mathbf{A}^2)$ by sending $(u_1, \ldots, u_r) \in \mathbf{A}^r$ to the ideal generated by the maximal minors of $\widetilde{A}$ evaluated at that point.

So replace any number of zero entries in A by parameters $u_1, \ldots, u_r$, thus obtaining a new matrix $\widetilde{A}$ with entries in $\widetilde{R} = k[x, y; u_1, \ldots, u_r]$ (for an example, see the matrix A_{ij} in theorem 5.1). The free resolution then becomes

$$0 \to \bigoplus_{p=1}^{m} \widetilde{R}(k_p, l_p) \xrightarrow{\widetilde{A}} \bigoplus_{q=1}^{m+1} \widetilde{R}(k_{q-1}, l_q) \to \widetilde{R} \to \widetilde{R}/\widetilde{I} \to 0 \tag{3.13}$$

where $\widetilde{I}$ is the ideal generated by the maximal minors of $\widetilde{A}$ and $\widetilde{R}(\alpha, \beta) = k[x, y; u_1, \ldots, u_r]$ with Γ acting on x, y as above. To keep the resolution Γ-equivariant, one is forced to define

$$\gamma \cdot u_i = \lambda(\gamma)^{k_p - k_{q-1}} \mu(\gamma)^{l_p - l_q} u_i \quad \forall \gamma \in \Gamma \tag{3.14}$$

if u_i sits in entry (q, p) in $\widetilde{A}$. This construction defines a family π over $\mathbf{A}^r$,

$$\begin{array}{ccc} V(\widetilde{A}) & \hookrightarrow & \mathbf{A}^2 \times_k \mathbf{A}^r \\ & {\scriptstyle\pi}\searrow & \downarrow \\ & & \mathbf{A}^r \end{array}$$

where $V(\widetilde{A})$ denotes the degeneration locus of $\widetilde{A}$, that is, $V(\widetilde{I})$. If the family is flat and with fibres of constant length n, this defines a morphism $\mathbf{A}^r \to \mathrm{Hilb}^n(\mathbf{A}^2)$.

Theorem 3.10. *Let I be a monomial ideal of colength n in $k[x, y]$, let $\widetilde{A}$ be a matrix constructed from I as above and let $Z = V(\widetilde{A}) \subset \mathbf{A}^2 \times_k \mathbf{A}^r$. Fix a generic one-parameter subgroup $\psi : \mathbf{G}_m \to \Gamma$ with negative weights on $k[\mathbf{A}^2]$. If the parameters u_i are situated in $\widetilde{A}$ in such a way that $\mathbf{G}_m$ acts on each u_i with* negative weights, *then the following holds.*

(i) The family $\pi : Z \to \mathbf{A}^r$ thus defined is flat and with fibres of constant length n.

(ii) If $\mathcal{B}$ is the set of monomials in $k[x, y]$ not in I, then $\mathcal{B}$ maps to a basis for the coordinate ring $k[Z_p]$ of the fibre over p for every $p \in \mathbf{A}^r$.

The proof is broken up into two lemmas. Keep the conditions of the theorem in the following.

Lemma 3.11. *The morphism $\pi : Z \to \mathbf{A}^r$ is finite. In fact, the monomials in $\mathcal{B}$ generate $k[Z]$ as a $k[\mathbf{A}^r]$-module.*

Proof. The subscheme $Z \subset \mathbf{A}^2 \times_k \mathbf{A}^r$ is defined by the ideal $\widetilde{I}$ in $k[\mathbf{A}^2] \otimes_k k[\mathbf{A}^r]$. View the elements of that ring as polynomials in x and y with coefficients from $k[\mathbf{A}^r] = k[u_1, \ldots, u_r]$. The generators of $\widetilde{I}$ are the maximal minors of $\widetilde{A}$ which can be written

$$f = x^k y^l + g \tag{3.15}$$

where the monomials $x^k y^l$ generate I and g is a polynomial in x and y with coefficients of *positive degree* in $k[\mathbf{A}^r]$, i.e. every term involves some parameter. In view of the Γ-equivariant free resolution (3.13), each such minor is semi-invariant under the Γ-action. This implies that each term in f has the same $\mathbf{G}_m$-weight. On the other hand the $\mathbf{G}_m$-weight is negative on all parameters, which implies that the leading term $x^k y^l$ has minimal weight among all monomials $x^i y^j$ in f. Hence any monomial divisible by $x^k y^l$ is congruent modulo $\widetilde{I}$ to some polynomial in $k[\mathbf{A}^2] \otimes_k k[\mathbf{A}^r]$ whose monomials have strictly larger weights.

Now $\mathcal{B}$ consists of the monomials not divisible by any of the leading terms $x^k y^l$. Furthermore, note that the $\mathbf{G}_m$-weight on a monomial $x^i y^j$ is increasing with decreasing i and j since $\mathbf{G}_m$ acts with negative weights on x and y. Thus every element in $k[\mathbf{A}^2] \otimes_k k[\mathbf{A}^r]$ not in the $k[\mathbf{A}^r]$-module generated by $\mathcal{B}$ is congruent modulo $\widetilde{I}$ to a polynomial in that module. This proves that $k[Z]$ is generated by $\mathcal{B}$. □

This lemma implies that $\mathcal{B}$ maps to a generating set of $k[Z_p]$ as a k-module for every fibre Z_p. Thus, once point (i) in the theorem is proved, such that $k[Z_p]$ has dimension n as a k-vector space, point (ii) follows from the lemma.

Lemma 3.12. *The morphism $\pi : Z \to \mathbf{A}^r$ is flat.*

Proof. Z is flat over a point $p \in \mathbf{A}^r$ if and only if, for every affine curve C and every morphism $f : C \to \mathbf{A}^r$ sending some basepoint $q \in C$ to p, the pullback $f^* Z \to C$ is flat over q. Moreover, this is the case if and only if no component of $f^* Z$ is supported in the fibre over q. Now f corresponds to a ring homomorphism

$$f^{\#} : k[\mathbf{A}^r] \to k[C]. \tag{3.16}$$

Then $f^* Z \subset \mathbf{A}^2 \times_k C$ is the degeneration locus of the matrix obtained by applying the map

$$\mathrm{id} \otimes f^{\#} : k[\mathbf{A}^2] \otimes_k k[\mathbf{A}^r] \to k[\mathbf{A}^2] \otimes_k k[C] \tag{3.17}$$

to the entries in $\widetilde{A}$. Thus $f^* Z$ is the determinantal subscheme of $\mathbf{A}^2 \times_k C$ defined by the maximal minors of this matrix, which implies that any component has codimension at most 2, hence dimension at least 1. By lemma 3.11, every fibre is zero-dimensional, so no component of dimension ≥ 1 is supported in the fibre over q. This proves flatness. □

So $\pi : Z \to \mathbf{A}^r$ is finite and flat. Then $\pi_* \mathcal{O}_Z$ is locally free, so the fibres have constant length. This concludes the proof of theorem 3.10.

Remark 3.13. Since the family Z in theorem 3.10 is Γ-invariant, the induced morphism $\varphi : \mathbf{A}^r \to \mathrm{Hilb}^n(\mathbf{A}^2)$ is Γ-equivariant, thus in particular $\mathbf{G}_m$-equivariant. Furthermore, with negative weights on the parameters, $t \cdot p$ approaches the origin when $t \to 0$ for any $p \in \mathbf{A}^r$, so $t \cdot \varphi(p)$ approaches the monomial ideal I. Hence the image of φ is contained in the cell surrounding I in the decomposition induced by $\mathbf{G}_m$.

Chapter 4

Cyclic groups

Let $G \subset \mathrm{SL}(2)$ be the cyclic group of order n generated by

$$\sigma = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \tag{4.1}$$

where ε is a primitive n'th root of unity. The simple singularity $\mathbf{A}^2/G$ is of type A_{n-1}.

Theorem 4.1. *Let $G \subset \mathrm{SL}(2)$ be as above and for $i = 1, \ldots, n$ define*

$$A_i(u,v) = \begin{pmatrix} x & u \\ y^{n-i} & x^{i-1} \\ v & y \end{pmatrix}. \tag{4.2}$$

(i) The family Z, defined by the ideal generated by the maximal minors of $A_i(u,v)$,

$$\begin{array}{ccc} Z & \hookrightarrow & \operatorname{Spec} k[x,y] \times_k \operatorname{Spec} k[u,v] \\ & \searrow^{\pi} & \downarrow \\ & & \operatorname{Spec} k[u,v] \end{array} \tag{4.3}$$

is flat over $\operatorname{Spec} k[u,v]$ and the coordinate rings of the fibres are isomorphic to $k[G]$ as $k[G]$-modules.

(ii) The morphisms

$$\varphi_i : \mathbf{A}^2 \to G\text{-Hilb}(\mathbf{A}^2), \qquad i = 1, \ldots, n \tag{4.4}$$

thus defined are open immersions, and their images form an open covering of G-Hilb$(\mathbf{A}^2)$.

Remark 4.2. By the uniqueness of the minimal resolution of $\mathbf{A}^2/G$, G-Hilb$(\mathbf{A}^2)$ is isomorphic (over $\mathbf{A}^2/G$) to the toric resolution. The construction described here, however, uses no reference to the toric resolution.

The strategy is as follows: The two-dimensional algebraic torus Γ acts on $\mathbf{P}^2$ and $\mathrm{Hilb}^n(\mathbf{P}^2)$ as described in chapter 3, keeping $\mathbf{A}^2$ resp. $\mathrm{Hilb}^n(\mathbf{A}^2)$ invariant. One may view G as a subgroup of Γ (sending σ to $\mathrm{diag}(1, \varepsilon, \varepsilon^{-1}) \in \Gamma \subset$

SL(3)), such that G acts on $\mathrm{Hilb}^n(\mathbf{P}^2)$ commuting with the action of Γ. Thus $\mathrm{Hilb}^n(\mathbf{P}^2)^G$ is Γ-invariant and therefore admits an induced cell decomposition for each generic one-parameter subgroup $\mathbf{G}_m \to \Gamma$. By the discussion in chapter 3, $\mathrm{Hilb}^n(\mathbf{A}^2)^G$ is a union of cells if the $\mathbf{G}_m$-action has negative weights on $k[\mathbf{A}^2]$.

In this way, each generic subgroup $\mathbf{G}_m \to \Gamma$ with negative weights on $k[\mathbf{A}^2]$ induces a decomposition of $\mathrm{Hilb}^n(\mathbf{A}^2)^G$ into locally closed affine spaces, and thus the same is true for each component. In particular, there is an induced decomposition of G-Hilb($\mathbf{A}^2$). By varying the one-parameter subgroup, and thus varying the decomposition, the affine charts in the theorem occur as two-dimensional, open cells.

4.1 Cell decompositions

The Γ-fixpoints in G-Hilb($\mathbf{A}^2$) are, using proposition 1.6, the G-invariant monomial ideals I such that $k[x,y]/I$ is the regular representation. Note that, since G is abelian, the irreducible representations of G are all of degree one, and they are given by

$$\sigma \mapsto \varepsilon^k, \quad k = 0, \ldots, n-1. \tag{4.5}$$

Furthermore, since I is monomial, a basis for $k[x,y]/I$ is given by (the images of) the monomials $x^i y^j$ not in I, and these elements are eigenvectors for σ with eigenvalues ε^{i-j}.

Proposition 4.3. *The set of Γ-fixpoints in G-Hilb($\mathbf{A}^2$) is the finite set consisting of the ideals*

$$I_i = (x^i, y^{n-i+1}, xy), \quad i = 1, \ldots, n. \tag{4.6}$$

Proof. By what is said above, the fixpoints are the monomial ideals I such that each value ε^k, $k = 0, \ldots, n-1$, occurs exactly once among the eigenvalues ε^{i-j} for the monomials $x^i y^j \notin I$. In particular, if I is a fixpoint, all but one invariant monomial belongs to I, and this monomial must be 1. So $xy \in I$, and if $x^i \in I$ is the smallest power of x in I, then $i \leq n$. Since the colength of I is n, the only possibility is $I = I_i$.

Conversely, the monomial basis for $k[x,y]/I_i$ is

$$1, x, \ldots, x^{i-1}, y^{n-i}, \ldots, y \tag{4.7}$$

and the eigenvalues for the σ-action are, in the same order,

$$1, \varepsilon, \ldots, \varepsilon^{i-1}, \varepsilon^i, \ldots, \varepsilon^{n-1} \tag{4.8}$$

so $k[x,y]/I_i$ is the regular representation. □

The next step is to construct an affine open neighbourhood of each fixpoint I_i: If one can find a one-parameter subgroup $\mathbf{G}_m \to \Gamma$ such that the cell containing I_i in the induced decomposition is two-dimensional, that cell would be open, being locally closed in a surface. By theorem 3.2, the cell is two-dimensional precisely if both weights in the induced action on $T_{I_i}(G\text{-Hilb}(\mathbf{A}^2))$ are positive. These weights can be found with the aid of proposition 3.7.

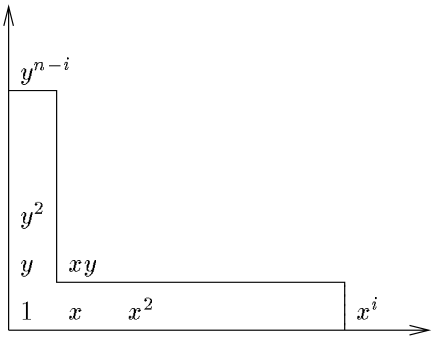

Figure 4.1: Torus fixpoints in G-Hilb($\mathbf{A}^2$)

Proposition 4.4. *The tangent space to the Γ-fixpoint I_i in G-*Hilb($\mathbf{A}^2$) *is given in the representation ring of Γ by*

$$T_{I_i}(G\text{-Hilb}(\mathbf{A}^2)) = \lambda^{-i}\mu^{n-i} + \lambda^{i-1}\mu^{i-n-1}. \tag{4.9}$$

Proof. The staircase picture of the ideal I_i is shown in figure 4.1. To avoid too many indices, let $I = I_i$.

G-Hilb($\mathbf{A}^2$) is a component of the G-invariant part of Hilb$^n(\mathbf{A}^2)$, which is smooth by proposition 3.9. By the same proposition, the tangent space at the fixpoint I is given by

$$T_I(G\text{-Hilb}(\mathbf{A}^2)) = T_I(\text{Hilb}^n(\mathbf{A}^2))^G. \tag{4.10}$$

Proposition 3.7 gives a basis of common eigenvectors for the action of Γ on $T_I(\text{Hilb}^n(\mathbf{A}^2))$ and the G-invariant subspace is spanned by the G-invariant eigenvectors.

In view of the inclusion $G \subset \Gamma$, the eigenvalues for $\sigma \in G$ acting on the eigenvectors corresponding to the terms $\lambda^{i-a_j}\mu^{b_i-j-1}$ and $\lambda^{a_j-i-1}\mu^{j-b_i}$ in proposition 3.7 are given by substituting $\lambda = \varepsilon$ and $\mu = \varepsilon^{-1}$:

$$\begin{aligned} \varepsilon^{(i-a_j)-(b_i-j-1)} &= \varepsilon^{-(a_j-j+b_i-i-1)} \\ \varepsilon^{(a_j-i-1)-(j-b_i)} &= \varepsilon^{a_j-j+b_i-i-1} \end{aligned} \tag{4.11}$$

Hence, these two eigenvectors are G-invariant at the same time, namely when

$$a_j - j + b_i - i - 1 \equiv 0 \pmod n. \tag{4.12}$$

But this expression is precisely the area of the hook determined by (i, j) as in figure 3.1. Comparing this with the picture of I in figure 4.1, there is only one hook with area congruent to n, namely the whole staircase. The summand in proposition 3.7 corresponding to this hook is precisely $\lambda^{-i}\mu^{n-i} + \lambda^{i-1}\mu^{i-n-1}$. □

Corollary 4.5. *For each fixpoint I_i there exists a generic one-parameter subgroup $\psi : \mathbf{G}_m \to \Gamma$, inducing negative $\mathbf{G}_m$-weights on $k[\mathbf{A}^2]$, such that the cell U_i containing I_i in the induced decomposition is two-dimensional.*

Proof. Define ψ by $\lambda \circ \psi(t) = t^a$ and $\mu \circ \psi(t) = t^b$ for two negative integers a and b. Then, by the previous proposition, the two weights for the induced action on the tangent space at I_i are positive if and only if

$$\begin{aligned} a(-i) + b(n-i) &> 0 \\ a(i-1) + b(i-n-1) &> 0. \end{aligned} \tag{4.13}$$

This is equivalent to

$$\frac{n-i}{i} < \frac{a}{b} < \frac{n-i+1}{i-1} \tag{4.14}$$

and such a, b can obviously be found. Furthermore, there is enough freedom to choose them outside any finite number of hyperplanes. □

Thus the existence of open affine neighbourhoods U_i around each fixpoint I_i is proved. The morphisms φ_i of theorem 4.1 will be constructed such that the images are U_i. So, after showing that they actually form an open covering of G-Hilb($\mathbf{A}^2$), the next step is to construct the morphisms φ_i explicitly.

Proposition 4.6. *The open sets U_i form an open covering of G-*Hilb($\mathbf{A}^2$)*.*

Proof. Let Y be the closed set G-Hilb$(\mathbf{A}^2) \setminus \bigcup U_i$. Recall that $I \in U_i$ if and only if $t \cdot I$ approaches I_i as $t \to 0$, where $t \in \mathbf{G}_m$ acts according to the chosen subgroup in corollary 4.5. Then each U_i is Γ-invariant, because for any $\gamma \in \Gamma$ one has

$$\begin{aligned} \lim_{t \to 0} t \cdot (\gamma \cdot I) &= \lim_{t \to 0} \gamma \cdot (t \cdot \gamma) \\ &= \gamma \cdot \lim_{t \to 0} t \cdot \gamma \\ &= \gamma \cdot I_i = I_i \end{aligned} \tag{4.15}$$

using the continuity of the Γ-action when moving γ out of the limit. This implies that Y is invariant. If Y is nonempty, take any $I \in Y$ and some generic one-parameter subgroup $\mathbf{G}_m \to \Gamma$ with negative weights on $k[\mathbf{A}^2]$. Then $t \cdot I$ approaches a Γ-fixpoint supported in $\mathbf{A}^2$, which is in Y since Y is closed. But every fixpoint is in some U_i, so Y is empty. □

4.2 Affine charts

Recalling the construction in theorem 3.10, there are free Γ-equivariant resolutions of the Γ-invariant ideals I_i,

$$0 \to \begin{matrix} R(1, n-i+1) \\ \oplus \\ R(i,1) \end{matrix} \xrightarrow{A_i} \begin{matrix} R(0, n-i+1) \\ \oplus \\ R(1,1) \\ \oplus \\ R(i,0) \end{matrix} \to I_i \to 0 \tag{4.16}$$

where

$$A_i = \begin{pmatrix} x & 0 \\ y^{n-i} & x^{i-1} \\ 0 & y \end{pmatrix}. \tag{4.17}$$

Now insert the parameters u, v in the zero slots in the matrix, thus obtaining the matrix $A_i(u,v)$ of theorem 4.1 with entries in $\widetilde{R} = k[x,y;u,v]$ and the free resolution

$$0 \to \begin{array}{c} \widetilde{R}(1, n-i+1) \\ \oplus \\ \widetilde{R}(i,1) \end{array} \xrightarrow{A_i(u,v)} \begin{array}{c} \widetilde{R}(0, n-i+1) \\ \oplus \\ \widetilde{R}(1,1) \\ \oplus \\ \widetilde{R}(i,0) \end{array} \to I_i(u,v) \to 0 \tag{4.18}$$

where $I_i(u,v)$ is generated by the maximal minors of $A_i(u,v)$. To keep the resolution equivariant, one is forced to define $\gamma \cdot u = \lambda(\gamma)^i \mu(\gamma)^{i-n} u$ and $\gamma \cdot v = \lambda(\gamma)^{1-i}\mu(\gamma)^{n-i+1} v$. This defines an action of Γ on the parameter space $\mathbf{A}^2 = \operatorname{Spec} k[u,v]$.

Remark 4.7. Note that with this action, the weights of the induced action on $T_0(\mathbf{A}^2)$ coincides with the weights on $T_{I_i}(G\text{-Hilb}(\mathbf{A}^2))$ in proposition 4.4. This is promising in view of trying to realize the isomorphism $\mathbf{A}^2 \cong U_i$ by means of the family defined by $A_i(u,v)$. Moreover, the one-parameter subgroups in corollary 4.5 act with negative weights on the parameters u, v, so theorem 3.10 applies.

For the rest of this section let i be fixed and write $A(u,v) = A_i(u,v)$, $I = I_i$ and $U = U_i$. By theorem 3.10, $A(u,v)$ defines a Γ-equivariant morphism

$$\varphi : \mathbf{A}^2 \to \operatorname{Hilb}^n(\mathbf{A}^2). \tag{4.19}$$

By the same theorem, the monomials in $\mathcal{B}$, i.e. the monomials not in the ideal I, maps to a basis in $k[Z_p]$. Since the monomials are eigenvectors for the G-action, this implies that $k[Z_p] \cong k[x,y]/I \cong k[G]$ as $k[G]$-modules. Thus the image of φ is in $G\text{-Hilb}(\mathbf{A}^2)$.

Lemma 4.8. *The differential*

$$d\varphi_0 : T_0(\mathbf{A}^2) \to T_{\varphi(0)}(G\text{-Hilb}(\mathbf{A}^2)) \tag{4.20}$$

is an isomorphism.

Proof. It is enough to show that $d\varphi_0$ is injective, since the source and target schemes are both smooth and of the same dimension.

Letting $D = \operatorname{Spec} k[\varepsilon]/\varepsilon^2$, the Zariski tangent space $T_0(\mathbf{A}^2)$ is the set of morphisms $D \to \mathbf{A}^2$ sending the closed point in D to the origin, whereas $T_{\varphi(0)}(G\text{-Hilb}(\mathbf{A}^2))$ is the space of first order deformations of the subscheme of $\mathbf{A}^2$ defined by $\varphi(0)$. The differential map

$$\operatorname{Mor}_k(D, \mathbf{A}^2) \xrightarrow{d\varphi} \left\{ \begin{array}{ccc} W & \hookrightarrow & \mathbf{A}^2 \times_k D \\ & \searrow^{\rho} & \downarrow \\ & & D \end{array} \;\middle|\; \begin{array}{c} \rho \text{ is flat and} \\ k[W_s] \cong k[G] \\ \text{for all } s \in D \end{array} \right\} \tag{4.21}$$

is given by sending a morphism $\alpha : D \to \mathbf{A}^2$ to the pullback $Z \times_{\mathbf{A}^2} D$ of the family $Z \to \mathbf{A}^2$ defined by the matrix $A(u,v)$.

So let α be defined by the ring homomorphism

$$\alpha^\# : k[u,v] \to k[\varepsilon]/\varepsilon^2. \tag{4.22}$$

Such a map belongs to the Zariski tangent space at the origin if and only if $\alpha^{\#}(u) = a\varepsilon$ and $\alpha^{\#}(v) = b\varepsilon$ for some $a, b \in k$. The family $d\varphi(\alpha)$ is then the subscheme of $\mathbf{A}^2 \times_k D = \operatorname{Spec} k[x, y; \varepsilon]/\varepsilon^2$ defined by the degeneration locus of the matrix

$$\begin{pmatrix} x & a\varepsilon \\ y^{n-i} & x^{i-1} \\ b\varepsilon & y \end{pmatrix} \tag{4.23}$$

whose maximal minors are

$$x^i - a\varepsilon y^{n-i}, \quad xy, \quad y^{n-i+1} - b\varepsilon x^{i-1}. \tag{4.24}$$

Now the zero element of the Zariski tangent space $T_{\varphi(0)}(G\text{-Hilb}(\mathbf{A}^2))$ corresponds to the family defined by the ideal I_0 generated by $\varphi(0)$ in $k[\mathbf{A}^2 \times_k D] = k[x, y, \varepsilon]/\varepsilon^2$. Thus, if $d\varphi(\alpha) = 0$, the ideal generated by the minors above equals I_0. But then $a\varepsilon y^{n-i}$ and $b\varepsilon x^{i-1}$ are elements of the same ideal, which implies $a = b = 0$, so $d\varphi$ is injective at the origin. □

Proposition 4.9. *φ is étale.*

Proof. Let $V \subset \mathbf{A}^2$ be the set of points $p \in \mathbf{A}^2$ such that $d\varphi_p : T_p(\mathbf{A}^2) \to T_{\varphi(p)}(G\text{-Hilb}(\mathbf{A}^2))$ is surjective. By [Har77, Prop. 10.4], φ is étale if $V = \mathbf{A}^2$. But the complement W of V is closed, being the degeneration locus of the matrix $d\varphi$, and it is $\mathbf{G}_m$-invariant: Any $t \in \mathbf{G}_m$ acts as an automorphism on both schemes, and by the $\mathbf{G}_m$-equivariance of φ one obtains a commutative diagram

$$\begin{array}{ccc} T_p(\mathbf{A}^2) & \xrightarrow{d\varphi_p} & T_{\varphi(p)}(G\text{-Hilb}(\mathbf{A}^2)) \\ dt_p \downarrow \cong & & dt_{\varphi(p)} \downarrow \cong \\ T_{t\cdot p}(\mathbf{A}^2) & \xrightarrow{d\varphi_{t\cdot p}} & T_{\varphi(t\cdot p)}(G\text{-Hilb}(\mathbf{A}^2)) \end{array} \tag{4.25}$$

which shows that $d\varphi_p$ is surjective if and only if $d\varphi_{t\cdot p}$ is surjective. Being invariant, W is a union of orbits, but any $\mathbf{G}_m$-orbit contains the origin in its closure since $\mathbf{G}_m$ acts with negative weights on the parameters. But the origin is in V by the previous lemma, so W is empty. □

Corollary 4.10. *φ is an open immersion.*

Proof. By [Gro67, Théoréme 17.9.1], a morphism is an open immersion if and only if it is injective, flat and finitely presented. Being étale, φ is flat, so it remains only to check injectivity.

So take two points $p, q \in \mathbf{A}^2$ and assume $\varphi(p) = \varphi(q)$. Let $C \subset \mathbf{A}^2$ be the closure of the $\mathbf{G}_m$-orbits containing p and q, that is $C = \overline{\mathbf{G}_m p \cup \mathbf{G}_m q}$. This curve is mapped by φ to the curve $D = \overline{\mathbf{G}_m \varphi(p)}$, using the $\mathbf{G}_m$-equivariance of φ. Then the restriction $\varphi|_C : C \to D$ is proper by the valuation criterion, using the $\mathbf{G}_m$-action to take limit of curves. Thus $\varphi_* \mathcal{O}_C$ is coherent, so upper semicontinuity of the length of fibres holds for $\varphi|_C$. Now the fibre over the origin has length 1 by proposition 4.9 so a general fibre has also length 1. Then $p = q$ since otherwise the general fibre would have length 2.. □

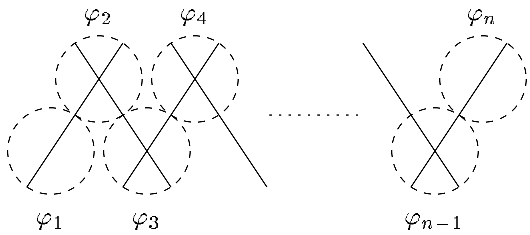

Figure 4.2: Affine covering of G-Hilb$(\mathbf{A}^2)$

i	u, v	Soc(R/I)	character
$i = 1$	$v = 0$	$\langle y^{n-1} \rangle$	ε
$1 < i < n$	$u = 0, v \neq 0$	$\langle x^i \rangle$	ε^i
$1 < i < n$	$u \neq 0, v = 0$	$\langle y^{n-i} \rangle$	ε^{n-i}
$i = n$	$u = 0$	$\langle x^{n-1} \rangle$	ε^{n-1}

Table 4.1: Soc(R/I) at smooth points of the exceptional fibre

Corollary 4.11. *The image of φ is U.*

Proof. The inclusion $\operatorname{im} \varphi \subset U$ follows from the $\mathbf{G}_m$-equivariance (see remark 3.13). Conversely, the image of φ is a neighbourhood of I_i, so if $t \cdot I$ approaches I_i as $t \to 0$, there exists some $t \in \mathbf{G}_m$ such that $t \cdot I \in \operatorname{im} \varphi$. Then, again by the equivariance of φ, I is also an element of $\operatorname{im} \varphi$. □

This concludes the proof of theorem 4.1. For φ is an open immersion by corollary 4.10, and by proposition 4.6 the sets U cover G-Hilb$(\mathbf{A}^2)$.

4.3 Remarks

Solving the equation $\varphi_i(u_i, v_i) = \varphi_{i+1}(u_{i+1}, v_{i+1})$ one finds the usual transition functions

$$u_{i+1} = u_i^2 v_i, \quad v_{i+1} = 1/u_i. \tag{4.26}$$

Figure 4.2 indicates the affine covering in theorem 4.1, suppressing the overlaps for clarity, where the black lines show the exceptional fibre. Note that $\varphi_i(u, v)$ is in the exceptional fibre E if and only if the support of the corresponding subscheme of $\mathbf{A}^2$ is the origin, which is equivalent to $uv = 0$ for $1 < i < n$, $v = 0$ for $i = 1$ and $u = 0$ for $i = n$. Thus one may find the representations Soc(R/I) explicitly. This is done in table 4.1 for all the smooth points of E, verifying the bijection of theorem 2.1(iii) in the A_n-case.

Moreover, theorem 4.1 implies that the conjectures in chapter 2 are true. Let $R = k[x, y]$ in the following.

Corollary 4.12 (of theorem 4.1). *Conjecture 2.8 is true in the A_n case, i.e. the tangent space to the exceptional fibre E at a point I is*

$$T_I(E) = \operatorname{Hom}_{k[G]}(I/\mathfrak{m}I, \operatorname{Soc}(R/I)) \tag{4.27}$$

Proof. The exceptional fibre is singular precisely at the points I where $\mathrm{Soc}(R/I)$ is reducible. At these points both sides of the equality are two-dimensional, thus equal the whole tangent space to G-Hilb($\mathbf{A}^2$), so there is nothing to prove.

So assume I is a nonsingular point of E. Let U_i be the affine chart containing I. Then $I = \varphi_i(u, 0)$ or $I = \varphi_i(0, v)$. The two cases are similar, so concentrate on the first case. There is a $\mathbf{G}_m$-action on U_i with fixpoint locus equal to the exceptional component E given in U_i by $v = 0$, namely

$$\left.\begin{aligned} x &\mapsto t^{n-i} \\ y &\mapsto t^i \end{aligned}\right\} \quad \text{for } t \in \mathbf{G}_m. \tag{4.28}$$

Hence

$$T_I(E) = \mathrm{Hom}_{R[G]}(I, R/I)^{\mathbf{G}_m} \tag{4.29}$$

and in view of the isomorphism $\mathrm{Hom}_{k[G]}(I, R/I) \cong \mathrm{Hom}_{k[G]}(I/\mathfrak{m}I, R/I)$ in equation (2.22) it is enough to show that an $R[G]$-homomorphism $\varphi : I \to R/I$ is $\mathbf{G}_m$-equivariant if and only if its image is in $\mathrm{Soc}(R/I)$. For this, use the decomposition of $k[x,y]/I$ as a G-representation given by the basis $\mathcal{B}$: φ is determined by the image of the generators for I, and the G-equivariance of φ demands

$$\varphi(x^i - uy^{n-i}) \in \langle y^{n-i} \rangle = \mathrm{Soc}(R/I) \tag{4.30}$$
$$\varphi(xy) \in \langle 1 \rangle \tag{4.31}$$
$$\varphi(y^{n-i+1}) \in \langle x^{i-1} \rangle \tag{4.32}$$

Then φ is $\mathbf{G}_m$-equivariant if and only if $\varphi(xy) = \varphi(y^{n-i+1}) = 0$, i.e. the image of φ is in $\mathrm{Soc}(R/I)$. □

Corollary 4.13 (of theorem 4.1). *Conjecture 2.13 is true in the A_n case, i.e. the multiplication map*

$$\mathfrak{m}/\mathfrak{m}^2 \otimes \mathrm{Soc}(R/I) \to I/(\mathfrak{m}I + \mathfrak{n}) \tag{4.33}$$

for I in the exceptional fibre, is a projection for $\mathrm{Soc}(R/I)$ *reducible.*

Proof. The representation $\mathrm{Soc}(R/I)$ is reducible precisely for $I = \varphi_i(0,0)$, with $1 < i < n$. In that case, fix the following bases.

$$\begin{aligned} \mathrm{Soc}(R/I) &= \langle x^{i-1}, y^{n-i} \rangle \\ I/(\mathfrak{m}I + \mathfrak{n}) &= \langle x^i, y^{n-i+1} \rangle \\ \mathfrak{m}/\mathfrak{m}^2 &= \langle x, y \rangle \end{aligned} \tag{4.34}$$

Then the multiplication map is given by

$$\begin{aligned} x \otimes x^{i-1} &\mapsto x^i \\ x \otimes y^{n-i} &\mapsto 0 \\ y \otimes x^{i-1} &\mapsto 0 \\ y \otimes y^{n-i} &\mapsto y^{n-i+1} \end{aligned} \tag{4.35}$$

and is thus a projection. □

Chapter 5

Binary dihedral groups

Let $G \subset \mathrm{SL}(2)$ be the binary dihedral group of order $4n$ generated by

$$\sigma = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.1}$$

for ε a primitive $2n$'th root of unity. The simple singularity $\mathbf{A}^2/G$ is of type D_{n+2}.

Theorem 5.1. *Let $G \subset \mathrm{SL}(2)$ be as above, and define*

$$A_{ij}(s,t,u,v) = \begin{pmatrix} x & 0 & s & 0 \\ y^{i+j-2n-2} & x & 0 & t \\ 0 & y^{2n-i} & x^{2n-j} & 0 \\ u & 0 & y & x^{i+j-2n-2} \\ 0 & v & 0 & y \end{pmatrix}$$

Also let S be the surface

$$S = V(su + sv + tv, uv + v^2 + (-1)^{n-1}) \subset \operatorname{Spec} k[s,t,u,v].$$

(i) Writing $V(M)$ for the determinantal subscheme defined by the maximal minors of a matrix M, the families

$$\begin{aligned} V\left(A_{n+2,n+1}(s,t,u,v)\right) &\subset \operatorname{Spec} k[x,y] \times_k S \\ V\left(A_{i,i}(0,t,(-1)^{n+1}t,0)\right) &\subset \operatorname{Spec} k[x,y] \times_k \operatorname{Spec} k[t], \\ & \qquad i = n+2, \dots, 2n \\ V\left(A_{n+1,n+1}(t,0,0,(-1)^{n+1}t)\right) &\subset \operatorname{Spec} k[x,y] \times_k \operatorname{Spec} k[t] \\ V\left(A_{n+2,n}(t,0,0,\pm\sqrt{(-1)^n})\right) &\subset \operatorname{Spec} k[x,y] \times_k \operatorname{Spec} k[t] \\ V\left(xy, x^{2n} + y^{2n} - t\right) &\subset \operatorname{Spec} k[x,y] \times_k \operatorname{Spec} k[t] \end{aligned}$$

are flat over the base schemes S resp. $\operatorname{Spec} k[t]$, the fibres are G-invariant and the coordinate rings of the fibres are isomorphic to $k[G]$ as $k[G]$-modules.

(ii) The morphisms thus defined, in the same order,

$$\begin{aligned}
\varphi &: S \to G\text{-Hilb}(\mathbf{A}^2) \\
\psi_i &: \mathbf{A}^1 \to G\text{-Hilb}(\mathbf{A}^2), \quad i = n+2, \ldots, 2n \\
\alpha &: \mathbf{A}^1 \to G\text{-Hilb}(\mathbf{A}^2) \\
\beta_\pm &: \mathbf{A}^1 \to G\text{-Hilb}(\mathbf{A}^2) \\
\gamma &: \mathbf{A}^1 \to G\text{-Hilb}(\mathbf{A}^2)
\end{aligned}$$

are locally closed immersions, and their images form a stratification of G-Hilb$(\mathbf{A}^2)$.

(iii) The exceptional fibre is given as follows within each stratum.

$\varphi(s,t,u,v)$	*for* $s+t=0, u=0, v=\pm\sqrt{(-1)^n}$ *(two lines)*
$\varphi(s,t,u,v)$	*for* $s=t=0, uv+v^2+(-1)^{n+1}=0$ *(a hyperbola)*
$\psi_i(t)$	*for all* t *(a line)*
$\alpha(0)$	*(a point)*
$\beta_\pm(0)$	*(two points)*
$\gamma(0)$	*(a point)*

Remark 5.2. The surface S in the theorem is isomorphic to $\mathbf{A}^1 \times (\mathbf{A}^1 \setminus \{0\})$. For this, replace the coordinate u by $w = u + v$ such that the two equations become

$$sw + tv = 0, \quad vw = (-1)^n. \tag{5.2}$$

Then projection onto the t and v coordinates gives an isomorphism with $\mathbf{A}^1 \times (\mathbf{A}^1 \setminus \{0\})$, since $w = (-1)^n/v$ and $s = (-1)^{n+1}tv^2$.

The strategy for obtaining this stratification can be sketched as follows. In contrast to the A_n case, the actions of G and Γ on $\mathrm{Hilb}^{4n}(\mathbf{A}^2)$ do not commute, so there is no torus action on G-Hilb$(\mathbf{A}^2)$ to be utilized. However, letting $H \subset G$ be the index 2 subgroup generated by σ, the actions of H and Γ do commute, so Γ acts on $\mathrm{Hilb}^{4n}(\mathbf{A}^2)^H$. So consider the inclusions

$$G\text{-Hilb}(\mathbf{A}^2) \subset \mathrm{Hilb}^{4n}(\mathbf{A}^2)^G \subset \mathrm{Hilb}^{4n}(\mathbf{A}^2)^H \tag{5.3}$$

and let $Y \subset \mathrm{Hilb}^{4n}(\mathbf{A}^2)^H$ be the component containing G-Hilb$(\mathbf{A}^2)$. Then Γ acts on Y, so every generic one-parameter subgroup $\mathbf{G}_m \to \Gamma$ with negative weights on $k[\mathbf{A}^2]$ induces a decomposition of Y into locally closed affine spaces. Unfortunately there are "too few" subgroups $\mathbf{G}_m \to \Gamma$ to obtain open cells around every Γ-fixpoint in Y. Instead one may fix a particular one-parameter subgroup and compute the induced decomposition Y. The τ-invariant parts of the cells form a stratification of Y^G. Thus one obtains a stratification of each component of Y^G and in particular of G-Hilb$(\mathbf{A}^2)$. This is the stratification in the theorem, for a particular choice of one-parameter subgroup.

5.1 A cell decomposition

Note that the irreducible representations of H are of degree one, given by

$$\sigma \mapsto \varepsilon^k, \quad k = 0, \dots, 2n-1. \tag{5.4}$$

By [IN99, Lemma 9.4], the isomorphism class of the $k[H]$-module $k[\mathbf{A}^2]/I$ is independent of the choice of I within each connected component of $\mathrm{Hilb}^{4n}(\mathbf{A}^2)^H$ (for any finite $H \subset \mathrm{GL}(2)$, hence in particular for the one considered here). Now $k[G] \cong k[H]^{\oplus 2}$ as $k[H]$-modules, so G-Hilb($\mathbf{A}^2$) is contained in some component of $\mathrm{Hilb}^{4n}(\mathbf{A}^2)^H$ where $k[\mathbf{A}^2]/I \cong k[H]^{\oplus 2}$. Thus define $Y \subset \mathrm{Hilb}^{4n}(\mathbf{A}^2)^H$ to be the union of all such components. In fact Y turns out to irreducible.

Throughout this section, fix a generic one-parameter subgroup $\mathbf{G}_m \to \Gamma$ with negative weights on $k[\mathbf{A}^2]$, but otherwise arbitrary. In the induced decomposition of Y there is a cell U for each Γ-fixpoint I, isomorphic to some affine space $\mathbf{A}^r$. To construct the isomorphism $\mathbf{A}^r \cong U$ explicitly, proceed as in section 3.2: First determine all Γ-fixpoints, that is, the monomial ideals in Y. Then construct a Γ-equivariant free resolution of each fixpoint I and "insert parameters" to obtain a morphism $\mathbf{A}^r \to \mathrm{Hilb}^{4n}(\mathbf{A}^2)$. Doing this with some care this can be made a locally closed immersion with image the cell U containing I.

Proposition 5.3. *The fixpoint set Y^Γ consists of the following ideals.*

$$I_{ij} = (x^i, x^{2n-j+2}y, x^2y^2, xy^{2n-i+2}, y^j) \quad \textit{for} \begin{cases} i = 2, \dots, 2n \\ j = 2n+2-i, \dots, 2n \end{cases} \tag{5.5}$$

$$J_i = (x^i, xy, y^{4n-i+1}) \qquad \textit{for } i = 1, \dots, 4n \tag{5.6}$$

Proof. As before, the fixpoints I are monomial ideals, and a basis for $k[x,y]/I$, which is an eigenvector basis for the H-action, is given by the monomials $x^iy^j \notin I$. The corresponding eigenvalues for σ are ε^{i-j}. Thus, $k[x,y]/I \cong k[H]^{\oplus 2}$ if and only if each ε^k occurs twice among the eigenvalues ε^{i-j}.

In particular there should be only two invariant monomials left in $k[x,y]/I$, first assume these are 1 and xy. Then $x^2y^2 \in I$. Let x^i and y^j be the minimal powers of x and y in I. Since x^{2n} and y^{2n} are invariants, one must have $i, j \le 2n$. Now note that x^{i-1} and y^{2n-i+1} are two monomials surviving in $k[x,y]/I$ with the same eigenvalues. Thus any other monomial with the same eigenvalue is in I, so $xy^{2n-i+2} \in I$. Switching the roles of x and y one finds $x^{2n-i+1}y \in I$. So $I \subset I_{ij}$ and this is in fact an equality since both ideals have colength $4n$. Conversely, one verifies that $k[x,y]/I_{ij} \cong k[H]^{\oplus 2}$ by writing out the eigenvalues of the surviving monomials.

Now assume $xy \in I$ and let x^i be the smallest power of x not in I. For I to have colength $4n$, I must be equal to J_i, and then $k[x,y]/J_i$ is isomorphic to $k[H]^{\oplus 2}$. □

When constructing free resolutions of the ideals above, it is not so clear how to "insert parameters" in a good way: In the A_n case there were just two zero entries in the matrices in the free resolutions, adequate for inserting two parameters. This time there are lots of zero entries. However, one may use remark 4.7 as a guide: The aim is to obtain a Γ-equivariant isomorphism $\varphi : \mathbf{A}^r \to U$ for each cell U. In particular the differential $d\varphi_0 : T_0(\mathbf{A}^r) \to T_{\varphi(0)}(U)$ must be an isomorphism of Γ-representations. Therefore one may use

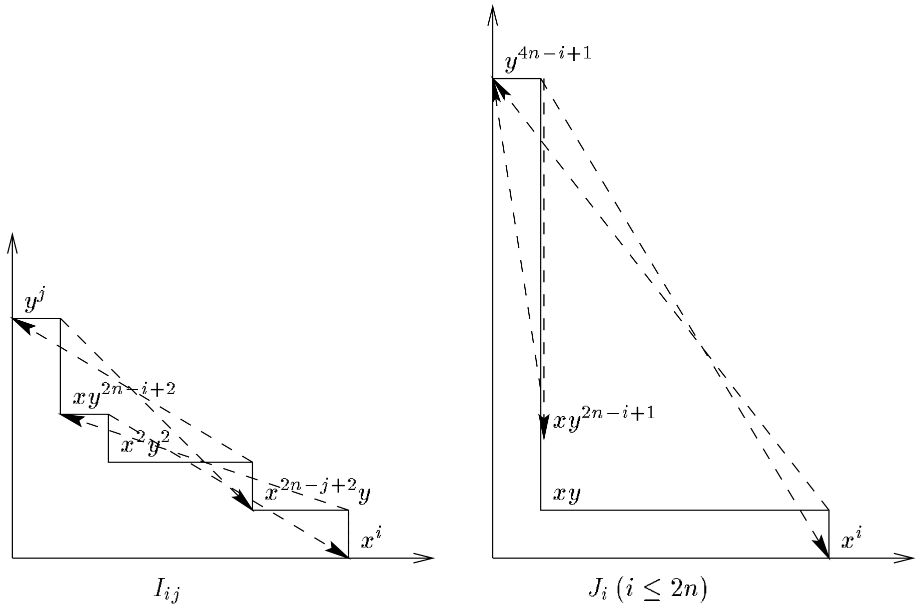

Figure 5.1: Monomial ideals in Y

proposition 3.7 to determine the Γ-action on the tangent spaces at each fixpoint I in Y and make sure Γ acts in the same way on $T_0(\mathbf{A}^r)$. In other words, one should insert parameters into entries in the matrix in the free resolution of I such that the induced Γ-action on the parameters (determined as in equation (3.14)) agrees with the action on $T_I(Y)$.

Proposition 5.4. *In the notation of proposition 5.3, the tangent spaces to the fixpoints I_{ij} and J_j in Y are given in the representation ring of Γ by*

$$\begin{aligned} T_{I_{ij}}(Y) &= \lambda^{i-2}\mu^{-2n+i-2} + \lambda^{1-i}\mu^{2n-i+1} \\ &\quad + \lambda^{2n-j+1}\mu^{1-j} + \lambda^{-2n+j-2}\mu^{j-2} \\ T_{J_i}(Y) &= \lambda^{i-1}\mu^{i-4n-1} + \lambda^{-i}\mu^{4n-i} \\ &\quad + \lambda^{-1}\mu^{2n-1} + \mu^{-2n} \qquad \text{for } i \leq 2n \\ T_{J_i}(Y) &= \lambda^{i-1}\mu^{i-4n-1} + \lambda^{-i}\mu^{4n-i} \\ &\quad + \lambda^{2n-1}\lambda^{-1} + \lambda^{-2n} \qquad \text{for } i > 2n \end{aligned} \tag{5.7}$$

Proof. As in the proof of proposition 4.4, the tangent space $T_I(Y)$ to a monomial ideal I is the H-invariant part of $T_I(\mathrm{Hilb}^{4n}(\mathbf{A}^2))$, which is spanned by H-invariant eigenvectors. Again, the eigenvectors corresponding to the terms $\lambda^{i-a_j}\mu^{b_i-j-1}$ and $\lambda^{a_j-i-1}\mu^{j-b_i}$ in proposition 3.7 are H-invariant if and only if

$$a_j - j + b_i - i - 1 \equiv 0 \pmod{n}. \tag{5.8}$$

and this is the area of the hook determined by (i,j). Referring to the picture of I_{ij} in figure 5.1, there are two hooks of area $2n$, and these are all the hooks with area congruent to 0. Similarly, for J_i there is one large hook of area $4n$

and a silly "hook" which is only a strip (with no bend) of area $2n$. The result follows from substituting the right values for i, j, a_j and b_i, as indicated by the arrows in figure 5.1. □

So let $\widetilde{R}(\alpha,\beta) = k[x,y;s,t,u,v]$ with Γ acting on x and y as in section 3.2, and consider the following Γ-equivariant free resolution, where the fixpoint I_{ij} is obtained by setting $s=t=u=v=0$.

$$0 \to \begin{array}{c} \widetilde{R}(1,j) \\ \oplus \\ \widetilde{R}(2,2n-i+2) \\ \oplus \\ \widetilde{R}(2n-j+2,2) \\ \oplus \\ \widetilde{R}(i,1) \end{array} \xrightarrow{A_{ij}(s,t,u,v)} \begin{array}{c} \widetilde{R}(0,j) \\ \oplus \\ \widetilde{R}(1,2n-i+2) \\ \oplus \\ \widetilde{R}(2,2) \\ \oplus \\ \widetilde{R}(2n-j+2,1) \\ \oplus \\ \widetilde{R}(i,0) \end{array} \to I_{ij}(s,t,u,v) \to 0 \tag{5.9}$$

Here the matrix $A_{ij}(s,t,u,v)$ is as defined in theorem 5.1 and $\gamma \in \Gamma$ acts by

$$\begin{aligned} \gamma\cdot s &= \lambda(\gamma)^{2n-j+2}\mu(\gamma)^{2-j}s & \qquad \gamma\cdot t &= \lambda(\gamma)^{i-1}\mu(\gamma)^{i-2n-1}t \\ \gamma\cdot u &= \lambda(\gamma)^{j-2n-1}\mu(\gamma)^{j-1}u & \qquad \gamma\cdot v &= \lambda(\gamma)^{2-i}\mu(\gamma)^{2n-i+2}v. \end{aligned} \tag{5.10}$$

Note that with this action, the tangent space $T_0(\operatorname{Spec} k[s,t,u,v])$ is isomorphic to $T_{I_{ij}}(Y)$ as a Γ-representation as wanted.

In the minimal resolutions of the fixpoints J_i, there are no entries in the matrices with correct Γ-weights. The trick is to take a non-minimal generating set by adding the monomials at the endpoints of the short arrows in figure 5.1. So in the case $i \le 2n$ consider the generating set

$$J_i = (y^{4n-i+1}, xy^{2n-i+2}, xy^{2n-i+1}, xy, x^i). \tag{5.11}$$

This leads to a non-minimal free resolution where one can indeed insert parameters to obtain the correct Γ-action as follows.

$$0 \to \begin{array}{c} \widetilde{R}(1,4n-i+1) \\ \oplus \\ \widetilde{R}(1,2n-i+2) \\ \oplus \\ \widetilde{R}(1,2n-i+1) \\ \oplus \\ \widetilde{R}(i,1) \end{array} \xrightarrow{B_i(s,t,u,v)} \begin{array}{c} \widetilde{R}(0,4n-i+1) \\ \oplus \\ \widetilde{R}(1,2n-i+2) \\ \oplus \\ \widetilde{R}(1,2n-i+1) \\ \oplus \\ \widetilde{R}(1,1) \\ \oplus \\ \widetilde{R}(i,0) \end{array} \to J_i(s,t,u,v) \to 0 \tag{5.12}$$

Here the map between the free modules is given by the matrix

$$B_i(s,t,u,v) = \begin{pmatrix} x & s & 0 & t \\ y^{2n-1} & 1 & 0 & 0 \\ u & y & 1 & 0 \\ 0 & 0 & y^{2n-i} & x^{i-1} \\ v & 0 & 0 & y \end{pmatrix} \tag{5.13}$$

and $\gamma \in \Gamma$ acts by

$$\begin{aligned} \gamma \cdot s &= \lambda(\gamma)\mu(\gamma)^{1-2n} s & \gamma \cdot t &= \lambda(\gamma)^{i}\mu(\gamma)^{i-4n} t \\ \gamma \cdot u &= \mu(\gamma)^{2n} u & \gamma \cdot v &= \lambda(\gamma)^{1-i}\mu(\gamma)^{4n-i+1} v. \end{aligned} \tag{5.14}$$

Similarly, for $i > 2n$ one should consider the non-minimal generating set $J_i = (y^{4n-i+1}, xy, x^{i-2n}y, x^{i-2n+1}y, x^i)$ and the free resolution

$$0 \to \begin{array}{c} \widetilde{R}(1, 4n-i+1) \\ \oplus \\ \widetilde{R}(i-2n, 1) \\ \oplus \\ \widetilde{R}(i-2n+1, 1) \\ \oplus \\ \widetilde{R}(i, 1) \end{array} \xrightarrow{B_i(s,t,u,v)} \begin{array}{c} \widetilde{R}(0, 4n-i+1) \\ \oplus \\ \widetilde{R}(1, 1) \\ \oplus \\ \widetilde{R}(i-2n, 1) \\ \oplus \\ \widetilde{R}(i-2n+1, 1) \\ \oplus \\ \widetilde{R}(i, 0) \end{array} \to J_i(s,t,u,v) \to 0 \tag{5.15}$$

where

$$B_i(s,t,u,v) = \begin{pmatrix} x & 0 & 0 & s \\ y^{4n-i} & x^{i-2n-1} & 0 & 0 \\ 0 & 1 & x & t \\ 0 & 0 & 1 & x^{2n-1} \\ u & 0 & v & y \end{pmatrix} \tag{5.16}$$

and $\gamma \in \Gamma$ acts by

$$\begin{aligned} \gamma \cdot s &= \lambda(\gamma)^{i}\mu(\gamma)^{i-4n} s & \gamma \cdot t &= \lambda(\gamma)^{2n} t \\ \gamma \cdot u &= \lambda(\gamma)^{1-i}\mu(\gamma)^{4n-i+1} u & \gamma \cdot v &= \lambda(\gamma)^{1-2n}\mu(\gamma) v. \end{aligned} \tag{5.17}$$

The degeneration locus of each of the matrices above, i.e. the closed subscheme of $\operatorname{Spec} k[x,y] \times_k \operatorname{Spec} k[s,t,u,v]$ defined by the ideal generated by the maximal minors, is a possibly non-flat family over $\operatorname{Spec} k[s,t,u,v]$. However, by theorem 3.10, the families defined by including only the parameters with negative $\mathbf{G}_m$-weights are flat and with fibres of constant length. So for each Γ-fixpoint I_0 (equal to I_{ij} or J_i), let r be the number of parameters with negative $\mathbf{G}_m$-weights and let $Z \subset \mathbf{A}^2 \times_k \mathbf{A}^r$ be the flat family over $\mathbf{A}^r$ defined by the corresponding matrix. Also recall that the set of monomials $\mathcal{B}$ outside I_0 maps to a basis in the coordinate ring of every fibre Z_p. Since the monomials are eigenvectors for the H-action, this proves that $k[Z_p] \cong k[Z_0] \cong k[H]^{\oplus 2}$ for every $p \in \mathbf{A}^r$. Thus the family defines a morphism $\mathbf{A}^r \to Y$. Moreover, by the Γ-equivariance of this morphism, the image is contained in the cell U surrounding the Γ-fixpoint considered (see remark 3.13). In sum, this construction gives morphisms

$$\varphi : \mathbf{A}^r \to U \tag{5.18}$$

equivariant under the Γ-actions on parameters defined above.

Proposition 5.5. *$\varphi : \mathbf{A}^r \to U$ is étale.*

Proof. Note that r is precisely the number of positive weights on $T_{\varphi(0)}(Y)$, which is the dimension of U according to theorem 3.2. Thus $d\varphi_0 : T_0(\mathbf{A}^r) \to T_{\varphi(0)}(U)$ is an isomorphism if it is injective, and this is enough to show étaleness arguing as in proposition 4.9.

So let $D = \operatorname{Spec} k[\varepsilon]/\varepsilon^2$ and let $\alpha \in \operatorname{Mor}(D, \mathbf{A}^2)$ be defined by the ring homomorphism $\alpha^\# : k[\mathbf{A}^r] \to k[\varepsilon]/\varepsilon^2$. Then α corresponds to a point in the Zariski tangent space to $\mathbf{A}^r$ at 0 if $\alpha^\#$ sends each parameter to some scalar times ε. Let I_0 be the ideal generated by $\varphi(0)$ in $k[\mathbf{A}^2 \times_k D] = k[x, y; \varepsilon]/\varepsilon^2$.

First consider the case where φ is defined by the matrix $A_{ij}(s, t, u, v)$, with some of the parameters s, t, u, v possibly zero. Then $d\varphi_0(\alpha)$ is the subscheme of $\mathbf{A}^2 \times_k D$ defined by the ideal generated by the maximal minors of the matrix obtained by applying $\alpha^\#$ to the parameters in $A_{ij}(s, t, u, v)$, that is

$$
\begin{aligned}
f_1 &= y^j + d\varepsilon x^{i-2} y^{i+j-2n-2} + c\varepsilon x^{2n-j+1} y \\
f_2 &= x y^{2n-i+2} + d\varepsilon x^{i-1} \\
f_3 &= x^2 y^2 \\
f_4 &= x^{2n-j+2} y - a\varepsilon y^{j-1} \\
f_5 &= x^i + b\varepsilon x y^{2n-i+1} + a\varepsilon x^{i+j-2n-2} y^{j-2}.
\end{aligned}
\tag{5.19}
$$

Then $d\varphi_0(\alpha) = 0$ if and only if the ideal thus obtained equals I_0, which is generated by the leading terms of the f_i. The minors f_2 and f_4 show that $d\varepsilon x^{i-1}$ and $a\varepsilon y^{j-1}$ are elements of I_0, thus $a = d = 0$. Similarly, f_1 and f_5 show that $b = c = 0$.

Now consider the case where $i \leq 2n$ and $B_i(s, t, u, v)$ defines the morphism φ. Then the family corresponding to $d\varphi_0(\alpha)$ is defined by the minors

$$
\begin{aligned}
f_1 &= y^{4n-i+1} - c\varepsilon y^{2n-i+1} - d\varepsilon x^{i-1} \\
f_2 &= x y^{2n-i+2} \\
f_3 &= x y^{2n-i+1} - a\varepsilon y^{4n-i} \\
f_4 &= xy - a\varepsilon y^{2n} \\
f_5 &= x^i - a\varepsilon x^{i-1} y^{2n-1} - b\varepsilon y^{4n-i}
\end{aligned}
\tag{5.20}
$$

and I_0 is generated by the leading terms. Then $d\varphi_0(\alpha) = 0$ if and only if $I_0 = (f_1, \ldots, f_5)$. Then f_3 gives $a = 0$, f_5 gives $b = 0$ and f_1 gives $c = d = 0$.

The case $i > 2n$ is similar. □

Corollary 5.6. *φ is an isomorphism $\mathbf{A}^r \cong U$.*

Proof. As in corollary 4.10, $\varphi : \mathbf{A}^r \to U$ is an open immersion, using étaleness and Γ-equivariance. By the Γ-equivariance, the image is U as in corollary 4.11. □

This ends the construction of the decomposition of Y.

5.2 Stratification of G-Hilb($\mathbf{A}^2$)

Now fix the one-parameter subgroup $\psi : \mathbf{G}_m \to \Gamma$ defined by $\lambda \circ \psi(t) = t^a$ and $\mu \circ \psi(t) = t^{a+\delta}$ with $a \ll \delta < 0$. The induced decomposition of Y is given

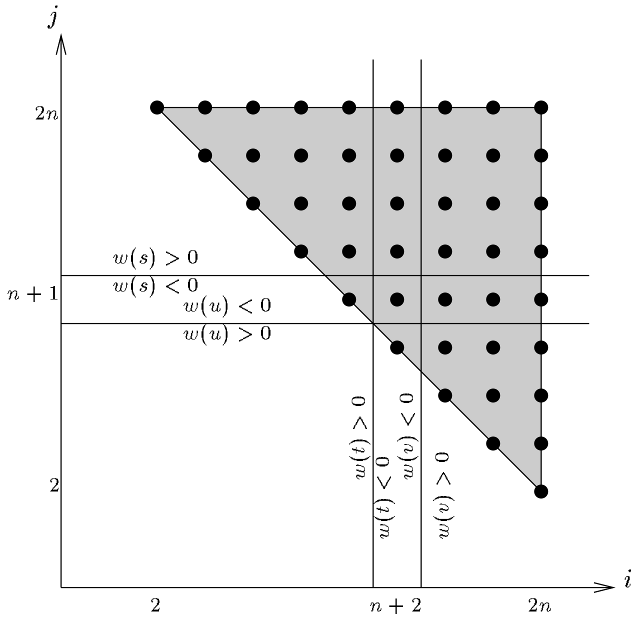

Figure 5.2: $\mathbf{G}_m$-weights for the action on Y

by the matrices in the previous section, one just has to decide the sign of the $\mathbf{G}_m$-weights on the parameters for each fixpoint. The particular one-parameter subgroup chosen has the advantage that this is rather easy.

The $\mathbf{G}_m$-weights $w(s), w(t), w(u), w(v)$ of the parameters may be calculated by substituting $\psi(t)$ for γ in the expressions (5.10), (5.14) and (5.17) defining the Γ-action in the various cases. First, for the fixpoints I_{ij} one finds

$$\begin{aligned} w(s) &= 2a(n-j+2)+\delta(2-j) \\ w(t) &= 2a(-n+i-1)+\delta(i-2n-1) \\ w(u) &= 2a(-n+j-1)+\delta(j-1) \\ w(v) &= 2a(n-i+2)+\delta(2n-i+2). \end{aligned} \tag{5.21}$$

Taking s as an example, the weight is negative if $n-j+2>0$ and positive if $n-j+2<0$. If $n-j+2=0$, then the weight is given by the δ-term, and then $\delta(2-j)=-\delta n>0$ so the weight is positive. In this way, the set of pairs (i,j) indexing the ideals I_{ij}, and hence the cells in the decomposition of Y, is divided into chambers according to the sign of each of the four $\mathbf{G}_m$-weights. This is shown in figure 5.2, where the dots are the allowed indices (i,j) and the horizontal and vertical lines show where the weights change sign. From the figure the dimension of the cells can be read off as the number of negative weights. In particular, the four-dimensional cell is the one containing $I_{n+2,n+1}$ and, with two exceptions, the τ-invariant fixpoints (on the diagonal $i=j$) lie in two-dimensional cells.

For the fixpoints J_i the situation is simpler. First, for $i\leq 2n$, substitution into equation (5.14) shows that the $\mathbf{G}_m$-weights are positive on s and t and negative on u and v. For $i>2n$, equation (5.17) gives positive $\mathbf{G}_m$-weights on

u and v and negative weights on s and t, with the sole exception $i = 2n + 1$, where $w(u)$ is negative also.

Now for the stratification, that is, determining the τ-invariant part of each cell. Note that although G-Hilb$(\mathbf{A}^2)$ is a component of Y^G, there might be other components. However, by [Nak99, Theorem 4.4], all such extra components are isolated points, so if $I \in Y^G$ belongs to a stratum of dimension ≥ 1, it is automatic that $I \in G$-Hilb$(\mathbf{A}^2)$. For zero-dimensional strata, one must check whether $k[x,y]/I$ is the regular representation or not. All cells belonging to the same chamber in figure 5.2 can be treated simultaneously, thus one obtains a stratification of the calculations!

In the following, for each Γ-fixpoint I_0, let $\mathcal{B}$ be the set of monomials outside I_0, mapping to a basis in $k[x,y]/I$ for every ideal in the cell containing I_0 according to theorem 3.10.

The 4-dimensional cell: $i = n + 2$ and $j = n + 1$

The cell containing $I_{n+2,n+1}$ is defined by the matrix $A_{n+2,n+1}(s,t,u,v)$, whose maximal minors are

$$\begin{aligned} f_1 &= y^{n+1} + (u+v)x^n y - (tuv)x^{n-1} \\ f_2 &= xy^n + (v)x^{n+1} - (su)y^{n-1} \\ f_3 &= x^2y^2 - (su+sv+tv)xy + (stuv) \\ f_4 &= x^{n+1}y - (tv)x^n + (s)y^n \\ f_5 &= x^{n+2} + (s+t)xy^{n-1} - (stu)y^{n-2}. \end{aligned} \tag{5.22}$$

Let I be a τ-invariant ideal defining the fibre over a point (s,t,u,v). Then

$$\tau f_3 - f_3 = 2(su+sv+tv)xy \in I \tag{5.23}$$

showing $su + sv + tv = 0$ since $xy \in \mathcal{B}$. Similarly,

$$\begin{aligned} vf_1 + (-1)^n \tau f_2 = {} & (uv + v^2 + (-1)^{n+1})x^n y \\ & - (tuv^2 + (-1)^n su)x^{n-1} \in I \end{aligned} \tag{5.24}$$

and since $x^n y$ and x^{n-1} belong to $\mathcal{B}$, the coefficients are zero. In particular $uv + v^2 + (-1)^{n+1} = 0$. Thus, the two equations

$$\begin{aligned} su + sv + tv &= 0 \\ uv + v^2 + (-1)^{n+1} &= 0 \end{aligned} \tag{5.25}$$

are necessary conditions for I to be τ-invariant. Conversely, this is sufficient, since then

$$\begin{aligned} \tau f_1 &= (-1)^n (u+v) f_2 \\ \tau f_2 &= (-1)^n (v) f_1 \\ \tau f_3 &= f_3 \\ \tau f_4 &= (-1)^{n+1} y f_2 + (-1)^n (v) f_4 \\ \tau f_5 &= (-1)^n y f_1 + (-1)^{n+1}(u+v)x^{n-2} f_3. \end{aligned} \tag{5.26}$$

This defines the morphism φ in theorem 5.1.

The exceptional fibre is as follows in this cell: The orbit defined by I is supported in the origin if and only if $stuv = 0$, which gives the two lines

$$s + t = 0, \quad u = 0, \quad v = \pm\sqrt{(-1)^n} \tag{5.27}$$

and the hyperbola

$$s = t = 0, \quad uv + v^2 + (-1)^{n+1} = 0. \tag{5.28}$$

Top left chamber: $i \leq n+1$ and $j \geq n+2$

These cells are defined by the matrices $A_{ij}(0, 0, u, v)$, so let I be a τ-invariant ideal generated by the minors

$$\begin{aligned} f_1 &= y^j + vx^{i-2}y^{i+j-2n-2} + ux^{2n-j+1}y \\ f_2 &= xy^{2n-i+2} + vx^{i-1} \\ f_3 &= x^2y^2 \\ f_4 &= x^{2n-j+2}y \\ f_5 &= x^i. \end{aligned} \tag{5.29}$$

Then $\tau f_5 = (-1)^i y^i \in I$ which is impossible since $y^i \in \mathcal{B}$, because $j > i$ in this chamber. So there are no invariant ideals in this chamber.

Bottom right chamber: $i \geq n+3$ and $j \leq n$

Again there are no τ-invariant ideals. The argument is entirely symmetric to the one for the top left chamber.

Top right chamber: $i \geq n+3$ and $j \geq n+2$

These cells are defined by the matrices $A_{ij}(0, t, u, 0)$, so let I be a τ-invariant ideal generated by the minors

$$\begin{aligned} f_1 &= y^j + (u)x^{2n-j+1}y \\ f_2 &= xy^{2n-i+2} \\ f_3 &= x^2y^2 \\ f_4 &= x^{2n-j+2}y \\ f_5 &= x^i + (t)xy^{2n-i+1} \end{aligned} \tag{5.30}$$

If $i > j$ or $i < j$, the elements τf_1 or τf_5 give impossible relations among the basis elements in $\mathcal{B}$. So assume $i = j$, then

$$f_5 - \tau f_1 = (t + (-1)^i u)xy^{2n-i+1} \in I \tag{5.31}$$

which shows, since $xy^{2n-i+1} \in \mathcal{B}$, that

$$t + (-1)^i u = 0. \tag{5.32}$$

Conversely, this condition is sufficient, since then

$$\begin{aligned} \tau f_1 &= f_5 & \tau f_4 &= (-1)^i f_2 \\ \tau f_2 &= -f_4 & \tau f_5 &= (-1)^i f_1 \\ \tau f_3 &= f_3. \end{aligned}$$

This defines the morphisms ψ_i for $i \geq n+3$ in theorem 5.1. Note that these cells are completely contained in the exceptional fibre.

Horizontal slice: $i \geq n+3$ and $j = n+1$

These cells are defined by the matrix $A_{ij}(s,t,u,0)$, so let I be a τ-invariant ideal generated by the minors

$$\begin{aligned} f_1 &= y^{n+1} + ux^n y \\ f_2 &= xy^{2n-i+2} - suy^{2n-i+1} \\ f_3 &= x^2y^2 - suxy \\ f_4 &= x^{n+1}y + sy^n \\ f_5 &= x^i + txy^{2n-i+1} + sx^{i-n-1}y^{n-1} - stuy^{2n-i}. \end{aligned} \tag{5.33}$$

Then $\tau f_2 = x^{2n-i+2}y - sux^{2n-i+1} \in I$ which is impossible since $x^{2n-i+2}y$ and x^{2n-i+1} belong to $\mathcal{B}$. So there are no τ-invariant ideals in this cell.

Vertical slice: $i = n+2$ and $j \geq n+2$

These cells are defined by the matrix $A_{ij}(0,t,u,v)$, so let I be a τ-invariant ideal generated by the minors

$$\begin{aligned} f_1 &= y^j + vx^n y^{j-n} + ux^{2n-j+1}y - tuvx^{2n-j} \\ f_2 &= xy^n + vx^{n+1} \\ f_3 &= x^2y^2 - tvxy \\ f_4 &= x^{2n-j+2}y - tvx^{2n-j+1} \\ f_5 &= x^{n+2} + txy^{n-1}. \end{aligned} \tag{5.34}$$

The element $\tau f_5 = (-1)^n y^{n+2} - tx^{n-1}y$ gives an impossible relation among elements in $\mathcal{B}$ unless $j = n+2$. Assuming this, one finds

$$\tau f_2 + f_4 = v(-1)^{n+1}x^{n+1} - tvx^{n-1} \in I \tag{5.35}$$

which shows that $v = 0$, since x^{n+1} and x^{n-1} are in $\mathcal{B}$. This case is then identical to the other cases on the diagonal $i = j$ in the "top right chamber", so the τ-invariant part is given by

$$v = 0, \quad t + (-1)^i u = 0 \tag{5.36}$$

and is entirely contained in the exceptional fibre. This defines the morphism ψ_i for $i = n+2$ of theorem 5.1.

The cell $i = n+1$ and $j = n+1$

This cell is defined by the matrix $A_{ij}(s,0,u,v)$, so let I be a τ-invariant ideal generated by the maximal minors

$$\begin{aligned} f_1 &= y^{n+1} + vx^{n-1} + ux^n y \\ f_2 &= xy^{n+1} + vx^n - suy^n \\ f_3 &= x^2y^2 - suxy - sv \\ f_4 &= x^{n+1}y + sy^n \\ f_5 &= x^{n+1} + sy^{n-1}. \end{aligned} \tag{5.37}$$

Then

$$f_1 + (-1)^n \tau f_5 = (v + (-1)^n s)x^{n-1} + ux^n y \in I \tag{5.38}$$

which is only possible if

$$u = 0, \quad v + (-1)^n s = 0. \tag{5.39}$$

Conversely, this is sufficient since then

$$\begin{aligned} \tau f_1 &= f_5 & \tau f_4 &= (-1)^{n+1} f_2 \\ \tau f_2 &= -f_4 & \tau f_5 &= (-1)^{n+1} f_1 \\ \tau f_3 &= f_3 \end{aligned}$$

and the exceptional fibre is given by the single point $s = u = v = 0$. This defines the morphism α in theorem 5.1.

The cell $i = n+2$ and $j = n$

This cell is defined by the matrix $A_{ij}(s,t,0,v)$, so let I be a τ-invariant ideal generated by the minors

$$\begin{aligned} f_1 &= y^n + vx^n \\ f_2 &= xy^n + vx^{n+1} \\ f_3 &= x^2y^2 - tvxy - sv \\ f_4 &= x^{n+2}y - tvx^{n+1} + sy^{n-1} \\ f_5 &= x^{n+2} + txy^{n-1} + sy^{n-2}. \end{aligned} \tag{5.40}$$

The element $\tau f_3 - f_3 = 2tvxy \in I$ shows that $tv = 0$ since $xy \in \mathcal{B}$. If $v = 0$ then $\tau f_1 = x^n \in I$, but $x^n \in \mathcal{B}$ so $v \neq 0$ and $t = 0$. Furthermore

$$f_1 - v\tau f_1 = (1 - (-1)^n v^2)y^n \in I \tag{5.41}$$

shows that $1 - (-1)^n v^2 = 0$ since $y^n \in \mathcal{B}$. Thus, necessary conditions for τ-invariance are

$$t = 0, \quad v = \pm\sqrt{(-1)^n} \tag{5.42}$$

and this is sufficient, since

$$\tau f_1 = (-1)^n v m_1 \tag{5.43}$$
$$\tau f_3 = f_3 \tag{5.44}$$
$$\tau f_5 = (-1)^{n+1} v x^{n-2} m_3 + (-1)^n y^2 m_1 \tag{5.45}$$

and $f_2 = x f_1$ and $f_4 = y f_5$. This defines the two morphisms $\beta_\pm$ in theorem 5.1.

The ideal is supported in the origin if and only if $sv = 0$, which gives the two points

$$s = t = 0, \quad v = \pm\sqrt{(-1)^n}. \tag{5.46}$$

The cells containing J_i with $i \leq 2n$

The cells are given by the matrix $B_i(0,0,u,v)$, so let I be a τ-invariant ideal generated by

$$\begin{aligned} f_1 &= y^{4n-i+1} - u y^{2n-i+1} - v x^{i-1} \\ f_2 &= x y^{2n-i+2} \\ f_3 &= x y^{2n-i+1} \\ f_4 &= xy \\ f_5 &= x^i. \end{aligned} \tag{5.47}$$

Then $\tau f_5 = (-1)^i y^i \in I$ which is impossible, since $y^i \in \mathcal{B}$. So there are no τ-invariant ideals in this cell.

The cells containing J_i with $i > 2n$

For $i > 2n+1$ the cells are defined by the matrix $B_i(s,t,0,0)$, and a similar argument to that above shows that there are no τ-invariant ideals.

If $i = 2n+1$ the cell is defined by $B_i(s,t,u,0)$, so let I be a τ-invariant ideal generated by the minors

$$\begin{aligned} f_1 &= y^{2n} - u x^{2n} + tu \\ f_2 &= xy - su \\ f_3 &= xy - su \\ f_4 &= x^2 y - sux \\ f_5 &= x^{2n+1} - s y^{2n-1} - tx \end{aligned} \tag{5.48}$$

where f_3 and f_4 of course are superfluous. Then $f_2 + \tau f_2 = 2su \in I$ showing $su = 0$. If $u = 0$ then $\tau f_1 = x^{2n} \in I$, but $x^{2n} \in \mathcal{B}$ so $s = 0$. In that case

$$x \tau f_1 + u y^{2n-1} f_2 - f_5 = t(u+1)x \in I \tag{5.49}$$

so $t = 0$ or $u = -1$. If $u = -1$ the ideal is invariant, since

$$\begin{aligned} \tau f_1 &= f_1 \\ \tau f_2 &= -f_2 \\ \tau f_5 &= x^{2n-1} f_2 - y f_1. \end{aligned} \tag{5.50}$$

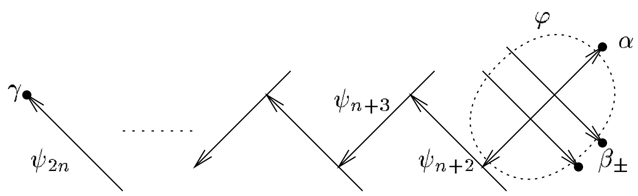

Figure 5.3: Stratification of G-Hilb($\mathbf{A}^2$)

Representation V	deg V	$\chi_V(\sigma)$	$\chi_V(\tau)$
V_0	1	1	1
V_1	1	1	-1
V_2	1	-1	1 for n even i for n odd
V_3	1	-1	-1 for n even $-i$ for n odd
W_r $(r = 1, \ldots, n-1)$	2	$\varepsilon^r + \varepsilon^{-r}$	0

Table 5.1: Character table for the binary dihedral group of order $4n$

If $t = 0$ and $u \neq -1$ then

$$\tau f_1 + u f_1 = (1-u^2)x^{2n} \in I \tag{5.51}$$

and since $x^{2n} \in \mathcal{B}$ and $u \neq -1$ one must have $u = 1$. And the ideal is then invariant.

So the invariant part of this cell has two components, namely the line $s = 0$ and $u = -1$, and the isolated point $s = t = 0$ and $u = 1$. But the isolated point may not belong to the component G-Hilb($\mathbf{A}^2$), and actually it does not. For let I be that point and let χ be the character of $k[x,y]/I$. If I belongs to G-Hilb($\mathbf{A}^2$), then $k[x,y]/I$ should be the regular representation, so $\chi(g) = 0$ for all $g \neq 1$. But using the monomial basis for $k[x,y]/I$ given above one verifies that $\chi(\tau) = 2$.

So an ideal in this cell is in G-Hilb($\mathbf{A}^2$) if and only if

$$s = 0, \quad u = -1. \tag{5.52}$$

In this case the generating set of the ideal reduces to $(xy, x^{2n} + y^{2n} - t)$. This defines the morphism γ of theorem 5.1. The ideal is supported at the origin if and only if $t = 0$.

These calculations conclude the proof of theorem 5.1.

5.3 Remarks

The stratification of G-Hilb($\mathbf{A}^2$) in theorem 5.1 is illustrated in figure 5.3, indicating the configuration of the exceptional components. A single arrow represents an $\mathbf{A}^1$ whereas the double arrow represents an $\mathbf{A}^1 \setminus \{0\}$. The two-dimensional cell given by φ is the dotted oval to the right, containing parts of

I	$\mathrm{Soc}(k[x,y]/I)$	representation
$\gamma(0)$	$\langle x^{2n}\rangle$	V_1
$\psi_{2n}(t), t \neq 0$	$\langle xy\rangle$	V_1
$\psi_i(t) \begin{cases} t \neq 0 \\ n+2 \leq i \leq 2n-1 \end{cases}$	$\langle x^{2n-i+1}y, xy^{2n-i+1}\rangle$	W_i
$\varphi(0,0,u,v) \begin{cases} uv + v^2 + (-1)^{n+1} = 0 \\ v \neq \pm\sqrt{(-1)^n} \end{cases}$	$\langle x^{n+1}, x^n y\rangle$	W_{n-1}
$\alpha(0)$	$\langle x^n y, xy^n\rangle$	W_{n-1}
$\varphi(s,-s,0,v) \begin{cases} s \neq 0 \\ v = \pm\sqrt{(-1)^n} \end{cases}$	$\langle vx^{n+1} + x^n y\rangle$	V_2 or V_3
$\beta_{\pm}(0)$	$\langle x^{n+1}y\rangle$	V_2 or V_3

Table 5.2: The irreducible representations $\mathrm{Soc}(k[\mathbf{A}^2]/I)$

three exceptional components. The rest of the exceptional fibre is contained in the one-dimensional strata given by ψ_i, with the exception of four points "at the edges". These are contained in one-dimensional strata that are left out for clarity, given by α, $\beta_{\pm}$ and γ.

Generators for all irreducible $\mathrm{Soc}(k[\mathbf{A}^2]/I)$ are shown in table 5.2, verifying the bijection in theorem 2.1(iii) in the D_n case. The representations V_i and W_i refer to the character table 5.1. A few comments are in order: In the last two rows there is a choice of sign which flips $\mathrm{Soc}(k[\mathbf{A}^2]/I)$ between the two representations V_2 and V_3. Which sign that gives which representation is dependent on whether n is even or odd. Furthermore, note that $W_i \cong W_{2n-i}$ such that the exceptional components given by $\psi_{n+2}, \dots, \psi_{2n-1}$ correspond to the representations $W_{n-2}, \dots, W_1$ in that order.

Corollary 5.7 (of theorem 5.1). *Conjecture 2.13 is true in the D_n case, i.e the multiplication map*

$$\mathfrak{m}/\mathfrak{m}^2 \otimes \mathrm{Soc}(k[x,y]/I) \to I/(\mathfrak{m}I + \mathfrak{n}) \tag{5.53}$$

for I in the exceptional fibre, is a projection when $\mathrm{Soc}(k[x,y]/I)$ is reducible.

Proof. Just write out the multiplication map in each case. To compute bases for the representations $I/(\mathfrak{m}I + \mathfrak{n})$ one should recall that the invariants of the binary dihedral group are

$$x^2y^2, \quad x^{2n} + y^{2n}, \quad xy(x^{2n} - y^{2n}) \tag{5.54}$$

so $\mathfrak{n}$ is generated by these. Also fix the basis $\mathfrak{m}/\mathfrak{m}^2 = \langle x, y\rangle$.

Case $I = \psi_{2n}(0)$: Writing out the generators one easily determines the bases

$$\mathrm{Soc}(k[x,y]/I) = \langle xy, x^{2n-1}, y^{2n-1}\rangle$$
$$I/(\mathfrak{m}I + \mathfrak{n}) = \langle x^{2n}, x^2y, xy^2\rangle.$$

Then the multiplication map is given by

$$x \otimes x^{2n-1} \mapsto x^{2n} \qquad x \otimes y^{2n-1} \mapsto 0 \qquad x \otimes xy \mapsto x^2y$$
$$y \otimes x^{2n-1} \mapsto 0 \qquad y \otimes y^{2n-1} \mapsto y^{2n} = x^{2n} \qquad y \otimes xy \mapsto xy^2.$$

and is thus a projection.

Case $I = \psi_i(0)$, $n+2 \le i < 2n$: Choose the bases

$$\begin{aligned} \mathrm{Soc}(k[x,y]/I) &= \langle x^{i-1}, y^{i-1}, x^{2n-i+1}y, xy^{2n-i+1} \rangle \\ I/(\mathfrak{m}I + \mathfrak{n}) &= \langle x^i, y^i, x^{2n-i+2}y, xy^{2n-i+2} \rangle. \end{aligned}$$

Then the multiplication map is given by

$$\begin{aligned} x \otimes x^{i-1} &\mapsto x^i & x \otimes x^{2n-i+1}y &\mapsto x^{2n-i+2}y \\ y \otimes x^{i-1} &\mapsto 0 & y \otimes x^{2n-i+1}y &\mapsto 0 \\ x \otimes y^{i-1} &\mapsto 0 & x \otimes xy^{2n-i+1} &\mapsto 0 \\ y \otimes y^{i-1} &\mapsto y^i & y \otimes xy^{2n-i+1} &\mapsto xy^{2n-i+2} \end{aligned}$$

and is thus a projection.

Case $I = \varphi(0,0,0,v)$, $v = \pm\sqrt{(-1)^n}$: Choose the bases

$$\begin{aligned} \mathrm{Soc}(k[x,y]/I) &= \langle x^{n+2}, x^n y, vx^n + y^n \rangle \\ I/(\mathfrak{m}I + \mathfrak{n}) &= \langle x^{n+1}y, x^{n+1} + vx^n y, xy^n + vx^{n+1} \rangle. \end{aligned}$$

Then the multiplication map is given by

$$\begin{aligned} x \otimes x^{n+1} &\mapsto 0 & x \otimes x^n y &\mapsto x^{n+1}y & x \otimes (vx^n + y^n) &\mapsto xy^n + vx^{n+1} \\ y \otimes x^{n+1} &\mapsto x^{n+1}y & y \otimes x^n y &\mapsto 0 & y \otimes (vx^n + y^n) &\mapsto y^{n+1} + vx^n y \end{aligned}$$

and is thus a projection.

These are all the reducible cases, since all other points I in the exceptional fibre are listed in table 5.2, where the socles are irreducible. □

The stratification in theorem 5.1 determines the tangent space to the exceptional fibre at all points I where $\mathrm{Soc}(k[\mathbf{A}^2]/I)$ is irreducible, except at the four points given by α, $\beta_\pm$ and γ. At these four points the theorem isn't strong enough to verify conjecture 2.8, but at every other point this can be done. To avoid too many calculations, consider only the points in the image of the ψ_i's.

Corollary 5.8 (of theorem 5.1). *Let I be a point in the image of ψ_i in theorem 5.1 such that* $\mathrm{Soc}(k[x,y]/I)$ *is irreducible. Then the tangent space to the exceptional fibre E at I is*

$$T_I(E) = \mathrm{Hom}_{k[G]}(I/\mathfrak{m}I, \mathrm{Soc}(k[x,y]/I)) \tag{5.55}$$

so conjecture 2.8 is true in this case.

Proof. Let I be the image of $p \in \mathbf{A}^1$ under ψ_i. The point p corresponds to a homomorphism $k[t] \to k$ sending t to some scalar a. A vector in the Zariski tangent space at p corresponds to a homomorphism $k[t] \to k[\varepsilon]/\varepsilon^2$ sending t to $a + b\varepsilon$ for some scalar b. The image under $d\psi$ is given by the family over $D = \mathrm{Spec}\, k[\varepsilon]/\varepsilon^2$ obtained by substituting $t \mapsto a + b\varepsilon$ in the matrix defining ψ_i, giving the ideal generated by

$$\begin{aligned} g_1 &= y^i + (-1)^{n+1}(a+b\varepsilon)x^{2n-i+1}y \\ g_2 &= xy^{2n-i+2} \\ g_3 &= x^2y^2 \\ g_4 &= x^{2n-i+2}y \\ g_5 &= x^i + (a+b\varepsilon)xy^{2n-i+1} \end{aligned} \tag{5.56}$$

Now the ideal $I = \psi_i(p)$ has generators $f_1, \ldots, f_5$ obtained by letting $\varepsilon = 0$ in the above expressions. Thus the element of $\mathrm{Hom}(I, k[x,y]/I)$ corresponding to the first order deformation above is given by

$$f_1 \mapsto bx^{2n-i+1}y, \quad f_5 \mapsto bxy^{2n-i+1} \tag{5.57}$$

sending the other generators to zero. Comparing with table 5.2 one finds that the image of this homomorphism is in $\mathrm{Soc}(k[x,y]/I)$. This shows the inclusion $T_I(E) \subset \mathrm{Hom}_{k[G]}(I/\mathfrak{m}I, \mathrm{Soc}(k[x,y]/I))$. Since the two vector spaces have the same dimension this is an equality. □

Chapter 6

Abelian subgroups of GL(2)

Let $G = C_{n,q} \subset \mathrm{GL}(2)$ be the cyclic group generated by

$$\sigma = \begin{pmatrix} \varepsilon_n^q & 0 \\ 0 & \varepsilon_n \end{pmatrix} \tag{6.1}$$

where $q < n$ are coprime integers and ε_n is a primitive n'th root of unity. Up to conjugation, these are all the abelian, finite, small subgroups of GL(2).

6.1 Fixpoints

As in the A_n-case, the actions of Γ and G on $\mathrm{Hilb}^n(\mathbf{A}^2)$ commute, so Γ acts on G-$\mathrm{Hilb}(\mathbf{A}^2)$. The discussion in chapter 4 gives a recipe for describing G-$\mathrm{Hilb}(\mathbf{A}^2)$ and in fact, everything works in this generalized setting.

The irreducible representations of G are of degree one, given by

$$\sigma \mapsto \varepsilon^k, \quad , k = 0, \ldots, n-1. \tag{6.2}$$

Thus the fixpoints for the Γ-action are the monomial ideals I of colength n such that every ε^k occurs exactly once among the eigenvalues ε^{qi-j} for the monomials $x^i y^j$ not in I. An example is shown in figure 6.1 in the case $n = 8, q = 5$: A monomial ideal is drawn as a staircase as before, but writing the number $qi - j$ in place of the monomial $x^i y^j$. The task is then to find all staircases such that every number $0, \ldots, n-1$ occurs exactly once inside the staircase. In the

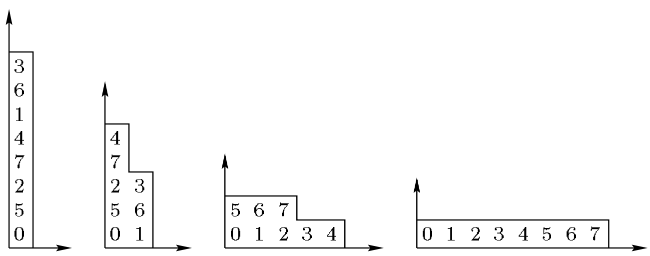

Figure 6.1: Torus fixpoints in the case $n = 8$ and $q = 5$

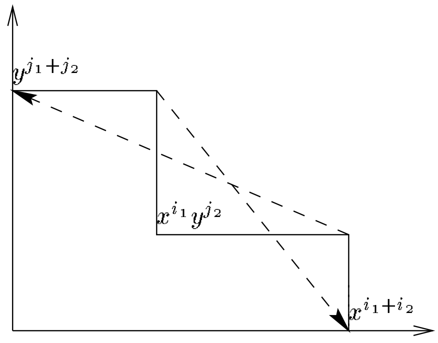

Figure 6.2: Torus fixpoints in general

example there are four such, shown in the figure. In general this leads to a combinatorial question which is not treated here.

Even without knowing the fixpoints explicitly, one can say just enough to continue the construction of G-Hilb$(\mathbf{A}^2)$.

Lemma 6.1. *The Γ-fixpoints in G-Hilb$(\mathbf{A}^2)$ can be written*

$$I = (y^{j_1+j_2}, x^{i_1}y^{j_2}, x^{i_1+i_2}) \tag{6.3}$$

for some integers i_1, i_2, j_1, j_2.

Proof. By proposition 2.7, the socle of $k[\mathbf{A}^2]/I$ is the sum of at most two irreducible representations. In the abelian case this means that $\operatorname{Soc} k[\mathbf{A}^2]/I$ is at most two-dimensional, thus there are at most two "steps" in the staircase picture. In other words there are three monomial generators, and they can be written in the above form. □

Such a torus fixpoint is depicted in figure 6.2. The two arrows indicate very natural candidates for the tangent space as a Γ-representation. However, without knowing explicitly i_1, i_2, j_1, j_2 it is somewhat hard to verify this directly. Nevertheless this guess turns out to be very true, see remark 6.5.

6.2 Affine charts

Following the construction of morphisms in section 3.2 one may consider, for each fixpoint I, the Γ-equivariant free resolutions

$$0 \to \begin{matrix} \widetilde{R}(i_1, j_1+j_2) \\ \oplus \\ \widetilde{R}(i_1+i_2, j_2) \end{matrix} \xrightarrow{A(u,v)} \begin{matrix} \widetilde{R}(0, j_1+j_2) \\ \oplus \\ \widetilde{R}(i_1, j_2) \\ \oplus \\ \widetilde{R}(i_1+i_2, 0) \end{matrix} \to I(u,v) \to 0 \tag{6.4}$$

where

$$A(u,v) = \begin{pmatrix} x^{i_1} & u \\ y^{j_1} & x^{i_2} \\ v & y^{j_2} \end{pmatrix} \tag{6.5}$$

and $\gamma \in \Gamma$ acts on u, v by

$$\begin{aligned}\gamma \cdot u &= \lambda(\gamma)^{i_1+i_2}\mu(\gamma)^{-j_1}u \\ \gamma \cdot v &= \lambda(\gamma)^{-i_2}\mu(\gamma)^{j_1+j_2}v.\end{aligned} \tag{6.6}$$

Lemma 6.2. *Let $I \in G$-Hilb$(\mathbf{A}^2)$ be a fixpoint as above, and let Γ act on* Spec $k[u,v]$ *as in equation 6.6. Then there exists a generic one-parameter subgroup $\psi : \mathbf{G}_m \to \Gamma$ with negative weights on $k[\mathbf{A}^2]$ such that the weights on u and v are negative.*

Proof. By direct computation: If $\lambda \circ \psi(t) = t^a$ and $\mu \circ \psi(t) = t^b$ with a and b two negative integers, then

$$\begin{aligned}t \cdot u &= t^{a(i_1+i_2)-bj_2}u \\ t \cdot v &= t^{-ai_j+b(j_1+j_2)}v\end{aligned} \tag{6.7}$$

showing that the weights are negative if and only if

$$\frac{j_1}{i_1+i_2} < \frac{a}{b} < \frac{j_1+j_2}{i_2} \tag{6.8}$$

and such a and b exist. □

By theorem 3.10 the matrix $A(u,v)$ defines a flat family over Spec $k[u,v]$ with fibres of constant length. Furthermore the basis $\mathcal{B}$ in that theorem consists of semi-invariant elements under the G-action, so the coordinate rings of the fibres are isomorphic to $k[G]$ as $k[G]$-modules. Thus a morphism

$$\varphi : \mathbf{A}^2 \to G\text{-Hilb}(\mathbf{A}^2) \tag{6.9}$$

is defined.

Lemma 6.3. *φ is étale.*

Proof. As before it is enough to show that $d\varphi_0$ is injective. Let $\alpha \in \mathrm{Mor}(D, \mathbf{A}^2)$ be given by the ring homomorphism $\alpha^{\#} : k[u,v] \to k[\varepsilon]/\varepsilon^2$ sending $u \mapsto a\varepsilon$ and $v \mapsto b\varepsilon$. Then $d\varphi_0$ sends α to the family in $\mathbf{A}^2 \times_k D$ defined by the minors

$$\begin{aligned}f_1 &= y^{j_1+j_2} - b\varepsilon x^{i_2} \\ f_2 &= x^{i_1}y^{j_2} \\ f_3 &= x^{i_1+i_2} - a\varepsilon y^{j_1}.\end{aligned} \tag{6.10}$$

If this equals the ideal generated by the initial terms then $a = b = 0$ since x^{i_2} and y^{j_1} is not in that ideal. □

Theorem 6.4. *Let $G \subset \mathrm{GL}(2)$ be a finite abelian subgroup. Then there are open immersions*

$$\varphi : \mathbf{A}^2 \to G\text{-Hilb}(\mathbf{A}^2) \tag{6.11}$$

for each fixpoint in 6.1, defined by the matrices $A(u,v)$ in equation (6.5)*, whose images form an open covering of G-Hilb$(\mathbf{A}^2)$.*

Proof. As in the A_n-case, the image of φ is the two-dimensional, open cell in the decomposition induced by the chosen subgroup $\mathbf{G}_m \to \Gamma$. And these cover G-Hilb$(\mathbf{A}^2)$. □

Remark 6.5. Since φ is Γ-equivariant under the action defined on u and v in equation (6.6), it follows that the tangent space at the fixpoint I is indeed as proposed in figure 6.2, that is $T = \lambda^{-i_1-i_2}\mu^{j_1} + \lambda^{i_1}\mu^{-j_1-j_2}$ in the representation ring of Γ.

One should note that this description coincides with the one constructed by Kidoh [Kid]. That description is based on the toric resolution of $\mathbf{A}^2/G$. In her work, the exponents i_1, i_2, j_1, j_2 are given explicitly by two continued fractions.

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